Nucleon Distribution in the Triton

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The triton bound state energy and wave function are derived from the pole of the solution of the inhomogeneous integral equation for the Faddeev component. In the present method, we can avoid a spurious or an unphysical solution. The Argonne v18 and the Paris potentials are adopted as a realistic two-nucleon local interaction. The three-nucleon force is not taken into account in the calculation. The nucleon distribution in the triton is investigated.

§ 1. Introduction

It is well known that the Faddeev equation in momentum space gives a good description for the three-nucleon (3N) scattering like the neutron-deuteron (n-d) elastic scattering and the deuteron breakup reaction induced by neutron. The Faddeev equation in continuum states is a set of inhomogeneous integral equations. However, the bound state problem was solved by a set of homogeneous integral equation by several authors.

The 3N bound state problem by the Faddeev equation with realistic 2N interactions has been investigated by many authors. For the solution of the problem, there are some methods; the coordinate representation, the momentum representation and the mixed representation. We consider only the momentum representation method in this paper, because the scattering problem of the 3N system is, in many cases, considered in momentum space. Beside the Faddeev equation approach, a variational calculation gives an excellent description for the triton bound state.

Tjon et al. have obtained 3N bound state energy by solving a set of inhomogeneous integral equations with negative energy for the Faddeev component, the form of which is the same as the case of scattering problem. In their pioneering work, only the S waves (singlet and triplet) have been taken into account in the calculation. That method of solution is based on iteration and the binding energy is determined from the position of the pole of the inhomogeneous integral equation, the pole of which is determined by the ratio of the successive terms in the iteration expansion. Let us call it the MT method. However, the method determining the eigenvalue (pole) does not lead to a correct binding energy even for a two-body problem in general when the two-body interaction includes a repulsive part. Hadjuk and Sauer, and Glöckle have already pointed out the above-mentioned fact. The methods of Refs. 6) and 10) are given at some length with the use of 2N model in § 3.

Hadjuk and Sauer, and Glöckle have solved a set of homogeneous integral equations to obtain 3N bound state wave functions and have modified the method of Ref. 6). In consequence, a spurious or an unphysical solution appeared besides the true ones. In order to eliminate the spurious or unphysical solution, they have paid a great deal of effort to modify the MT method. The spurious solution comes from
the MT method. However, to obtain the bound state energy, Glöckle has, in fact, used the Neumann series for a resolvent operator and the Padé method for the summation of the series. Therefore, his method gives a good description for the bound state energy. However, there is another method to obtain the unique solution of the homogeneous integral equation.

In $2N$ problem, it is well known that the $2N$ bound state can be obtained from the pole of the scattering equation (the Lippmann-Schwinger equation), which is an inhomogeneous integral equation, by the analytic continuation and the wave function is proportional to the residue at the pole. In a viewpoint of the analytic continuation, the $3N$ bound state should also be obtained from the inhomogeneous Faddeev equation and the wave function can be derived from the residue at the pole as well as $2N$ problem. So we use the inhomogeneous integral equation by Tjon et al. and the Padé method similar to Ref. 10 to obtain the real bound state energy and wave function. In order to have the wave function, we introduce the small imaginary part $\epsilon$ in the energy and take the residue.

The aim of this paper is to solve the triton bound state from the pole of the inhomogeneous Faddeev equation starting with the scattering state, to show that the residue of the solution at the pole satisfies the *homogeneous* integral equation and to investigate the proton and neutron densities in the triton.

In the next section, the outline to derive the triton wave function from the Faddeev equation is presented and it is shown that the wave function satisfies the *homogeneous* integral equation. In § 3, we treat the case of the $2N$ problem with a square well potential, in order to show that the present result coincides with the exact ones by the Schrödinger equation. In § 4, the present method is applied to the $3N$ problem with a realistic potential and we investigate the proton and neutron distributions of the triton. As the realistic potential, we adopt the Argonne (AV$_{14}$) and the Paris potentials. In the final section, a summary and the discussion of the present results are given.

§ 2. Faddeev equation and the triton wave function

First of all, we denote, as usual, the coordinates of $a$, $\beta$ and $\gamma$ components by $(1, (2, 3)), (2, (3, 1))$ and $(3, (1, 2))$, respectively. In $3N$ scattering problem, the initial asymptotic state vector in a certain component is denoted as $|\psi^{(i)}\rangle = |\Phi(i, (j, k))\rangle$, where the Roman letters $i$, $j$ and $k$ are one of the cyclic permutations of 1, 2 and 3. The state $|\Phi(i, (j, k))\rangle$ means that the particle $i$ is in a free state and the pair particles $j$ and $k$ are in the deuteron bound state. So the state $|\Phi(i, (j, k))\rangle$ should be antisymmetric with respect to the interchange of the particles $j$ and $k$. The symbols $\Phi^{(1)}$, $\Phi^{(2)}$ and $\Phi^{(3)}$ are used instead of the symbols $\Phi_1$, $\Phi_2$ and $\Phi_3$ in Ref. 2, respectively. From the above definition, we find that the state vectors $|\Phi^{(i)}\rangle$ $(i=1, 2$ and $3)$ have the following properties,

$$P_{ij}|\Phi^{(i)}\rangle = -|\Phi^{(j)}\rangle \ (k \neq i, j), \quad P_{ij}|\Phi^{(i)}\rangle = -|\Phi^{(j)}\rangle,$$  

(1)

where the symbol $P_{ij}$ means the exchange operator with respect to the particles $i$ and
j. These properties often appear in some kinds of state vectors.

Now, in order to make the complete antisymmetric state vector of the 3N system, we must solve the following Faddeev equation,

$$\begin{pmatrix}
|X^{(1)}(z)\rangle \\
|X^{(2)}(z)\rangle \\
|X^{(3)}(z)\rangle
\end{pmatrix}
=\begin{pmatrix}
\Phi^{(1)}(z) \\
\Phi^{(2)}(z) \\
\Phi^{(3)}(z)
\end{pmatrix}
+ G_0(z)
\begin{pmatrix}
0 & t_a(z) & t_b(z) \\
t_a(z) & 0 & t_b(z) \\
t_a(z) & t_a(z) & 0
\end{pmatrix}
\begin{pmatrix}
|X^{(1)}(z)\rangle \\
|X^{(2)}(z)\rangle \\
|X^{(3)}(z)\rangle
\end{pmatrix}$$

or

$$\begin{align}
|X^{(1)}(z)\rangle &= \Phi^{(1)}(z) + G_0(z)t_a(z)|X^{(2)}(z)\rangle + G_0(z)t_b(z)|X^{(3)}(z)\rangle,

|X^{(2)}(z)\rangle &= \Phi^{(2)}(z) + G_0(z)t_b(z)|X^{(3)}(z)\rangle + G_0(z)t_a(z)|X^{(1)}(z)\rangle,

|X^{(3)}(z)\rangle &= \Phi^{(3)}(z) + G_0(z)t_b(z)|X^{(1)}(z)\rangle + G_0(z)t_a(z)|X^{(2)}(z)\rangle,
\end{align}$$

where \( z = E + i\epsilon \). \( G_0 \) is the Green function of the free 3N system and the operators \( t_a, t_b \) and \( t_t \) are 2N \( t \) operators for the three pairs of particles \( (2, 3), (3, 1) \) and \( (1, 2) \), respectively. The vectors \( |X^{(n)}\rangle \) have also the same properties as the vectors \( |\Phi^{(n)}\rangle \) \((n=1, 2 \) and 3).

We can write down the above equations in the following form,

$$\begin{align}
|X^{(i)}(z)\rangle &= \Phi^{(i)}(z) + G_0(z)2\tilde{t}^{(i)}(z)|X^{(j)}(z)\rangle \\
&= |\Phi^{(i)}\rangle - G_0(z)2\tilde{t}^{(i)}(z)P_i |X^{(i)}(z)\rangle,
\end{align}$$

where

$$\begin{align}
2\tilde{t}^{(1)}(z) &= t_a(z)(1 - P_{2a}), \\
2\tilde{t}^{(2)}(z) &= t_a(z)(1 - P_{31}), \\
2\tilde{t}^{(3)}(z) &= t_r(z)(1 - P_{12}).
\end{align}$$

In order to obtain only the outgoing state, we must subtract the incident wave state \( |\Phi^{(i)}\rangle \) from the scattering state \( |X^{(i)}(z)\rangle \). Introducing the notation \( |\Psi^{(i)}(z)\rangle = |X^{(i)}(z)\rangle - |\Phi^{(i)}\rangle \) for \( i=1, 2 \) and 3, we have the more simplified form of the integral equations

$$(1 - K^{(i)}(z))|\Psi^{(i)}(z)\rangle = K^{(i)}(z)|\Phi^{(i)}\rangle$$

or

$$|\Psi^{(i)}(z)\rangle = K^{(i)}(z)(1 - K^{(i)}(z))^{-1}|\Phi^{(i)}\rangle,$$

where

$$K^{(i)}(z) = -G_0(z)2\tilde{t}^{(i)}(z)P_i.$$

At this stage, we can go on to the bound state problem, if the energy \( E \) is considered on the negative energy. In order to determine the bound state energy, Malfliet and Tjon introduced the parameter \( \lambda \) on \( K \) and expanded the right-hand side (rhs) of the above equation into the Neumann series,
where $|\Phi^{(i)}\rangle$ and $|\Psi^{(i)}\rangle$ are the eigenvectors of the total angular momentum $J$ and its component $M$. In their method, the angular momentum representation $|p_n, p_a, JM, a\rangle$ (see § 4) is made for each term of the equation and the ratio is taken as follows,

$$
\lim_{n \to \infty} \frac{\langle p_n, p_a, JM, a|n+1, J \rangle \langle p_n, p_a, JM, a|n, J \rangle}{\lambda(p_n, p_a, J, a, E)} = 1.
$$

But, any Padé approximation $P[n, m]$ can be used, if it converges. The bound state energy is determined from the condition $\mu=1$. The energy is independent of all $p_n, p_a$ and $a$. This is the MT method.

On the other hand, Glöckle have used the point-wise Padé method for the summation of the Neumann series $(1-K)^{-1}=1+K+K^2+\cdots$, because the Padé method is unrelated to the circle of convergence. If we write the angular momentum representation of Eq. (8) with $\lambda=1$ in the diagonal Padé approximation form $P[n, n]$, we have

$$
\langle p_n, p_a, JM, a|\Psi^{(i)}\rangle = \lim_{n \to \infty} P[n, n] = \frac{1}{\mu(p_n, p_a, J, a, E)}.
$$

But, any Padé approximation $P[n, m]$ can be used, if it converges. The bound state energy is determined from the condition $\mu(p_n, p_a, J, a, E)=0$. Such a zero point coincides with the zero point of the Fredholm determinant. Let us call this method the Padé method. Both the methods give the same bound state energy as long as the ground state is considered in a special case. This fact will be shown in an example of $2N$ bound state problem in the next section. In both the methods, the essential part is to search the singular point (pole) of the solution of the Faddeev integral equation, as is well known.

Once $r$ bound states with simple pole are determined, then the wave function can be written down as follows,

$$
\langle p_n, p_a, JM, a|\Psi^{(i)}\rangle = \sum_{n} \frac{C_n W^{(i)}_n}{E - E_n} + R(E),
$$

where the constants $C_n$ are arbitrary and $R$ is an entire function of $E$, which is unimportant at poles. After introducing a small imaginary part $ie$ to the bound state energy $E$, we again solve Eq. (6). Inserting the solution into Eq. (5), multiplying $Z-E_n$ to both sides and taking $\lim Z \to E_n$, we have the homogeneous integral equation $(1-K^{(i)}(z)) W^{(i)}_n = 0$ satisfied by the function $C_n W^{(i)}_n$. We can show the equation is equivalent to the homogeneous equation of Eq. (2) or (3). For example, in the case $i=1$, we have the equation

$$
W^{(i)}_n = K^{(i)} W^{(i)}_n
$$

$$
= - G_0(z) 2 \Gamma^{(i)} P_{12} W^{(i)}_n
= - G_0(z) t_a (1 - P_{23}) P_{12} W^{(i)}_n
= G_0(z) t_a (W^{(2)}_n + W^{(3)}_n),
$$
where we have made use of the property of Eqs. (1) and (7). Making the cyclic permutation 1, 2 and 3, and $\alpha$, $\beta$ and $\gamma$ for the above equation, we have just the homogeneous equation of Eq. (2) or (3). Then, normalizing the function, we have the antisymmetric state vector by $|W\rangle = |W^{(1)}\rangle + |W^{(2)}\rangle + |W^{(3)}\rangle$.

§ 3. Example of two-body bound state

First of all, we consider a simple square well $2N$ problem as an example, before the $3N$ problem. The scattering integral equation (the Lippmann-Schwinger equation) is derived in the following form,

\begin{equation}
 t = v + v g_0(z) t, \quad t = v \omega, 
\end{equation}

\begin{equation}
 (k-1)|\phi\rangle = k|\rho\rangle, 
\end{equation}

\begin{equation}
 |\phi\rangle = k(1-k)^{-1}|\rho\rangle, 
\end{equation}

where $|\phi\rangle = (\omega-1)|\rho\rangle$, $k = g_0(z)\omega$ and $z = \epsilon + i\epsilon$. These equations are the same form with Eqs. (5) and (6). In the scattering problem, $|\rho\rangle$ was chosen to be a solution of the asymptotic (or free) Hamiltonian. We still use the eigenket of the momentum. So the momentum representation of $|\rho\rangle$ should be chosen to be $\delta$ function. For simplicity, we consider a square well potential model to $v$, and only $S$ states. The depth and the range of the potential are 120 MeV and 2.9 fm, respectively. $\hbar^2/m = 41.47$ MeV fm$^2$ has been used in the present calculation. Of course this model can exactly be solved by the Schrödinger equation. In this model, two bound states appear at $e_1 = -86.946$ MeV and $e_2 = -2.998$ MeV corresponding to $1S$ and $2S$ states, respectively.

The energy eigenvalue of the integral equation should be on the negative real axis, because the potential $v$ is real. So we put $\epsilon = 0$, take $\epsilon < 0$ and solve the integral equation. The energy dependence of $\lambda(\rho, \epsilon)$ for this model by the MT method is displayed in Fig. 1. However, $\lambda$ is independent of $\rho$. The ground state energy surely coincides with the exact one. But, the $\lambda$ never goes to 1 at the $2S$ state energy $e_2$. This means that the condition $\lambda = 1$ gives only the lowest bound state energy of attractive potential without repulsive core.

Fig. 1. The energy dependence of $\lambda$ to a two-body square well potential is shown. $e_1 = -86.946$ MeV and $e_2 = -2.998$ MeV.
On the other hand, the energy dependence of $\mu(p, e)$ by the Padé method at $p=0.7593$ fm$^{-1}$ is displayed in Fig. 2. The zeros of $\mu(p, e)$ exactly coincide with the exact ones. So the Padé method is reliable to obtain the bound state wave functions. The function $w_n(p) = \lim_{\epsilon \to 0} (z - e_n) \phi(p)$ should be derived from the integral equation (14). The 1S and 2S state energy eigenvalues of the potential by the Padé method are obtained by interpolation.

The imaginary part of $(z - e_n) \phi_n(p)$ should theoretically be zero for all $p$ in the $\Delta$-region. However, in numerical calculation, $\epsilon$ cannot be put to zero. We assumed $\epsilon = 10^{-4}$ MeV, then the imaginary part of $(z - e_n) \phi_n(p)$ is, of course, not zero, but very small for all $p$ comparing with the real part. Then, taking the real part of it and normalizing the function, we have the real wave function $w_n(p)$. The 1S state wave function $w_1(p)$ in momentum space is displayed in Fig. 3, comparing with the exact one by the Schrödinger equation. $w_1(p)$ has been normalized in terms of the condition $\int_{-\infty}^{\infty} |w_1(p)|^2 dp = 1$. The agreement of the present result is excellent. The 2S function also agrees, although the result is not shown. The wave functions of the $S$ states in coordinate space of course coincide with the exact ones.

§ 4. Triton wave function

In order to expand the triton wave function into the partial wave form, we use the basic vector, as is defined in Ref. 2), in the $\alpha$ component

$$|l_1, \ast \rangle |l_2, \ast \rangle |l_3, s_1 \rangle (l_2, s_2) j_2, JM, (i_1, i_2) i, \tau \rangle$$
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\[ \equiv |(l, s_l)j_l, (l_a, s_a)j_{l_a}, JM, (i, i_a)i, r, *, *\rangle, \]

where the vector \( |l, *\rangle \) is defined depending on the representation,

\[ \langle l, r|l, r'\rangle = \delta(r - r'), \]

\[ \langle l, p|l, p'\rangle = \delta(p - p'), \]

\[ \langle l, r|l, p\rangle = i^l \sqrt{\frac{2}{\pi}} (rp)j_l(rp). \]

Then, the vectors \( |\Phi^{(1)}\rangle \) and \( |\Psi^{(i)}(z)\rangle \) which are the eigenvectors of the total angular momentum \( J \) and its component \( M \) can be expanded as follows,

\[ |\Phi^{(1)}\rangle = \int_0^\infty dp_1 \int_0^\infty dp_a \sum_a C_{1,a} \delta(p_1 - p_1) \delta(p_a - p_a)|JM, a\rangle, \]

\[ |\Psi^{(i)}(z)\rangle = \int_0^\infty dp_1 \int_0^\infty dp_a \sum_a \Psi^{(i)}(p_1, p_a, J, a, z)|JM, a\rangle, \]

where the function \( \Psi^{(i)}(p_1, p_a, J, a, z) \) is independent of \( M \) in the angular momentum representation, because of the Wigner-Eckart theorem. The coefficients \( C_{1,a} \) are arbitrary constants independent of the momenta \( p_1 \) and \( p_a \). So we can choose them to be unity. We must keep the condition \( l_a + i_a + t_a = \text{odd} \). The other vectors, \( |\Phi^{(2,3)}\rangle \) and \( |\Psi^{(2,3)}(z)\rangle \), are also expanded in the same way.

The binding energy of the triton was obtained by the Padé method. Once the binding energy was determined from the Neumann series Eq. (8), we introduce the small imaginary part \( i\epsilon \) in the energy as was done in § 2. Substituting Eqs. (19) and (20) into Eq. (6), we can solve \( \Psi^{(i)}(p_1, p_a, J, a, z) \), except for the normalization constant. Multiplying \( i\epsilon \), taking the real part of it and normalizing the real wave function, we can make the normalized state vector \( |W^{(i)}\rangle \). The antisymmetrized state vector for the triton is made as \( |W\rangle = |W^{(1)} + W^{(2)} + W^{(3)}\rangle \), as shown in § 2.

In order to investigate the convergence on the channel numbers, we have performed the 26 \( (j_a < 3) \), 34 \( (j_a \leq 4) \) and 42 \( (j_a \leq 5) \) channel calculations.

In \( n-d \) elastic scattering calculation, the short range part of the nuclear force was sensitive. For example, the analyzing powers \( A_y \) and \( iT_{11} \) were sensitive on \( LS(t = 1) \) force\(^{(13,14)} \) of the Paris potential. So, we modify the short range part of the odd \( LS(t = 1) \) potential to the same as in Ref. 13). Moreover, we newly introduce the multiplying constant \( \lambda_{AV}^{(1)} = 1.056 \) on the eleventh term of singlet even central potential. This modification gives rise to the \( 2N \) scattering length \( ^1a_{nn} = -23.75 \) fm, which was desirable to explain the final state interaction peak in the \( D(n, p)2n \) reaction\(^{(3,15)} \). We refer, hereafter, this potential to a modified Paris potential and display the binding energies of the triton for the AV\(_{14}\), the Paris and the modified Paris potentials in Table I.

**Table I.** Binding energies (in MeV) for the AV\(_{14}\), the Paris and the modified Paris potentials.

<table>
<thead>
<tr>
<th>number of channel</th>
<th>AV(_{14})</th>
<th>Paris</th>
<th>modified Paris</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>7.639</td>
<td>7.436</td>
<td>7.806</td>
</tr>
<tr>
<td>34</td>
<td>7.654</td>
<td>7.449</td>
<td>7.820</td>
</tr>
<tr>
<td>42</td>
<td>7.661</td>
<td>7.455</td>
<td>7.824</td>
</tr>
</tbody>
</table>
As seen from Table I, the binding energies for the 42 channel calculation are still $0.004 - 0.007$ MeV larger than the 34 channel ones. The present modified Paris potential gives about 0.37 MeV larger binding energy than the original one. Such a
difference was almost unrelated to the used channel numbers. This fact implies that the short range part of the nuclear force plays an important role in the 3N calculation. The 42 channel calculation gives considerably good convergent results. If we need more exact ones, the 50 channel calculation ($j_\pi \leq 6$) must be performed. Unlike the $n-d$ scattering problem, the convergence of the Faddeev calculation by partial wave decomposition is very slow. Unfortunately, the different methods$^{8,16,17}$ of calculation give a little different binding energies for the same potential.

Fig. 4. The proton distributions $\rho_p(r)$ in the triton by 42 channel calculation are shown. The solid, the dotted and the dashed curves show the results by the AV18, the Paris and the modified Paris potentials, respectively.

Fig. 5. The neutron distributions $\rho_n(r)$ in the triton by 42 channel calculation are shown. The crosses with error bar are the experimental data quoted from Ref. 19). The meaning of curves are the same as in Fig. 4.
We display the components of the wave function for the 42 channel calculation in Table II. The present results are consistent with the one by Bömelburg and Glöckle,\textsuperscript{18} although the potentials and the numbers of channel used are different.

As seen from Table II, there are three states for $l_1 = 0$. These states are the main components of the triton wave function. The other states $l_1 > 0$ are less than 1.2%, most of them come from the antisymmetrization of the wave function. The small differences of the components of the wave function lead to a large difference of binding energies.

Next, we display the proton density $\rho_p(r)$ in Fig. 4 and the neutron density $\rho_n(r)$ in Fig. 5 by 42 channel calculation. $r$ implies the nucleon position radius measured from the center of mass of the triton. The experimental data, which were analyzed from the world data by Friar et al.,\textsuperscript{19} in Fig. 5 are quoted from Ref. 19. Both the densities $\rho_p(r)$ and $\rho_n(r)$ have been normalized to unity.

We find from Figs. 4 and 5 that $\rho_p(r)$ is a little greater than $\rho_n(r)$ for small $r$ region ($r \leq 1.3$ fm). The behaviour of both densities for the small $r$ region depends on the potentials used. The central dip of the experimental data, as was shown in Fig. 5, was not reproduced for $\rho_n(r)$. But we get considerably larger $\rho_n(r)$ for the small $r$ region than the ones of Ref. 19. This is still left as a puzzle. Both the densities show almost the same behaviour, unrelated to potentials, for the large $r$ region ($r \geq 1.3$ fm).

\section*{§ 5. Summary and discussion}

In the previous section, we have shown the derivation of the triton wave function as well as $2N$ one. As shown in Table I, the 42 channel calculation gives considerably good convergence for binding energies. The modification of the Paris potential gives a sizable binding energy than the original one. The difference is unrelated to the numbers of channel used. As shown in Table II, in the triton, the three states with $l_1 = 0$ occupy about 90% and give main contribution. This means that $^1S_0$ and $^3S_1 - ^3D_1$ states of $2N$ interaction are main ones, as is well known. The calculated binding energies are different for each potential used, and largely deviated from the experimental one 8.482 MeV. $^3D_1$ state is diminished in $3N$ calculation. The higher partial wave component than $l_1 = 0$ comes from the antisymmetrization of the triton wave function. The large difference of theoretical binding energies and experimental one means that a $2N$ potential is still unsettled.

The exact charge form factor can be calculated from $\rho_p$ by the relativistic electron scattering theory.\textsuperscript{20}

In the present method, the original inhomogeneous Faddeev equation is deformed, but not modified. Only the energy was assumed to be negative one. The triton wave function can exactly be calculated from the residue of the Faddeev solution as well as the $2N$ bound state ones. A spurious or an unphysical solution did not appear. Therefore, the Faddeev equation used in the scattering problem can still be applied to the triton bound state. The $3N$ bound state can be obtained from the residue at the pole of the inhomogeneous Faddeev equation without any modification.
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