On the basis of Fermi acceleration in oblique shock waves, the spectral indices of accelerated particles are obtained. We adopt the test particle approximation and assume that particle distribution is isotropic in the local fluid frames. We calculate the spectral index from the average energy gain and the escape probability per one cycle incorporating magnetic mirror effects based on the adiabatic approximation. Our results apply to any value of shock speed and any obliqueness as far as the shock is subluminal. For a given obliqueness and the compression ratio of 4, the index decreases first with an increase of the shock speed, reaches a minimum value below 2.0, and then increases to infinity as the shock speed approaches the subluminal limit. As for the dependence on the obliqueness, the index decreases with an increase of the obliqueness reaching a minimum value around 1.84 in a highly oblique region and it increases to infinity at the subluminal limit.

§ 1. Introduction

Theory of the first order Fermi acceleration in shock waves of supernova remnants has been widely applied to explain cosmic rays below $10^{15}$ eV. The theory predicts a power law spectrum, of which index $\alpha$ is determined only by the compression ratio $r$ under simplified but reasonable assumptions. Even though the essential properties such as the observed power-law spectrum and energetics of cosmic rays are well explained by this picture, many problems are still addressed (see review of Drury, and Blandford and Eichler). While the Fermi acceleration brings about a universal spectral index $\alpha=2.0$ for typical astronomical shock waves ($r=4.0$), a variety of spectral indices of relativistic electrons have been reported from radio observations in supernova remnants. Another issue is the maximum energy of accelerated protons; although the maximum energy may be identified with the ‘knee’ energy $10^{15} - 10^{16}$ eV in the observed cosmic ray spectrum, the shock acceleration is too slow to accelerate protons to that energy in the lifetime of supernova remnants.

Several authors have proposed that an oblique shock is the key to solve such discrepancies. For oblique shocks, in which the direction of magnetic field is oblique to the shock normal, the motion of a charged particle is more complicated than that for parallel shocks, in which the field lies along the shock normal. A charged particle may be transmitted through the shock or reflected from it. In addition, its gyration center is shifted by drift motions caused by electromagnetic fields in the shock rest frame. Jokipii applied a transport theory and investigated mainly an effect of the drift along the shock front. He suggested that the rate of energy gain becomes as much as 100 times larger than that in the parallel shock. Although Jokipii discussed in the shock rest frame, it is more convenient to work in another frame called the ‘de-Hoffmann-Teller frame’ (see § 2 in this paper; hereafter HT frame). Drury discussed the oblique shock concisely using the HT frame in his review and suggested...
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that the field obliqueness does not affect the shape of the spectrum of accelerated particles, as far as \( \cos \phi_{\text{up}} \gg u_s/v \), where \( v \) is the velocity of a particle, \( u_s \) the speed of the shock, and \( \phi_{\text{up}} \) field angle referred to the shock normal in the upstream rest frame. Ostrowski\(^9\) investigated the acceleration in the oblique shock based on the individual particle picture as Bell\(^1\) had done for the parallel shock. The spectral index was calculated from the mean energy gain per one cycle and from the escape probability. In his paper, the conservation of magnetic moment (see § 2 in this paper) was applied at particles' crossing the shock front. Restricting the arguments within a first order of the fluid velocity in the HT frame, he concluded that the spectral indices of accelerated particles with various obliqueness are all the same as those for the parallel shock and that the acceleration time scale becomes shorter as the obliqueness of shock waves increases. On the other hand, Kirk and Heavens\(^10\) reached a different conclusion as far as the spectral index concerns. They solved the Boltzmann equation for the phase-space distribution function of particles in the HT frame assuming a pitch angle diffusion. They numerically obtained the distribution function and concluded that the spectrum becomes flatter monotonously with the obliqueness \( \phi_{\text{up}} \).

Several authors applied Monte Carlo simulation to examine an effect of the oblique magnetic fields in shock acceleration. Monte Carlo simulation is one of the advantageous methods to investigate the acceleration in the oblique shock by means of obtaining the phase-space distribution function explicitly. Making use of the Monte Carlo simulation, Takahara and Terasawa\(^11\) showed that in the oblique shocks the spectral index is smaller than that for the parallel shock and the acceleration time becomes shorter as the obliqueness increases. One of their objectives was to examine non-adiabatic effects when particles cross the shock fronts, but they found that the effect was not so large. Ostrowski\(^12,13\) investigated an effect of perturbation of magnetic field and found that the perturbation produces a non-monotonous change of spectral index. Baring et al.\(^14\) analyzed how injection and acceleration efficiency varies with the Mach number and the obliqueness. They reached a conclusion that the acceleration efficiency might be too low in oblique shock to explain cosmic ray production in most sources. A recent Monte Carlo simulation\(^15\) resulted in almost the same conclusions with earlier work of Takahara and Terasawa\(^11\) about the spectral indices and the acceleration time.

Although various analytic and numerical treatments have been applied to the issue of the acceleration in the oblique shock as mentioned above, it is still an open problem how flat the spectral index can be with \( u_s \) and \( \phi_{\text{up}} \). The spectral index is important to study a detail of acceleration mechanism of cosmic rays and to interpret various observational results. In this paper, we extend the analytic method of Ostrowski\(^9\) to full order of the fluid speed in the HT frame for subluminal shocks. We focus our attention on the spectral index which is calculated from the mean energy gain per one cycle and the escape probability. We obtain analytically the spectral index from them as a function of the shock speed \( u_s \), the obliqueness \( \phi_{\text{up}} \), and the compression ratio \( r \). We assume that a particle crosses the shock front adiabatically in the HT frame and that the particle is scattered elastically in the local fluid frames. We consider the particle distribution to become isotropic in the local fluid frames by a certain isotropic scattering, but do not specify the scattering process.
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unlike Kirk and Heavens\textsuperscript{10} who applied the pitch angle scattering. In § 2, we describe the de-Hoffmann-Teller frame (the HT frame) and the treatment of magnetic mirror effects, resulting in the conservation of magnetic moment and energy of the particle at the shock crossing. In § 3.1, we express the anisotropy in the phase-space distribution function in the HT frame. In §§ 3.2 and 3.3, we obtain the mean energy gain per one cycle and the escape probability, respectively by integrating the distribution function over the magnetic moment. The spectral indices for various shock speeds and obliquenesses are presented in § 4. When we obtain the results, the compression ratio, in reality the velocity ratio in the shock rest frame, is fixed at 4 which is typical for astronomical shock wave. Discussion and conclusions follow in subsequent sections.

§ 2. Basic assumptions

We adopt the test particle approximation in which energetic particles do not affect the shock dynamics and in which the shock is regarded as an infinitesimal discontinuity on a scale of particle's mean free path. Procedures we present in this section are basically the same as those of Ostrowski\textsuperscript{9} and Kirk and Heavens\textsuperscript{10}. We summarize mainly the de-Hoffmann-Teller frame and the magnetic mirror effect to ease the understanding of our investigation.

2.1. Frames of references

Consider a shock front traveling at a speed of $u_s$ into a plasma at rest with magnetic field of strength $B_{up}$ and of angle $\phi_{up}$ referred to the shock normal. We adopt the coordinate system such that the shock plane is embedded in the $y$-$z$ plane, the shock front moves to the minus $x$ axis, and magnetic field lies in the $x$-$y$ plane. We make a Lorentz transformation by $-u_s$ along the $x$ axis first. This frame, denoted by superscript $s$, is the shock rest frame. In this frame, crossing points of the magnetic field and the shock front move along the $y$ axis with a speed of $u_s\Gamma_s\tan\phi_{up}$, where $\Gamma_s=1/\sqrt{1-u_s^2/c^2}$. Oblique shocks can be conveniently classified into two types. If $u_s^2/c^2<\cos\phi_{up}$ (then $u_s\Gamma_s\tan\phi_{up}<c$), the shock is called subluminal, while if $u_s^2/c^2>\cos\phi_{up}$ (then $u_s\Gamma_s\tan\phi_{up}<c$), it is called superluminal. For the subluminal shock, we can make another Lorentz transformation toward minus $y$ axis by $u_s\Gamma_s\tan\phi_{up}$. The resulting frame is called the de-Hoffmann-Teller frame\textsuperscript{16} (hereafter referred to as the HT frame). As shown in the following, we can express all physical quantities in the HT frame with three parameters: $u_s$, $\phi_{up}$ and $r$. From now on we work in the HT frame, unless noted otherwise.

In the HT frame, electric field does not appear and the fluid motion is parallel to the magnetic field both in the upstream and downstream. The fluid speed in the upstream is given by

$$u_1=\frac{u_s}{\cos\phi_{up}}.$$  \hfill (1)

From the Rankine-Hugoniot relations, we find that the downstream plasma moves with a speed of
In this paper, subscripts of 1 and 2 indicate physical quantities in the upstream and the downstream, respectively. We customarily use the velocities $\beta_1$ and $\beta_2$ normalized by the speed of light.

Magnetic fields are also obtained from Lorentz transformation and Rankine-Hugoniot relations. We assume that the energy density of the magnetic field is much smaller than the kinetic energy of the plasma, and in consequence the magnetic field does not influence dynamics of the shock wave. The field strength in the upstream proves to be the same with that in the upstream rest frame; $B_1 = B_{up}$. The field strength of downstream is given by

$$B_2 = B_{up}\sqrt{1 + (r^2 - 1)\Gamma_0^2 \sin^2 \Phi_{up}},$$

and we define the ratio

$$b = \frac{B_2}{B_1} = \sqrt{1 + (r^2 - 1)\Gamma_0^2 \sin^2 \Phi_{up}}.$$

For superluminal shocks, we do not have the reference frame in which electric field globally disappears. We restrict our study to the subluminal case. If $u_0 \leq 0.3$, as is the case for supernova remnants, most shocks are regarded as subluminal except for an extraordinary oblique case.

### 2.2. Magnetic mirror effects

Now we consider the behavior of a particle in a varying magnetic field. If the space variation of the field is small compared to $vT$, where $v$ and $T$ denote the velocity and the gyro-period of the particle motion, respectively, it is well known that the magnetic moment, $p_\perp^2/B$, of the particle is adiabatically conserved, where $p_\perp$ denotes the perpendicular component of the particle momentum to the magnetic field. According to the simulation of non-adiabatic effects, conservation of the magnetic moment is approximately, or statistically, correct even if the magnetic field is discontinuous at the shock front. Thus, we assume in this paper the conservation of the magnetic moment at the shock crossing; we have

$$\frac{p_1^2(1-\mu_1^2)}{B_1} = \frac{p_2^2(1-\mu_2^2)}{B_2},$$

where $\mu$ is the pitch angle cosine. Owing to the absence of electric field, the energy of particle is also conserved. Then Eq. (5) is rewritten as

$$1 - \mu_1^2 = b(1 - \mu_2^2).$$

Since the magnetic field in the downstream is stronger than that of the upstream for oblique shocks, $B_2 > B_1$ (Eq. (4)), particles with a pitch angle smaller than $\mu_0$ in the upstream are reflected. The critical value of the pitch angle cosine is obtained from Eqs. (4) and (6) as
For reflected particles, the pitch angle cosine after reflection becomes
\[ \mu' = -\mu. \] (8)

On the other hand, a transmitted particle from the upstream to the downstream changes its pitch angle cosine to
\[ \mu_2 = \sqrt{1 - b(1 - \mu^2)} \] (9)
by Eq. (6). When a particle impinges from the downstream to the shock front, it is always transmitted to the upstream and its pitch angle cosine is changed to
\[ \mu_1 = -\sqrt{1 - \frac{1}{b}(1 - \mu_2^2)}. \] (10)

§ 3. Method

In this section, we calculate the mean energy gain per one cycle and the escape probability to obtain the spectral index based on the method of Bell\(^{11}\) who investigated the case of parallel shock and of Ostrowski\(^{9}\) who extended it to oblique shocks but restricted the argument within a first order of the fluid speeds. We develop this method to full order of the fluid speeds. First we postulate that the phase-space distribution function is isotropic in the local fluid frames, and calculate the mean energy gain per one cycle and the escape probability in the following subsection.

3.1. Distribution of \( \mu \) in the HT frame

A basic assumption in our consideration is that the phase-space distribution functions of particles in the local fluid frames are isotropic as a result of a certain isotropic scattering, although we do not specify the scattering process. Then the distribution function becomes anisotropic in the HT frame. The effects of the anisotropy become significant as the shocks become relativistic (see, e.g., Peacock\(^{17}\)). In the oblique shock of our interest, since the speed of the fluid in the HT frame increases with obliqueness as seen in Eqs. (1) and (2), we have to take into account the anisotropy in the HT frame. The distribution in pitch angle is especially important because the magnetic mirror effect works in a different way upon particles with different \( \mu \), as seen in § 2.2. In this subsection, we investigate the anisotropy in the HT frame using an invariance of the phase-space distribution function.

We start with the isotropic phase-space distribution function in the local fluid frame both for upstream and downstream. We can write the distribution function near the shock front as
\[ f_s(p) = g_s(p), \] (11)
where subscript \( f \) denotes quantities in the local fluid frame. Then the number density in the local fluid frame is obtained by

\[ \mu_0 = \sqrt{1 - \frac{1}{b}}. \] (7)
From the invariance of the phase-space distribution function, the distribution function in the HT frame equals that of the fluid frame, i.e.,

$$f(p) = f(p_n).$$  \( (13) \)

Using Eq. (11) and a Lorentz transformation, we can write as

$$f(p) = g(\gamma u, p(1 - \beta u \mu)),$$  \( (14) \)

where \( \beta u \) is the fluid speed normalized by the speed of light and \( \gamma u \) is the Lorentz factor of the fluid. We assume that particles are relativistic. The number density in the HT frame is obtained by

$$n = \int f(p) d^3 p = 2\pi \int_0^\infty g(p_n) \frac{p_n^2}{\gamma u^2(1 - \beta u \mu)^2} \frac{\partial f(p, \mu)}{\partial(p, \mu)} dp_n d\mu$$

$$= 2\pi \int_0^\infty g(p_n) p_n^2 dp_n \int_{-1}^1 \frac{d\mu}{\gamma u^3(1 - \beta u \mu)^3}$$

$$= \gamma u n_0.$$  \( (15) \)

In the last equality of Eq. (15), we apply Eq. (12).

Consider a net flux parallel to a magnetic field (or along the fluid motion). Integrating over momentum from 0 to infinity and over pitch-angle cosine from -1 to 1, we obtain

$$F_{\parallel} = \int \frac{c^2 p_{\parallel} f(p)}{E} d^3 p = 2\pi c \int_0^\infty g(p_n) p_n^2 dp_n \int_{-\mu}^{\mu} \frac{d\mu}{\gamma u^3(1 - \beta u \mu)^3}$$

$$= n_0 \gamma u c \beta u.$$  \( (16) \)

The second step of Eq. (16) makes us confirm a probability that a particle has a pitch-angle cosine in a range of \( [\mu, \mu + d\mu] \) is

$$dP(\mu) \propto \frac{\mu d\mu}{(1 - \beta u \mu)^3}.$$  \( (17) \)

This probability distribution is the same as that indicated by Peacock\(^{17}\) and is a consequence of isotropic scattering in the local fluid flame.

For the upstream, we normalize the probability for \( 0 \leq \mu \leq 1 \) for a later convenience and obtain for pitch angle cosine distribution of impinging particles from upstream

$$dP(\mu) = 2(1 - \beta_1)^2 \frac{\mu_1 d\mu_1}{(1 - \beta_1 \mu_1)^3}.$$  \( (18) \)

In a limit of \( \beta_1 \to 0 \), this reduces to

$$dP_0(\mu) = 2\mu_1 d\mu_1.$$  \( (19) \)

For the downstream, we normalize the probability for \( -1 \leq \mu \leq 0 \) and obtain for pitch
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angle cosine distribution of impinging particles from downstream

\[ dP_2(\mu_2) = -2(1+\beta_2)^2 \frac{\mu_2 d\mu_2}{(1-\beta_2 \mu_2)^3}. \]  

(20)

In a limit of \( \beta_2 \to 0 \), this reduces to

\[ dP_2^0(\mu_2) = -2\mu_2 d\mu_2. \]  

(21)

The limiting forms of these probabilities are as expected for non-relativistic shocks.

Using Eqs. (7) and (18), we can obtain the transmission probability of a particle impinging from the upstream as

\[ P_t = \int_{\mu_0}^{1} dP_1(\mu_1) = 2(1-\beta_1)^2 \int_{\mu_0}^{1} \frac{\mu_1 d\mu_1}{(1-\beta_1 \mu_1)^3} = \frac{1-\mu_0^2 + 2\beta_1 \mu_0 (\mu_0 - 1)}{(1-\beta_1 \mu_0)^2}. \]  

(22)

For the reflection probability, we have

\[ P_{ref} = \int_{\mu_0}^{1} dP_1(\mu_1) = 2(1-\beta_1)^2 \int_{\mu_0}^{1} \frac{\mu_1 d\mu_1}{(1-\beta_1 \mu_1)^3} = \mu_0^2 \frac{(1-\beta_1)^2}{(1-\beta_1 \mu_0)^2}. \]  

(23)

In the limit of \( \beta_1 \to 0 \) they reduce to

\[ P_{t}^0 = 1 - \mu_0^2 \]  

(24)

and

\[ P_{ref}^0 = \mu_0^2, \]  

(25)

respectively, which are the same as those are obtained by Ostrowski.9

3.2. Energy gain per one cycle

Consider a particle which approaches the shock front from the upstream. To obtain the average energy gain per one cycle we compare the incident energy with the resultant energy after one cycle. This cycle consists of two separate paths. One path is that the particle is reflected at the shock and is scattered in the upstream to be directed to the shock front again, as shown in Fig. 1(a). We call this path the 'reflected' case. In the other path, the particle is transmitted to the downstream through the shock, is scattered in the downstream to be pointed to the shock front, is transmitted through the shock again, and is scattered in the upstream to be directed to the shock front, as shown in Fig. 1(b). We call this path the 'transmitted' case. In the transmitted case, the particle escape to infinity in the downstream with a finite probability. We investigate the escape probability in the next subsection. We here consider only particles returning to the shock front. Averaging over their pitch angle distribution for these two paths, the mean energy gain is obtained.

First we analyze the reflected case. Suppose that the particle has an energy \( E_A \) and a pitch-angle cosine \( \mu_A \) with \( 0 < \mu_A < \mu_0 \) when it impinges on the shock front from the upstream. The particle is consequently reflected at the shock front because of the magnetic mirror effect. According to the discussion in § 2.2, its energy and pitch angle after the reflection (denoted by the subscript \( A' \)) are related to the initial values as
respectively. Then the particle scattered in the upstream to be directed to the shock front as shown in Fig. 1(a). In scattering, the energy of the particle does not change in the local fluid frame, i.e., we obtain the energy of the particle when it impinges on the shock front second time as

$$E_{\gamma}^f = E_{\alpha}^f,$$  \hspace{1cm} (28)

where superscript $f$ denotes a quantity in the upstream fluid frame. From the Lorentz transformations, we have

$$E_{\gamma} = \gamma(1 - \mu_{\gamma}\beta_1)E_{\alpha} = \gamma(1 + \mu_{\alpha}\beta_1)E_{\alpha},$$  \hspace{1cm} (29)

and for the reflected particle

$$E_{\beta}^r = \gamma(1 - \mu_{\beta}\beta_1)E_{\beta}.$$  \hspace{1cm} (30)

Thus we obtain the energy gain $\Delta E_{\text{ret}} = E_{\beta} - E_{\alpha}$ and the fractional gain per one cycle as

$$\frac{\Delta E_{\text{ret}}}{E_{\alpha}} = \frac{1 + \mu_{\beta}\beta_1}{1 - \mu_{\beta}\beta_1} - 1,$$  \hspace{1cm} (31)

where we use Eqs. (26) through (30). Using the pitch-angle distribution given by Eq. (18), we average the fractional gain over $\mu_{\alpha}$ and $\mu_{\beta}$ to obtain

$$\left\langle \frac{\Delta E_{\text{ret}}}{E} \right\rangle_\mu = \int_{\mu_{\alpha}}^{\mu_{\beta}} d\mu_{\alpha} \int_{\mu_{\beta}}^{\mu_{\beta}} d\mu_{\beta} \frac{dP_1(\mu_{\alpha})}{d\mu_{\alpha}} \frac{dP_1(\mu_{\beta})}{d\mu_{\beta}} \frac{\Delta E_{\text{ret}}}{E_{\alpha}},$$

$$= 4(1 - \beta_1)^4 \int_{\mu_{\alpha}}^{\mu_{\beta}} d\mu_{\alpha} \int_{\mu_{\beta}}^{\mu_{\beta}} d\mu_{\beta} \frac{\mu_{\alpha}}{(1 - \beta_1\mu_{\alpha})^3} \frac{\mu_{\beta}}{(1 - \beta_1\mu_{\beta})^3} \left( \frac{1 + \mu_{\alpha}\beta_1}{1 - \mu_{\beta}\beta_1} - 1 \right),$$

$$= 4(1 - \beta_1)^4 \int_{\mu_{\alpha}}^{\mu_{\beta}} \frac{\mu_{\alpha}(1 + \mu_{\alpha}\beta_1)}{(1 - \beta_1\mu_{\alpha})^3} d\mu_{\alpha} \int_{\mu_{\beta}}^{\mu_{\beta}} \frac{\mu_{\beta}}{(1 - \beta_1\mu_{\beta})^4} d\mu_{\beta} - P_{\text{ret}}.$$  \hspace{1cm} (32)

Next we examine the transmitted case. Consider the particle which has an energy $E_{\alpha}$ and a pitch angle $\mu_{\alpha}$ with $\mu_{\beta} < \mu_{\alpha} < 1$ in the upstream and is consequently transmitted to the downstream. According to the discussion in § 2.2, its energy and pitch angle after the transmission (denoted by the subscript C) are related to the initial values as

$$E_{\gamma} = E_{\alpha}$$  \hspace{1cm} (33)

and

$$\mu_{\gamma} = -\mu_{\alpha},$$  \hspace{1cm} (27)
and

$$\mu_c(\mu_a) = \sqrt{1 - b(1 - \mu_a^2)}, \quad (34)$$

respectively. Equation (34) is obtained from Eq. (9). The particle is scattered to be pointed to the shock front. In the scattering, the energy of the particle is conserved in the local fluid frame. Then we have

$$E_c^I = E_b^I,$$  \quad (35)

where subscript D denotes quantities of the particle when it impinges to the shock front from the downstream and superscript f denotes a quantity in the downstream fluid frame. Its energy is related to the initial energy through Lorentz transformations as

$$E_b = \frac{1 - \mu_c \beta_2}{1 - \mu_b \beta_2} E_c = \frac{1 - \mu_c(\mu_a) \beta_2}{1 - \mu_b \beta_2} E_A.$$  \quad (36)

Then the particle is transmitted from the downstream to the upstream and have an energy and a pitch angle denoted by subscript A' as

$$E_{A'} = E_b$$  \quad (37)

and

$$\mu_{A'}(\mu_c) = -\sqrt{1 - \frac{1}{b}(1 - \mu_c^2)}, \quad (38)$$

respectively. The particle keeps the same way as the reflected particle after there. Therefore, we have

$$E_b^I = E_{A'}^I,$$  \quad (39)

where f represents the upstream fluid frame, and

$$E_b = \frac{1 - \mu_{A'} \beta_1}{1 - \mu_b \beta_1} E_{A'} = \frac{1 - \mu_c(\mu_0) \beta_1}{1 - \mu_b \beta_1} E_D.$$  \quad (39)

We can attain the fractional energy gain for the transmitted particle in one cycle from Eqs. (36) and (39) as

$$\frac{\Delta E_{tr}}{E_A} = \frac{1 - \mu_a(\mu_0) \beta_1}{1 - \mu_b \beta_1} \frac{1 - \mu_c(\mu_0) \beta_2}{1 - \mu_b \beta_2} - 1.$$  \quad (40)

Averaging over $\mu_a$, $\mu_0$ and $\mu_b$, we obtain the average energy gain as

$$\langle \frac{\Delta E_{tr}}{E} \rangle = \frac{d\mu_a}{d\mu_a} \int_{\mu_a}^{0} \frac{d\mu_b}{d\mu_b} \int_{\mu_b}^{1} \frac{d\mu_c}{d\mu_c} \frac{dP_a(\mu_a)}{d\mu_a} \frac{dP_b(\mu_0)}{d\mu_0} \frac{dP_b(\mu_b)}{d\mu_b} \frac{\Delta E_{tr}}{E_A}$$

$$= -8(1 - \beta_1)(1 + \beta_2)^2 \int_{\mu_a}^{1} d\mu_a \int_{\mu_b}^{0} d\mu_b \int_{\mu_b}^{1} d\mu_b$$

$$\times \frac{\mu_a}{(1 - \beta_1 \mu_a)^3} \frac{\mu_0}{(1 - \beta_2 \mu_0)^3} \frac{\mu_b}{(1 - \beta_1 \mu_b)^3}$$
Summing up the average energy gain of the reflected particles (32) and the transmitted particles (41), we obtain the average energy gain per one cycle $\Delta E=\xi E$ as

$$\xi=\left\langle \frac{\Delta E}{E} \right\rangle = \left\langle \frac{\Delta E_{\text{tr}}}{E} \right\rangle + \left\langle \frac{\Delta E_{\text{ref}}}{E} \right\rangle = 4(1-\beta_1)^4 B[A_1-2(1+\beta_2)^2 A_2 D]-1,$$  \hfill (42)

where

$$A_1=A_1(u_0, \Phi_{\text{up}}, r)=\int_{\mu_0}^{\mu_0} \frac{\mu_0(1+\mu_0 \beta_1)}{(1-\beta_1 \mu_0)^3} d\mu_0,$$  \hfill (43)

$$A_2=A_2(u_0, \Phi_{\text{up}}, r)=\int_{\mu_0}^{\mu_0} \frac{\mu_0(1-\mu_0 \beta_2)}{(1-\beta_1 \mu_0)^3} d\mu_0,$$  \hfill (44)

$$B=B(u_0, \Phi_{\text{up}}, r)=\int_{\mu_0}^{\mu_0} \frac{\mu_0 d\mu_0}{(1-\beta_1 \mu_0)^3}$$  \hfill (45)

and

$$D=D(u_0, \Phi_{\text{up}}, r)=\int_{\mu_0}^{\mu_0} \frac{\mu_0(1-\mu_0 \beta_2)}{(1-\beta_2 \mu_0)^3} d\mu_0.$$  \hfill (46)

An integration of $A_1$ and $B$ are relatively easy to accomplish. We take partial fraction expansions of the integrands and integrate separately. The results are

$$A_1(u_0, \Phi_{\text{up}}, r)=\frac{P_{\text{ref}}}{2(1-\beta_1)^3} - \frac{1}{\beta_1^2} \ln(1-\mu_0 \beta_1) + \frac{3\mu_0^2 \beta_1 - 2\mu_0}{2\beta_1(1-\mu_0 \beta_1)^2},$$  \hfill (47)

and

$$B(u_0, \Phi_{\text{up}}, r)=\frac{3-\beta_1}{6(1-\beta_1)^3}.$$  \hfill (48)

As for $A_2$ and $D$, mathematical expressions are more complicated, and we show the manipulation in the Appendix. The results are

$$A_2(u_0, \Phi_{\text{up}}, r)=\frac{1}{2(1-\beta_1)^3} \left[ P_{\text{tr}} + \frac{\beta_2}{\beta_1^2} \left( 1 - 2\beta_1 + \frac{1-\beta_1}{1-\beta_1^2 \mu_0^2} \right) \right]$$

$$- \frac{1}{\sqrt{1-\mu_0^2}} \frac{\beta_2}{\beta_1^2} \ln \left( \frac{\mu_0}{1+\sqrt{1-\mu_0^2}} \right) - \frac{3\mu_0^2 \beta_1^2 - 2}{2(1-\beta_1^2 \mu_0^2)^{3/2} \sqrt{1-\mu_0^2}} \frac{\beta_2}{\beta_1^3}$$

$$\times \ln \left[ \frac{\mu_0(1-\beta_1)}{|1-\beta_1 \mu_0^2 + \sqrt{1-\beta_1 \mu_0^2} \sqrt{1-\mu_0^2}|} \right].$$  \hfill (49)
and

\[
D(\mu_s, \Phi_{up}, r) = -\frac{3 + \beta_2^2}{6(1 + \beta_2^2)} + \frac{\beta_1}{6\beta_2^2} \left\{ \frac{5\beta_2^2 \mu_0^2 + 2}{1 - \mu_0^2} \left( \frac{\beta_2^2 \mu_0^2}{1 - \mu_0^2} + 1 \right)^2 \left( \mu_0^2 - \frac{1}{1 + \beta_2^2} \right) \right. \\
\left. - \frac{3\beta_2^2 \mu_0^2 + 2}{\beta_2^2 \mu_0^2 + 1} \left[ \mu_0 - \frac{1}{(1 + \beta_2^2)^2} \right] + 2 \left[ \mu_0 - \frac{1}{(1 + \beta_2^2)^3} \right] \right\}
\]

On the other hand, since the downstream fluid moves away from the shock, mean motion of particles also directs away from the shock. Consider a plane parallel to the shock front in the downstream. The escape probability for a particle impinging on the shock front from the upstream is expressed by

\[
P_{esc} = P_{tr} \frac{F_2^{\text{net}}}{F_2},
\]

where \(F_2\) is the flux through that plane due to particles moving to the positive \(x\) axis and \(F_2^{\text{net}}\) is the net flux through that plane (obtain from Eq. (16)); they are given by

\[
F_2 = \cos \Phi_2 \int_0^{\gamma_2 \beta_2} dp_2 \int_0^{\gamma_2 \beta_2} d\mu_2 \frac{c^2 p_2}{E_2} \frac{\partial \Pi_{\parallel}}{\partial \gamma_2 \beta_2} (p_2, \mu_2)
\]

\[
= \frac{1}{4} n_{e2} c \cos \Phi_2 \frac{\tau_2}{\gamma_2^3 (1 + \beta_2)^2},
\]

where \(\Phi_2\) is field angle referred to the shock normal in the HT frame, and

\[
F_2^{\text{net}} = \cos \Phi_2 F_{2\parallel} = n_{e2} \cos \Phi_2 \gamma_2 c \beta_2,
\]

respectively. Substituting Eqs. (22), (52) and (53) for Eq. (51), we obtain

\[
P_{esc} = \frac{1 - \mu_0^2 + 2\beta_1 \mu_0 (\mu_0 - 1)}{(1 - \beta_1 \mu_0)^2} \frac{4\beta_2}{(1 + \beta_2)^2}.
\]

### § 4. Results

In the previous section, we have obtained the proportional coefficient \(\xi\) for the average energy gain and the escape probability \(P_{esc}\) for each cycle. As is well known...
(briefly summarized in Gaisser, if $\xi$ and $P_{esc}$ are independent of the energy of particles, the differential energy spectrum of the particles can be written as a power law spectrum $N_p(E)dE \propto E^{-\alpha}dE$, where

$$\alpha = -\frac{\ln(1-P_{esc})}{\ln(1+\xi)} + 1.$$  

(55)

In this section, we calculate the spectral index using Eq. (55) and represent resultant spectral indices. First we give a result for a non-relativistic case and in the next subsection we describe in detail its general expression for subluminal oblique shocks.

4.1. Non-relativistic and quasi-parallel shock limit

When $u_s$ and $\Phi_{up}$ are sufficiently small, we can safely adopt the approximation that the fluid speeds in the HT frame are non-relativistic. We call it the non-relativistic and quasi-parallel shock limit. In this limit, $\xi$ and $P_{esc}$ can be expanded by the speeds of the fluids in the HT frame, and so can the spectral index.

Taking the lowest order terms of Eqs. (31) and (41), we obtain the energy gain of the non-relativistic and quasi-parallel shock limit. Thus we have

$$\left< \frac{\Delta E_{\text{ret}}}{E} \right> = 4\beta_1 \int_0^{\mu_o} d\mu_A \int_0^{1} d\mu_B \mu_A \mu_B (\mu_A + \mu_B)$$

$$= \frac{2}{3} \beta_1 (\mu_0^3 + \mu_0^2)$$  

(56)

and

$$\left< \frac{\Delta E_{\text{tr}}}{E} \right> = 8 \int_0^{\mu_o} d\mu_A \int_0^{1} d\mu_B \int_0^{1} d\mu_C \mu_A \mu_B \mu_C [(\mu_B - \mu_A)\beta_1 + (\mu_B - \mu_C)\beta_2]$$

$$= \frac{2}{3} \left[ \beta_1 (2 - \mu_0^3 - \mu_0^2) - 2\beta_2 (1 - \mu_0^2) \right].$$  

(57)

Then the lowest order term of the energy gain per one cycle is

$$\xi = \left< \frac{\Delta E}{E} \right> = \frac{4}{3} \beta_1 - \frac{4}{3} \beta_2 (1 - \mu_0^2).$$  

(58)

Next take the lowest order of Eq. (54) again and we obtain

$$P_{esc} = 4(1 - \mu_0^2)\beta_2.$$  

(59)

At the lowest order, since the energy gain and the escape probability are linear with respect to the fluid speeds in the HT frame, they are also small quantities. Hence, we can expand Eq. (55) as

$$\alpha \approx \frac{P_{esc}}{\xi} + 1.$$  

(60)

Substituting Eqs. (58) and (59) to Eq. (60), we obtain the spectral index in the non-relativistic and quasi-parallel shock limit as
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Fig. 2. The spectral index plotted against logarithm of the normalized shock speed $u_s/c$ for a fixed obliqueness $\cos \Phi_{up}$; (a) 1.0 (parallel), (b) 0.75, (c) 0.5, (d) 0.25 and (e) 0.1.

$$\alpha \approx \frac{(1-\mu^2)\beta_2}{\beta_1-\beta_2(1-\mu^2)} + 1 = \frac{r+2}{r-1}. \quad (61)$$

This form is the same as the canonical one for parallel shocks as suggested by Drury$^3$ and Ostrowski.$^9$

4.2. General results

Substituting Eqs. (42) and (54) to Eq. (55), we obtain the spectral index as a function of the shock speed $u_s$, the field obliqueness $\Phi_{up}$, and the compression ratio $r$. In this paper, we fix the compression ratio $r$ at 4.0, which corresponds to the strong shock limit for mono-atomic gas.

We show the shock speed dependence of the index in Figs. 2(a)~(e). The figures display the spectral index versus the shock speed for $-3.0 \leq \log u_s/c \leq 0.0$ for obliqueness of 1.0, 0.75, 0.5, 0.25 and 0.1 in Figs. 2(a), (b), (c), (d) and (e) respectively. Parallel shock corresponds to $\cos \Phi_{up} = 1.0$ and the obliqueness increases as $\cos \Phi_{up}$ decreases. Figure 2(a) shows that the index is close to 2.0 in the low shock speed limit, that the
spectrum becomes flat as the shock speed becomes high, and that the index reaches asymptotically 1.0 as the shock speed approaches \( c \). When the shock is parallel and its speed approaches the speed of light, the escape probability in Eq. (54) attains a finite value, 0.64, because of the fixed compression ratio (in reality the velocity ratio) in our model. Contrary to the escape probability, the energy gain becomes infinite at this limit, then the index approaches 1.0 from Eq. (55). For a given obliqueness, as seen in Figs. 2(b)~(e), one can see that the index is also close to 2.0 in the non-relativistic shock limit and that the index decreases slowly with an increase of the shock speed. The spectral index, then, attains a minimum at a certain value of the shock speed. A magnitude of the shock speeds at the minimum shifts to a smaller value as the obliqueness increases. The minimum values of the indices become smaller with the obliqueness except for both nearly parallel shocks and extremely oblique shocks. Above the shock speed at the minimum, the index increases rapidly to infinity as the shock speed approaches the subluminal limit, \( \frac{u_s}{c} = \cos \Phi_{up} \). Since the resultant spectrum becomes extremely steep, the Fermi acceleration does not take place in this limit.

To examine the above results in more detail, we show in Figs. 3(a)~(c) the energy gain, the escape probability and the transmission probability as a function of \( \log \frac{u_s}{c} \) for \( \cos \Phi_{up} = 0.25 \), which corresponds to Fig. 2(d). As seen in Figs. 3(b) and (c), for non-relativistic shock speeds, an increase of the escape probability with \( \frac{u_s}{c} \) is suppressed by a slow increase of the transmission probability from the upstream. Since the energy gain increases more rapidly as a function of \( \frac{u_s}{c} \) than the escape probability, the index decreases according to Eq. (55). As the shock speed increases further, the transmission probability and the escape probability increase rapidly compared to the energy gain, and both reach 1.0 as \( \frac{u_s}{c} \) approaches \( \cos \Phi_{up} \). Then the increase of the shock speed leads to an extreme large value of the resultant spectral index. This is because \( u_s \) approaches \( c \) for oblique shocks. On the other hands, \( u_s \) approaches \( c/4 \) for the parallel shock, so that the behavior of the spectral index for the parallel shock is distinct from that for oblique shocks.
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2.1

\(0\)

2.05

~

1.95

1.

1.85

0.2 0.4 0.6 0.8 2.1

\(0\)

2.05

a

1.

1.9

1.85

\(0\)

0.2 0.4 0.6 0.8

Fig. 4. The spectral index plotted against the obliqueness \(\cos \phi_{\text{up}}\) for a fixed shock speed \(u_s/c\); (a) 0.01, (b) 0.03 and (c) 0.1.

2.5

~

1.5

\(0\)

0.5

0.2 0.4 0.6 0.8

Fig. 5. (a) The average energy gain per one cycle, (b) the escape probability and (c) transmission probability plotted against the obliqueness \(\cos \phi_{\text{up}}\) for a fixed shock speed \(u_s/c = 0.1\).

Next we present the obliqueness dependence of the spectral index in Figs. 4(a)~(c). The spectral indices are plotted as a function of \(\cos \phi_{\text{up}}\). The shock speed is fixed at 0.01, 0.03 and 0.1 in Figs. 4(a), (b) and (c), respectively. Starting from the parallel shock, in which the index has a value \(\approx 2.0\), the index increases to reach 2.0 at around \(\cos \phi_{\text{up}} \approx 0.95\). As \(\cos \phi_{\text{up}}\) decreases more, the index decreases gradually with an increase of the obliqueness. The index then, reaches a minimum of \(\approx 1.84\) for \(-2.0 \leq \log u_s/c \leq -1.0\). The obliqueness corresponding to the minimum index is more oblique for smaller shock velocity. The index goes to infinity as \(\cos \phi_{\text{up}}\) approaches the subluminal limit; \(u_s/c\). Our results are different from Kirk and Heavens' results\(^{10}\) in which they found a monotonous decrease of the index. The resultant index of \(u_s\) does not become far below 2.0, while Kirk and Heavens' index approaches 1.0 at the subluminal limit. These differences are discussed in the next section.

To examine the results of Figs. 4(a)~(c) in more detail, we show the energy gain, the escape probability and the transmission probability as a function of \(\cos \phi_{\text{up}}\), respectively in Figs. 5(a)~(c), for \(u_s/c = 0.1\) which corresponds to Fig. 4(c). For obliqueness: \(0.95 \leq \cos \phi_{\text{up}} \leq 1.0\), the escape probability increases rapidly with the
obliqueness compared to the energy gain because of a relatively large transmission probability from the upstream, as shown in Fig. 5(c). The index, therefore, increases with the obliqueness. For obliqueness \( \cos \theta_{\text{up}} \leq 0.95 \), smaller transmission probability reduces the increase of the escape probability as seen in Figs. 5(b) and (c), while the energy gain increases slightly faster than the escape probability. This is the reason for the slow decrease of the index there. Finally the index increases because the escape probability increases more rapidly than that of the energy gain, as seen in Figs. 5(a)–(c) as the transmission probability approaches 1.0 again.

We have also investigated a range of shock speeds and obliquenesses beyond that represented in Figs. 2 and 4. For an extremely oblique shock, behavior of the index with \( \log u_s/c \) is similar to that in Fig. 2, but the index at the minimum increases to be just below 2.0. Since the extremely oblique shock reaches the subluminal limit at a lower shock speed, the energy gain becomes smaller, then the index increases. For an extremely slow speed shock, behavior of the index with the obliqueness is similar to Fig. 4, but the index at minimum becomes larger and just below 2.0 because the energy gain becomes smaller for a slower shock. When the shock is very close to the parallel one, the index reaches asymptotically 1.0 at its minimum. For an extremely high speed shock, the index also approaches 1.0 again at its minimum. For each case, the small index at minimum originates from a remarkably large energy gain, which is reminiscent of the relativistic limit for the parallel shock.

§ 5. Discussion

We obtained spectral indices of the accelerated particles for any values of the shock speed and obliquenesses for subluminal shock. An extremely flat spectrum with the index reaching 1.0 does not appear except for an relativistic and nearly parallel shock. In the subluminal limit, \( u_s/c = \cos \theta_{\text{up}} \), the spectral index becomes close to infinity; no statistical acceleration takes place. The resultant indices show \( 1.84 \leq \alpha \leq 2.0 \) for a reasonable shock speed and obliqueness.

Kirk and Heavens\(^{10}\) have obtained the spectral index of accelerated particles in the oblique shocks by using a method based on the transport equation; their result is different from ours. Their spectral indices decrease monotonously with the obliqueness to reach 1.0 in the subluminal limit of \( u_s/c = \cos \theta_{\text{up}} \), while our results have the minimum of about 1.84 in the index and the spectrum becomes very steep as in the subluminal limit. This difference may be ascribed to differences in the treatment of diffusion. They obtained the pitch-angle cosine distribution, assuming the pitch angle diffusion. The distribution in upstream obtained by them had a peak around \( \mu = 0 \) in the HT frame and only a small fraction of particles have pitch angle cosine large enough to be transmitted to the downstream. Such a distribution makes the escape probability smaller because of a decrease of transmitted particles from the upstream, and then the spectra become flat. On the other hand, we merely assume that the distribution of pitch angle cosine is isotropic in the local fluid frames. As a result, a larger fraction of particles have pitch angle cosine large enough to be transmitted to the downstream. Such a distribution makes the escape probability larger because of an increase of transmitted particles from the upstream, and then the
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spectra become steep. Although we do not specify any scattering process in this paper, we presume that our assumption is applicable to scattering such as a large angle scattering. Our results are compatible with the notion that no statistical acceleration occurs for quasi-perpendicular shocks, while their results are not. The difference arises from the difference of the pitch angle distribution. Monte Carlo simulation is now under investigation to examine this problem.

Radio observations of supernova remnants show synchrotron emission produced by nonthermal electrons, which are most likely accelerated in shock waves. The synchrotron index, $s$, is related to the energetic particle index, $a$, by $s=(a-1)/2$. The spectral index created by our model makes the synchrotron index of $0.42\leq s\leq 0.5$, which does not explain a variety of the observed index $0.3\leq s\leq 0.7$. If the energetic electrons are accelerated by Fermi mechanism in shock waves of supernova remnants, effects such as geometry of spherical shocks or magnetic field perturbation may play a role. Apart from this problem, the tendency in dependence of the synchrotron index on the obliqueness may be observationally seen. We expect that the synchrotron index of highly oblique region is observed to be smaller than that of less oblique regions if our model is correct. Although no conclusive evidence has been reported for appreciable spatial variations of the index, observations based on our hypothesis for supernova remnants expanding into homogeneous galactic magnetic-field are important.

We may apply the above argument to the $\gamma$-ray astronomy. We expect that a progress in $\gamma$-ray observations will enable us in a near future to detect high energy $\gamma$-rays from supernova remnants, reflecting characteristics of the accelerated protons. The high energy $\gamma$-rays created by decay of neutral pions from nuclear interaction have the same spectral index with high energy protons. If the $\gamma$-ray emission from supernova remnants is observed, the spectral index dependence on obliqueness may possibly be examined.

We have restricted our argument within the subluminal case. When both the shock speed and the obliqueness are so high, a superluminal shock, $u_s/c > \cos \Phi_w$, is realized. Our procedure no longer can be adapted and another approach is required. Such a condition may be realized in the AGNs. We expect that Monte Carlo simulation is an effective method in this case. It is assured that much still remains to be done for a study of acceleration mechanism in the superluminal shock.

The acceleration time is as much important as the spectral index for the shock acceleration. The time scale of acceleration and the size of the accelerator determine theoretically the maximum accelerated energy. In the parallel and non-relativistic cases, the acceleration time $t_{acc}$ is evaluated by using the fluid speed $u$, a diffusion coefficient $D$ and a fractional energy gain per one cycle $\xi$ as $t_{acc}\approx 4D/(cu\xi)$ based on a study of a diffusion-convection equation. Ostrowski developed this formulation to adjust the oblique shock but he covers only a first order of the fluid speed in the HT frame. The diffusion-convection equation is not appropriate to full range of the oblique shock because the fluid speeds become near to the speed of light in highly oblique ranges. Moreover, cross field diffusion is crucial for oblique shocks. Thus, we do not discuss this problem in this paper. It seems appropriate to obtain the distribution function by relying on Monte Carlo simulation. Then the accelerations...
tion time can be estimated from the time dependence of the energy distribution function. Monte Carlo simulation under investigation will be address the acceleration time, too.

§ 6. Conclusion

We have obtained the spectral indices of accelerated particles as a function of the shock speed $u_s$, the field obliqueness $\Phi_{up}$, and the compression ratio $r$, extending the analysis of Ostrowski to full order of the fluid speeds in the HT frame. Particle distributions are assumed to be isotropic in the local fluid frames both in the upstream and the downstream. In the de-Hoffmann-Teller frame, we calculate the average energy gain and the escape probability per one cycle including magnetic mirror effects under the adiabatic approximation. The spectral index is obtained analytically from these two quantities. At the non-relativistic and quasi-parallel shock limit, the index approaches the canonical value. For a given obliqueness, the index decreases first with an increase of the shock speed, and attains minimum below 2.0 for $r=4.0$. Then it increases to infinity as the shock speed approaches the limit determined by subluminal condition. For a fixed shock speed, the index increases slightly around at $\cos \Phi_{up} \approx 0.95$ to be 2.0. As the obliqueness increases, the index decreases gradually and then it reaches to the minimum value $\approx 1.84$, for $r=4.0$ in a highly oblique region. At the subluminal limit, the resultant spectrum becomes very steep as obliqueness and Fermi acceleration does not occur. This is compatible with notion that no statistical acceleration occurs for quasi-perpendicular (superluminal) shocks. We conclude that in our model the resultant indices have only narrow range of $1.84 \leq \alpha \leq 2.0$ for reasonable shock speed and obliqueness.

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Appendix

We explain here how we integrate Eqs. (44) and (46) using mathematic formulas. Consider Eq. (44) first, we divide the integrant into two parts as follows:

$$A_2 = \int_{\mu_0}^{1} \frac{\mu_\alpha d\mu_\alpha}{(1-\beta_1 \mu_\alpha)^3} - \beta_1 \int_{\mu_0}^{1} \frac{\mu_\alpha \sqrt{1 - b + b\mu_\alpha^2}}{(1-\beta_1 \mu_\alpha)^3} d\mu_\alpha .$$

The first integration is easily carried out. For the second integral, we transform $\mu_\alpha$ such that

$$x = 1 - \beta_1 \mu_\alpha .$$
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Hence the second term of Eq. (62) becomes

$$\frac{1}{\sqrt{1-\mu_0^2}} \beta_1^3 \int_{-\beta_1 \mu_0}^{1-\beta_1} \left( \frac{1}{x^3} - \frac{1}{x^2} \right) \sqrt{x^2-2x+1-\beta_1^2 \mu_0^2} \, dx. \quad (64)$$

We introduce mathematic formulas for the integral of square-root of a quadratic function. Let us define

$$I[m, n] = \int x^m (\sqrt{Ax^2 + Bx + C})^n \, dx. \quad (65)$$

This integral has the asymptotic form of

$$I[m, n] = \frac{1}{(m+1)C} \left\{ x^{m+1}(\sqrt{Ax^2 + Bx + C})^{n+2} \right. \quad (66)$$

$$- \frac{(2m+n+4)B}{2} \left( m + n + 3 \right) I[m+2, n] \right\}.$$

When \( n=1 \), as a case of Eq. (44), this asymptotic form turns out to be

$$I[-1, 1] = H(x) + \frac{B}{2} I_r(x) + C I_0(x) \quad (67)$$

and

$$I[-2, 1] = - \frac{H(x)}{x} + A I_r(x) + \frac{B}{2} I_0(x), \quad (68)$$

respectively, where

$$H(x) = \sqrt{Ax^2 + Bx + C}, \quad (69)$$

$$I_r(x) = \frac{1}{\sqrt{A}} \ln |2Ax + B + 2H(x)| \quad (70)$$

and

$$I_0(x) = \frac{1}{\sqrt{C}} \ln \left| \frac{x}{Bx + 2C + 2\sqrt{CH(x)}} \right|. \quad (71)$$

The integral in Eq. (64) is written by

$$I[-3, 1] - I[-2, 1] = \int \left( \frac{1}{x^3} - \frac{1}{x^2} \right) \sqrt{Ax^2 + Bx + C} \, dx$$

$$= - \frac{1}{2C} \left( \left[ \left( \frac{B}{2} - 2C \right)x + C \right] \frac{H(x)}{x^4} \right.$$

$$+ 2AC I_r(x) + \left[ \frac{1}{4} B^2 + (B-A)C \right] I_0(x) \left\}. \quad (72)$$

Substituting

$$A = 1, \quad B = -2, \quad C = 1 - \beta_1^2 \mu_0^2, \quad (73)$$
we can get Eq. (49).

Consider Eq. (46) next. Divide the integrant into two parts as

$$D = \int_{-1}^{0} \frac{\mu_0 d\mu_0}{(1-\beta_2 \mu_0)^4} - \beta_2 \int_{-1}^{0} \frac{\mu_0}{(1-\beta_2 \mu_0)^4} (-1) \sqrt{1 - \frac{1}{b} + \frac{1}{b} \mu_0^2} d\mu_0. \quad (74)$$

The first integral is easily carried out again. For the second integral, we transform $$\mu_0$$ to $$y$$ by

$$y = 1 - \beta_2 \mu_0. \quad (75)$$

Then we obtain

$$-\sqrt{1 - \mu_0^2} \frac{\beta_2}{\beta_1^3} \int_{1+\delta_1}^{1} \left( \frac{1}{y^4} - \frac{1}{y^3} \right) \sqrt{y^2 - 2y + 1 + \frac{\beta_2^2 \mu_0^2}{1 - \mu_0^2}} dy. \quad (76)$$

From the asymptotic form of (66), we can see that the integral is related to (69) and (70) by

$$I[-4, 1] - I[-3, 1] = \int \left( \frac{1}{y^4} - \frac{1}{y^3} \right) \sqrt{Ay^2 + By + C} \ dy$$

$$= -\frac{1}{3C'} \left\{ \left[ Ay^2 + By + C - \frac{3(B+2C')(By+2C')}{8C'y^2} \right] H(y) \right. \right.$$  

$$\left. \left. - \frac{3(B+2C')(B^2 - 4AC')}{16C'} J_0(y) \right\} \right., \quad (77)$$

where $$A$$ and $$B$$ are the same with Eq. (73) and $$C'$$ is

$$C' = 1 + \frac{\beta_2^2 \mu_0^2}{1 - \mu_0^2}. \quad (78)$$

Thus we can calculate Eq. (50).

References