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Analysis of the Rainfall-Runoff Transformation Process

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Considerable effort has been expended on the theoretical aspects of the applicability of the unsteady flow equations to natural overland flow. The major difficulty when these equations are used to simulate watershed flow lies in estimating the friction factor for natural situation. In this paper a generalized friction factor is introduced and a kinematic relationship which describes the »kinematic state« of the watershed surface is proposed. The resulting equations for unsteady free surface flow is a kinematic wave approximation. These equations are solved analytically and the solution obtained in the form of integral series.

Introduction

Because the relation between rainfall and runoff is one of the most important problems in hydrology, it has been the subject of many studies in these recent years. So there are available in hydrological litterature a wide spectrum and reasonably accurate mathematical models for rainfall-runoff modelling. These models have been presented and commented on by Dooge (1973). On one hand, most of these models were lumped and did not give any information about the physics of the transformation process involved; on the other hand the one dimensional gradually varied flow equations, which constitute the basic equations for unsteady overland flow, were always simplified when applied to watershed flow. Thus hydrologists have introduced a pletora of models, based on the simplification of the flow equations »each of which reflects a small portion of hydrologic reality, together with gross empirical simplifying assumptions based on the personal judgement of the individual modeller« (Natale and Todini 1977).

The major difficulty when flow equations are used to simulate watershed flow lies in estimating the friction resistance for natural situation. In this paper a generalized friction factor is introduced and a rational criteria for the choice of the simplified equations is proposed.

The Rough Surface Flow

One of the watershed flow characteristics is that water moves through a medium strewed up with vegetation, stones and other roughness elements. Boundary roughness here is much greater than that encountered in ordinary hydraulic structures. As noted by Woolhiser (1975), at low rates of flow, the roughness elements protude through the free water surface, and at high rates of flow, the boundary geometry may change in time and distance because of erosion. This is a confirmation of Wooding's analyses (1965), who suggested to define the depth as the volume of water per unit area (averaged over an area larger, compared with the dimension of the irregularities). However, since on vegetated surfaces the plant leaves and stems may offer more resistance to flow than the soil roughness, the geographical areas needed will be so large as to be topographically homogeneous.

Physically speaking, the transformation process in the watershed flow contains various sequential phases such as infiltration of rain water which forms the production process, surface flow, prompt subsurface flow, and underground response which forms the transformation process and the stream flow which is the propagation process. Each of these main processes can be described by appropriate flow equations. The transformation process is better described by the partial differential equations of gradually varied unsteady flow, known as De St Venant Equations

$$\frac{\partial H}{\partial t} + \frac{\partial Q}{\partial x} \equiv q(x,t) \tag{1}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + g\frac{\partial H}{\partial x} = g(I_0 - J) - q\frac{U}{H} \tag{2}$$

where $q = q_r - q_e$, and q_r is the rainfall rate, q_e is the rainfall loss (and will be described later). U is the local mean velocity, and H, the local depth; I_o is the slope of the watershed, J is the friction slope, and Q is the discharge resulting from the transformation process.

If Eqs. (1) and (2) should be used to describe flow over and through the watershed (Fig. 1), the basic principles underlying the theory of hydrodynamics must be applied. One of these principles states that a fluid, moving relative to a solid boundary exerts a force on the boundary. The first is shear stress which gives

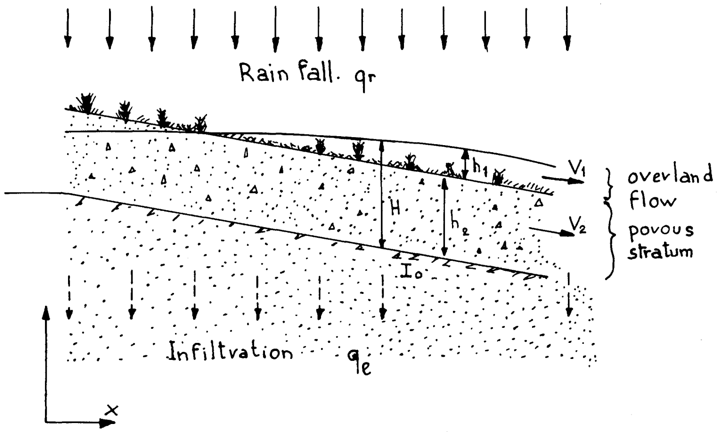

Fig. 1. Definition sketch of natural overland flow.

rise to forces tangential to the surface. The second is pressure variation along the surface that causes forces normal to the surface (Bear 1972). The vector sum of the normal and tangential surface forces integrated over the entire surface of the considered body give a resultant force. The component of this force in the direction of the velocity is called drag force. The total drag force is the sum of the frictional and pressure components and is defined by the Stokes' Equation (Bear 1972)

$$F_0 = C_D \frac{V^2}{2} S \tag{3}$$

where C_D is the total drag coefficient and is a function of Reynolds number $C_D = 24/Re$. S is the frontal area normal to the velocity V.

In this study it is assumed that roughness, plant leaves and stems are uniformly distributed, and their influence on flow per unit length can be simulated by that of spherical grain. Then if the drag coefficient per unit length is defined by

$$C_1 \equiv \frac{\eta}{Re^{mo}} \tag{4}$$

η- will characterize the watershed surface, and *mo* – will be a coefficient varying slowly from 1 to 0 as the flow varies slowly from laminar to turbulent. Wooding (1965) has noted that for vegetated catchment the flow regime may vary between laminar and turbulent. Nevertheless, a value of $m_o = 1$ will be assumed hereafter for simplicity.

Thus, it appears that if the flow functions U and H should be determined by the continuity and dynamic equations (Eqs.(1) and (2)), the form of the friction slope J must be specified by other parameters such as η and Re, (the Reynolds number). It will be proved that η is a function of the Froude number. Then the roughness influence can be better appreciated if a flow through it is compared to flow through a medium chosen as reference, i.e. if the similitude parameters are defined and calculated.

In hydraulic studies, normal flows are generally accepted as reference. Then the following normalizing quantities are defined:
U_o-normal velocity, H_o-normal depth, L_o-a normal linear dimension of the watershed.

At this juncture if the general flow equations (continuity and dynamic) are reduced to the dimensionless form by expressing the variables in terms of dimensionless parameters which describe the scale of the water flow compared to the normal flow, it will appear that the two universal parameters which are meaningfull for the watershed flow are Reynolds number *Re* and Froude number *Fr*. Hence, along with the continuity and the dynamic equations, it is necessary to have two additional relationships to describe the watershed flow. These relationships must take into account the specific characteristic of the flow through the medium, and thus must allow for an objective evaluation of η; moreover they must adequately describe the influence of viscosity forces. These relationships must have the following forms

$$Re = f_0(Re_0, \eta, \Gamma) \tag{5}$$

$$Fr = f_0'(Fr_0, \eta, \Gamma) \tag{6}$$

where Γ is a parameter which describes the frictional forces. However, it is more convenient to eliminate Re_o and Fr_o, the Reynolds number and Froude number for normal flow accepted as reference and to write these relationships in the following forms

$$\Gamma = f_1(Re, Fr) \tag{7}$$

$$\eta = f_1'(Re, Fr) \tag{8}$$

Estimation of the Friction Factor

Considering the problem shown in Fig. 1, it appears that the transformation process in consideration involves a typical catchment flow in the sense given by Wooding (1965) and whose depth is h_1; an interflow i.e. flow in the porous surface stratum h_2, with high permeability specified by Ishihara (1971), and lastly after the soil moisture storage is satisfied, a ground-water flow resulting from ground water recharge h_3. For simplicity it is assumed hereafter that $h_3 = 0$ and then q_e, the deep infiltration rate is considered as the loss. Overland catchment flow and interflow are considered.

Overland Catchment Flow

For channel flow, the friction slope (or energy gradient) is usually evaluated by

$$J_1 \equiv \frac{V_1^2}{C^2 R} \tag{9}$$

where C is the Chezy coefficient
R is hydraulic radius
V_1 is the water velocity

The catchment flow is a rough surface flow. When one accounts for the additional drag coefficient introduced by the roughness, the friction slope for overland flow will be

$$J_1 \equiv \left(\frac{1}{C^2 R} + \frac{C_1}{2g}\right) V_1^2 \tag{10}$$

Interflow Component

It is assumed that the porous surface stratum can be simulated by a porous medium made of spherical grains each of diameter d_0. Then the number of such spheres in a representative elemental volume of porous medium that forms a cylinder of length $d\ell$ in the direction of the flow and cross section dS can be evaluated from the equation (Bear 1972).

$$\eta_0 \equiv \frac{(1-P)\, dS\, d\ell}{\beta\, d_0^3} \tag{11}$$

where P is the porosity of the medium

β is a coefficient which accounts for the geometry of the grain. For a spherical grain $\beta = \pi/6$ and grains other than spherical $\beta \neq \pi/6$

For a single sphere, the Stokes' Equation for drag is given by Eq. (3), written in the form

$$F_0 \equiv C_D\, \rho \frac{V_\ell^2}{2} So \tag{12}$$

where V_ℓ is the local velocity and So is the surface of the sphere. Taking into account the effect of neighbouring spheres, a coefficient λ is introduced (Bear 1972). Then Eq. (12) may be written as

$$F_0 = \lambda \mu d_0 \bar{V}_\ell \tag{13}$$

Taking $\bar{V}_\ell$ as the average velocity of flow around the particles (grains) in the main direction, it may be defined as the ratio of specific flux (apparent velocity) to the porosity (Bear 1972, Polubarinova-Kochina-1962)

$$\bar{V}_\ell = \frac{V_2}{P} \tag{14}$$

The total drag force is given by

$$F_t = \eta_0 F_0 \tag{15}$$

Now, assuming that the body forces in the motion equations are due to gravity, the total drag is divided by the weight of water to give

$$J_2 \equiv \frac{\eta_0 F_0}{\rho g P \, dA \, d\ell} \tag{16}$$

By inserting the value of η, F_0, and V_2, and evaluating $So = \lambda d_0^2/12$ the following equation is obtained

$$J_2 \equiv \frac{1-P}{P^3} \frac{\lambda}{24 g \beta d_0} C_D V_2^2 \tag{17}$$

This relation is interesting in that it brings out the particular importance of the Reynolds number for the friction slope and gives an explanation to the deviation from Darcy's law and thus gives the fundament of the non-Darcian flow extensively studied in recent years (Bear 1972; Chevauteau et Thirriot 1967).

The Generalized Friction Factor

Assume now that the catchment flow and interflow have the same direction, then the whole watershed system can be considered as a layered porous medium where flow is parallel to layers, V_1 and V_2 are the upper layer and underlayer flow velocities respectively and their corresponding discharges per unit width are Q_1 and Q_2; h_1 and h_2 being the respective thickness and $H = h_1 + h_2$ the total thickness. The total discharge is $Q = Q_1 + Q_2$.

If one wishes to consider an equivalent velocity such that the same discharge Q will be conducted through the same aquifer of thickness H such that

$$Q = UH \tag{18}$$

under a given gradient J, then the velocity U must have the following form

$$U = V_1 \left(1 - \frac{h_2}{H}\right) + V_2 \frac{h_2}{H} \tag{19}$$

By comparing this relationship with the relations for V_1 and V_2 and taking the gradient to be the same for both flows the following equation is obtained

$$U^2 = \left\{ \left(\frac{2gC^2R}{2g + C_1 C^2 R} \right) \left(1 - \frac{h_2}{H}\right)^2 + \frac{24 g \beta d_0 P^3}{\lambda (1-P) C_D} \left(\frac{h_2}{H} \right)^2 + \right.$$
$$\left. + 2 \left(\frac{2gC^2R}{2g + C_1 C^2 R} \right)^{\frac{1}{2}} \left(\frac{24 g \beta d_0 P^3}{\lambda (1-P) C_D} \right)^{\frac{1}{2}} \left(1 - \frac{h_2}{H}\right) \left(\frac{h_2}{H} \right) \right\} J \tag{20}$$

Introducing a quantity Γ such that

$$\Gamma \equiv 8g\,H\frac{J}{U^2} \qquad (21)$$

and combining with Eq. (20) gives

$$\Gamma \equiv 8gH\{(\frac{2gC^2R}{2g+C_1\,C^2R})(1-\frac{h_2}{H})^2 + \frac{24g\beta d_0P^3}{\lambda(1-P)C_D}(\frac{h_2}{H})^2 +$$

$$+ 2(\frac{2gC^2R}{2g+C_1\,C^2R})^{\frac{1}{2}}(\frac{24g\beta d_0P^3}{\lambda(1-P)C_D})^{\frac{1}{2}}(1-\frac{h_2}{H})(\frac{h_2}{H})\}^{-1} \qquad (22)$$

Comparison with Earlier Relations for Friction Factor

As noted by Woolhiser (1975), most investigators made the assumption that the Darcy-Weisbach resistance Law $V^2 = [(8\,g\,/\,\Gamma_o)\,I_o\,H]$ is appropriate and then measurement of depth, discharge and slope are used to estimate the friction factor.

For laminar flow over rough surface, the following relation which takes rain drop impact into account has been proposed (Woolhiser 1975)

$$\Gamma = \frac{\bar{K}_0}{Re} + \frac{a}{Re}I^b \qquad (23)$$

Where $\bar{K}_o$ is a parameter without rainfall and a and b are empirical coefficients. $\bar{K}_o$ is related to the characteristics of the surface, a is on the order of 10 and $b \equiv 1$.

When the following values $h_2 = O$, $R = h_1$ for overland and $C^2 = 8\,g\,/\,\Gamma_o$ are assumed in Eq. (22), a value of the friction factor is obtained for a rough surface flow

$$\Gamma = \Gamma_0 + 4C_1h_1 \qquad (24)$$

Where Γ_o is the Darcy-Weisbach friction factor for smooth surface. This equation is analog to Eq. (23).

Nevertheless in Eq. (24), the cumulative precipitation is considered and the raindrop impact is not analysed separately. The state of the watershed surface described adequately by a parameter C_1. Eq. (24) is therefore a rigourous derivation of Eq. (23).

The equivalent Chezy coefficient and Manning coefficent can be defined respectively as

$$C_{ws} = C(1+\alpha_{ws})^{-\frac{1}{2}}$$

$$n_{ws} = n_0(1+\alpha_{ws})^{\frac{1}{2}}$$

where $\alpha_{ws} \equiv 4C_1 h_1/\Gamma_0^{\bullet}$. When $\Gamma_0 = 24/Re$, $\alpha_{ws} = \eta h_1/6$

For a smooth surface $\eta = 0$ and $\alpha_{ws} = 0$, then $C_{ws} = C$; $n_{ws} = n_o$; $\Gamma = \Gamma_o$
When $h_1 = 0$, we have

$$\Gamma \equiv h_2 \frac{1-P}{P^3} \frac{\lambda}{3\beta d_0} C_D \tag{25}$$

Assume d_o to be some length dimension of the porous matrix. For example if d_o should be a length dimension representing the elemental channel of the porous medium, then the Reynolds number will be defined through porous medium $Re \equiv V_2 d_o/\nu$ (ν being the kinematic viscosity of the fluid). However in porous medium studies, it is customary to employ some representative dimension of the grains for d_o. Bear (1972) mentioned that the mean grain diameter is often taken as the length dimension d_o. Sometimes d_{10} or d_{50} are used. Collin (1961) Suggested $d_o = (^k/p)^{1/2}$ where k is the permeability and p the porosity. Ward (1964) used $k^{1/2}$. In all cases the Darcy-Weisbach friction factor Γ has been defined for porous media as

$$\Gamma_0 = \frac{2gd_0 J}{V_2^2} \tag{26}$$

then d_o is defined by

$$d_0 \equiv \frac{\Gamma_0 V_2^2}{2gJ} \tag{27}$$

Nevertheless, Fanning using analogy with overland flow gives another definition for the friction factor in porous medium

$$\Gamma_0 \equiv \frac{2gRJ}{V_2^2} \tag{28}$$

Continuing this analogy, one can define $h_2 = R = d_o/4$ then the friction factor for porous medium is given by

$$\Gamma = \frac{\lambda}{12\beta} \frac{1-P}{P^3} C_D \tag{29}$$

i.e. the friction factor for porous medium depends on the porosity and the drag coefficient.

When $C_D = 24/Re$, then

$$\Gamma \equiv \frac{2\lambda}{\beta} \frac{1-P}{P^3} \frac{1}{Re}$$

which has the form of Fanning friction factor.

Thus it is proved that Eq. (22) gives a more general form of the friction factor and is valid for channel flow, overland flow and flow through porous media.

Estimation of the Kinematic Relation

Considering a smooth surface flow (Chezy's flow) as reference for overland flow, now, with normal depth H_o and normal velocity $V_o^2 = C^2 R I_o$ it follows from Eq. (10) that

$$C_1 \equiv \frac{2g(\alpha^2-1)}{C^2 R} \tag{30}$$

where $\alpha = V_o / V_1$; or introducing Froude numbers

$$C_1 = \frac{2J}{H_0}\left(\frac{1}{Fr_1^2} - \frac{1}{Fr_0^2}\right) \tag{31}$$

Fr_1 and Fr_o are Froude numbers for the rough surface and smooth surface flow respectively. For the normal flow $J = I_o$ and

$$C_1 = \frac{2I_0}{H_0}\left(\frac{1}{Fr_1^2} - \frac{1}{Fr_0^2}\right) \tag{32}$$

Parameter C_1 has dimension (L^{-1}). If reduced to dimensionless form by introducing the reference length L_o, we have

$$C_0 = C_1 L_0 = 2(K_1 - K_0) \tag{33}$$

where K_1 and K_o are the kinematic parameter for rough surface flow and smooth surface flow respectively: $(K_i = I_o L_o / H_o Fr_i^2)$. Combining Eq. (33) and the definition of C_1 (Eq. 4) gives

$$\eta = 2\left(\frac{ReK_1}{L_0} - \frac{ReK_0}{L_0}\right) \tag{34}$$

This parameter has again a dimension of (L^{-1}). It is convenient for the purposes of analyses to express Eq. (34) in a dimensionless form. Thus one obtains

$$K_1 = \frac{L_0}{2}\frac{\eta}{Re} + K_0 \tag{35}$$

$L_o\ \eta / 2$ is a dimensionless quantity characterizing the roughness of the medium and $L_o\ \eta / 2Re$ is equivalent to a kinematic number. Therefore, Eq. (35) will be called the kinematic relationship. It expresses the relationship between the roughness of the surface and the kinematic parameter and thus it can be used to determine the friction slope.

The Simplified Dynamic Equation

For the purposes of analyses, write the dynamic equation in the following form

$$J \equiv I_0 - \left(\frac{\partial H}{\partial x} + \frac{U}{g}\frac{\partial U}{\partial x} + \frac{1}{g}\frac{\partial U}{\partial t}\right) \doteq \frac{q}{g}\frac{U}{H} \tag{36}$$

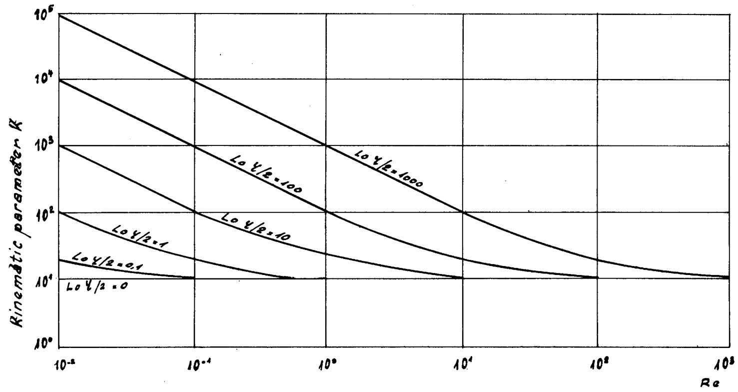

Fig. 2. Relationship between the kinematic equation and the Reynolds number.

This equation has been used to classify gradually varied flow problems into kinematic wave problems and dynamic wave problems. From Eq. (36), it can be seen that discharge (or velocity) is a function of the dynamic equation. When it is expressed as a function of depth alone, it may be derived from the continuity equation. The latter approach is called the kinematic wave approach and the former, the dynamic wave approach.

The fundament of the kinematic approach has been widely described by Wooding (1965), Eagleson (1970), Woolhiser (1975) Miller and Cunge (1975) etc. The gist of the theory is that, the kinematic approximation applies if Froude number is less than two: $Fr < 2$, and the dynamic approach if $Fr < 1$; when $Fr > 2$, the kinematic approximation breaks down.

Woolhiser and Liggett (1967) have proved that kinematic approximation is valid for $K_o = (L_o I_o / H_o Fr_o) > 10$. Following these authors, the condition $Fr_o \leqslant 2$, $1 < Fr_1 < 2$ and $K_o > 10$ will be assumed to hold throughout the discussion.

From Eq. (33), the condition for Reynolds number is

$$\frac{2\eta H_0}{3I_0} < Re < \infty \tag{37}$$

The lower limit of this double inequality depends on the definition of $L_o\eta/2$. Setting $K_o = 10$, Fig. (2) shows the relationship between the kinematic equation (Eq. 35), the Reynolds number Re, and the value of $L_o\eta/2$ describing the watershed surface. In its general form, the kinematic relation (Eq. (35)) is similar to the friction factor obtained for the rough surface flow (Eq.(24)). $L_o\eta/2 = 0$ describes the smooth surface flow. In this case, the lower limit of the Reynolds number is zero. When $L_o\eta/2 = 1{,}000$ the kinematic relation is identical to the Fanning friction factor. However, the lower limit of the Reynolds number is $Re \simeq 33$. It is assumed in the following, that the Reynolds number induced by the equivalent velocity U is larger than this value.

Now if

$$K_1 > \frac{L_0 \eta}{2} \frac{1}{Re} + K_0 \tag{38}$$

the kinematic approximation is applied, then $J = I_o$ and

$$U \equiv \left(\frac{8g}{\Gamma} I_0 R\right)^{\frac{1}{2}} \tag{39}$$

There is a unique relationship between depth and discharge and depth will be normal for uniform flow at that discharge (Woolhiser 1975). $R \simeq H$ stands for hydraulic radius and is approximated by the depth. Γ is the friction factor and is evaluated by Eq. (22). Since the friction factor is a function of depth, a coefficient α (H) can be defined

$$\alpha(H) = \left(\frac{8gI_0}{\Gamma(H)}\right)^{\frac{1}{2}} \tag{40}$$

then the discharge is expressed as a unique relation of depth

$$Q = \alpha(H)\, H^{\frac{3}{2}} = \psi(H) \tag{41}$$

Eq. (41) is used in kinematic approximation theory assuming $\alpha = c\ st$ and instead of exponent 3/2 an exponent m, which accounts for flow variation from laminar (m=3) to turbulent (m=5/3) (Eagleson 1970). Though exponent m accounts for the variability of the roughness, the influence of the friction factor is underestimated when α is assumed to be constant. Eqs. (22), (24) and (25) show that α is constant only for smooth surface flow. The Kinematic relationship (Eq. (35)) accounts for the variation of the flow from laminar to turbulent (Fig. 3)).

If

$$K_1 < \frac{L_0 \eta}{2} \frac{1}{Re} + K_0 \tag{42}$$

the kinematic approximation will not remain valid any longer. The character of the flow will be governed by the Reynolds number and the parameter $L_0\eta/2$.

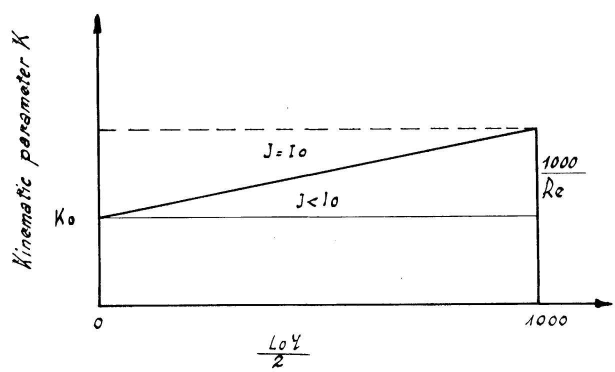

Fig. 3. Schematic representation of the Kinematic relationship $Re \equiv cst.$

When $L_0\eta/2 = 0$ (Smooth surface flow), J is evaluated by Eq. (36) taking $\Gamma = \Gamma_0$. When $L_0\eta/2 > 0$ the value of J will depend on the Reynolds number and the Froude number explicitely introduced by coefficient C_1, and J will be evaluated from Eqs. (21) and (22).

Thus it can be concluded that the kinematic Eq. (35) describes entirely the watershed flow, and along with Eq. (22) determines the transformation process in the watershed.

Analytical Solution of the Watershed Flow

When Eq. (41) is combined with Eq. (1), a system of equations which describes adequately the watershed flow is obtained. This system of equations has been solved for $\alpha = cst$ by the methods of characteristics (Eagleson 1970, Overton and Meadow 1976) and by various numerical methods using finite difference schemes (Woolhiser 1975). An analytical approach is considered hereafter for a more general case $\alpha = \alpha\ (H)$.

In applied hydrology, when distributed parameter models are used to describe the watershed behaviour, the major problem is the areal information availability. Information on the areal variability of hydrologic parameters can be gathered only by means of dense networks of measurement stations. Because of the large cost involved, hydrological information are available, only in form of areal average values. Thus the accuracy of the distributed model is affected by the lack of data specifying the boundary and the initial condition, rainfall input distribution etc.

To match the model with the practice, various methods are used to transform the gradually varied flow equation into a lumped hydrological model. A detailed review of these methods is given by Miller and Cunge (1975). The approximated scheme described by Muzik (1974) is adopted here after, (Fig. 4).

Consider the distributed parameter $q\ (x,t)$ and $Q\ (x,t)$ indicating the rate of rainfall excess and the discharge to be lumped into a finite number of input and output variables respectively. q_i and Q_i, (where $i = 1,2 \ldots n$). For each pair of input-output the continuity equation is considered in the »hydrologic form« (Dooge 1973, Thirriot 1974), by integrating Eq. (1) over a correspondent watershed reach to obtain

$$\begin{aligned} \frac{dy_1}{dt} &= \phi_1 - \psi_1 \\ \frac{dy_2}{dt} &= \phi_2 - (\psi_2 - \psi_1) \\ &\cdots\cdots\cdots\cdots \\ \frac{dy_n}{dt} &= \phi_n - (\psi_n - \psi_{n-1}) \end{aligned} \qquad (43)$$

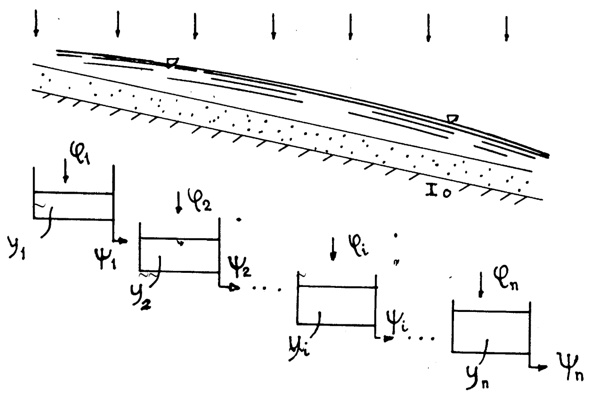

Fig. 4. Definition sketch of overland flow simulation by interacting reservoirs.

where

$$y_i = \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} H_i(x,t)\,dx$$

$$\psi_i - \psi_{i-1} = \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} \frac{\partial Q_i}{\partial x}\,dx$$

$$\phi = \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} q(x,t)\,dx$$

In matrix notation Eq. (43) can be written in the form

$$\frac{dY}{dt} = \varphi - \Psi \qquad (44)$$

Eq. (44) is the continuity equation and equivalent to Eq. (1). The dynamic equation is given by

$$\Psi = \Psi[Y(t);t] \qquad (45)$$

Noting that Eq. (41) expresses the discharge as a unique relation of depth, $\psi_i(y_i)$ can be developed in Taylors' series in the neighbourhood of $y_i\ (t_o) = 0$

$$\psi_i(y_i(t);t) = \sum_{j=1}^{m} a_{ij}(t)\,y_i^j\,(t) \qquad (46)$$

Eq. (44) then transforms into

$$\frac{dY}{dt} = \sum_{j=1}^{m} A_j Y_j + \varphi \qquad (47)$$

where

$$\frac{dY}{dt} = \begin{pmatrix} \frac{dy_1}{dt} \\ \cdot \\ \cdot \\ \cdot \\ \frac{dy_n}{dt} \end{pmatrix} ; \quad Y_j = \begin{pmatrix} y_1^j \\ \cdot \\ \cdot \\ \cdot \\ y_n^j \end{pmatrix} ; \quad \varphi = \begin{pmatrix} \phi_1 \\ \cdot \\ \cdot \\ \cdot \\ \phi_n \end{pmatrix}$$

$$A_j = \begin{pmatrix} -a_{1j} & 0 & \vdots & 0 & 0 & \vdots & 0 & 0 \\ a_{1j} & -a_{2j} & \vdots & 0 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & a_{i-1j} & -a_{ij} & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 0 & 0 & \vdots & a_{n-1j} & a_{nj} \end{pmatrix}$$

Reservoir approach is of widespread use in hydrological modelling (Miller et al – 1975). Therefore though this system can be solved numerically without too much difficulty, it is worthwhile to derive an approximative but analytical solution which gives more insight and permits a theoretical comparison with various existing reservoir model.

In terms of the system approach theory Eq. (47) is a non-linear vector differential equation which describes the state $Y(t)$ of the considered watershed and Eq. (45) expresses the output as a non-linear function of this state. The system of Eqs. (45) and (47) can be solved using the local inverse theory. (Halme and Orava 1972).

Define the following operator

$$\Lambda(Y) = [Y(t_0);\ \frac{dY}{dt} - \sum_{j=1}^{m} A_j Y_j] \qquad (48)$$

$\Lambda(y)$ is a polynomial operator having homogeneous components

$$D_1(Y) = [Y(t_0);\ \frac{dY}{dt} - A_1 Y_1] \qquad (49)$$

$$D_j(Y) = [0;\ A_j Y_j] \qquad (50)$$

The invertibility of polynomial operator has been discussed by Halme et al (1972) for a large class of non-linear equations. It is easily observed that for $D_1(Y)$ the inverse $D_1^{-1}(Y)$ exists and with regard to the set of pair formed by initial condition $Y(t_o)$ and input function can be represented in the form

$$D_1^{-1}(Y) = Y_L(t) = \Phi(t,t_0)Y(t_0) + \int_{t_0}^{t} \Phi(t,\xi)\varphi(\xi)d\xi \qquad (51)$$

where in the general case $\Phi\ (t,t_o)$ is the $n \times n$ fundamental matrix of the system. The fundamental matrix for Eq. (45) and (47) can be evaluated by means of the Laplace transform. Suppose for simplicity in the reasoning that $A_1 = cst$ and denote the Laplace variable by »s«, then taking the Laplace transform of (49) yields after some calculation, the following form for the fundamental matrix

$$L[\Phi] = \left[\begin{array}{cc:cc}
\frac{1}{s+a_{11}} & 0 & 0 & 0 \\
\frac{a_{11}}{(s+a_{11})(s+a_{21})} & \frac{1}{s+a_{11}} & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\prod_{i=1}^{n-2} a_{i1}}{\prod_{i=1}^{n-1}(s+a_{i1})} & \frac{\prod_{i=2}^{n-2} a_{i1}}{\prod_{i=2}^{n-1}(s+a_{i1})} & \frac{1}{s+a_{(n-1)1}} & \\
\frac{\prod_{i=1}^{n-1} a_{i1}}{\prod_{i=1}^{n}(s+a_{i1})} & \frac{\prod_{i=2}^{n-1} a_{i1}}{\prod_{i=1}^{n}(s+a_{i1})} & \frac{a_{n-1}}{(s+a_{(n-1)1})(s+a_{n1})} & \frac{1}{s+a_{n1}}
\end{array}\right]$$

It can be proved (Athans and Falb 1966; Sevely et al 1973), that the fundamental matrix verifies the following properties

a) $\Phi(t_0,t_0) = I$

b) $\frac{d}{dt}\Phi(t,t_0) = A_1(t)\Phi(t,t_0)$

c) $A_1(t) = \frac{d}{dt}\Phi(t,t_0)\big|_{t=t_0}$

d) $\Phi(t,t_0) = Y_{unf}(t)\,Y^{-1}(t_0)$ where Y_{unf} is a matrix which columns are solution of the unforced system.

e) $\Phi(t_1,t_3) = \Phi(t_1,t_2)\Phi(t_2,t_3)$

f) $[\Phi(t,t_0)]^{-1} = \Phi(t_0,t)$

g) $\det(\Phi(t,t_0)) = \exp\left[\int_{t_0}^{t} \text{trace } A_1(\xi)d\xi\right]$

where $\text{trace } A_1(\xi) = \sum_{i=1}^{n} a_{ii}(\xi)$

When $A_1 = cst$

$$\Phi(t,t_0) = \Phi(t-t_0) = \exp[A_1(t-t_0)]$$

By definition of the system $Y(t_o) = 0$, and in general it is possible to define a system such that $Y(t_o) = 0$ or to introduce a new variable $Y^* = Y(t) - Y(t_o)$ such that $Y^*(t_o) = 0$; then, $Y_L(t)$ can be written in the following form for time invariant system

$$Y_L(t) \equiv \int_{t_0}^{t} \Phi(t-\xi)\varphi(\xi)\, d\xi \tag{52}$$

where $Y_L(t)$ is a column matrix element of which has the form

$$y_{Lk}(t) \equiv \sum_{\nu=1}^{k} \int_0^t \kappa_{\nu,1}(t,\xi)\phi_\nu(\xi)d\xi$$

and

$$\kappa_{\nu,1}(t,\xi) = \sum_{i=\nu}^{k} \prod_{i=\nu}^{k-1} a_{i1}\exp[-a_{i1}(t-\xi)] \tag{53}$$

Now $D_j^{-1}(Y)$ is obtained from the knowledge of Eq. (52) which expresses the linear component; the vector column Y_j are formed, and the ν^{th} element is obtained in the following form

$$y_\nu^j(t) = \sum_{i_1=1}^{\nu} \cdots \sum_{i_j=1}^{\nu} \int_0^t \cdots \int_0^t \kappa_{i_1 \ldots i_j}(t,\xi_{i_1} \ldots \xi_{i_j}) \phi_{i_1}(\xi_{i_1})\phi_{i_2}(\xi_{i_2}) \ldots \phi_{i_j}(\xi_{i_j})d\xi_{i_1} \ldots d\xi_{i_j} \tag{54}$$

When this expression is used to calculate Ψ the resulting equation is a column matrix, elements of which can be rearranged in the form of integral series

$$\bar{\Psi}_k(t) = \sum_{\nu=1}^{k} (\sum_{i_1=1}^{k} \cdots \sum_{i_j=1}^{k} \int_0^t \cdots \int_0^t \kappa_{\nu,i_1 \ldots i_j}(t,\xi_{i_1} \ldots \xi_{i_j}) \phi_{i_1}(\xi_{i_1}) \ldots \phi_{i_j}(\xi_{i_j})d\xi_{i_1} \ldots d\xi_{i_j}) \tag{55}$$

Eq. (55) is a generalized system approach solution for overland rainfall-runoff transformation process. The advantage of Volterra series representation has been discussed elsewhere (Afouda 1978). A theoretical comparison with some reservoir approaches is only made here after.

When $n = 1$, $m = 1$, the single linear reservoir approach is obtained which is widespread use in hydrology. $\kappa(t)$ reduces to

$$\kappa(t) \equiv a(t)\exp\left(-\int_0^t a(\xi)d\xi\right)$$

for time varying model, and to a simple exponential expression when $a = cst$. Coefficient a is often taken as an empirical coefficient. The general approach presented here allows for the calculation of a from Eq. (41) and Eq. (22). When the friction factor is presented in a simplified form, Eq. (24), the following form is obtained

$$a \equiv \frac{3}{2} U \left(\frac{1 + \frac{2}{3}\alpha_{ws}}{1 + \alpha_{ws}} \right)$$

If $n \geqslant 2$, $m = 1$, a cascading linear reservoir approach is obtained. This case has been investigated in laboratory by Muzik (1974). When Eq. (24) is used for the friction factor $A_1 = cst$, and if $\varphi_1 \neq 0$, $\varphi_i = 0$, $(i = 2, 3 \ldots n)$, the Dooge approach (1959) is obtained. When the Dooge boundary conditions are used, $\kappa(t)$ will have the following form

$$\kappa(t) \equiv \sum_{i=1}^{n} \left(\frac{1}{a_{i1}}\right)^{n-2} \left[\prod_{\nu=1}^{n} \left(\frac{1}{a_{i1}} - \frac{1}{a_{\nu,1}}\right)\right]^{-1} e^{-a_{i1}t} \qquad (56)$$

In the particular case of the Nash conditions $a_{11} = a_{21} = \ldots a$

$$\kappa(t) \equiv \frac{1}{(n-1)!} (at)^{n-1} e^{-at} \qquad (57)$$

If $n = 1$, $m > 2$, a single non linear reservoir approach is obtained. The solution for this approach has been proved to have the form of Volterra series. (Afouda 1976, 1978), and $\kappa(t)$ are appropriate combination of exponential.

The case when $n \to \infty$, $m < \infty$ is of interest. Considering the linear component

$$y_{Lk}(t) \equiv \sum_{\nu=1}^{k} \int_0^t \kappa_{\nu,1}(t,\xi)\phi_\nu(\xi)d\xi$$

It can be easily seen from the definition of the system that if $n \to \infty$, this linear component will tends towards

$$H(x,t) = \int_0^t \kappa_\infty(x,t,\xi)q(x,\xi)d\xi \qquad (58)$$

The task now is to evaluate $\kappa_\infty(x,t,\xi)$. From the definition of the linear component,

$$\kappa_\infty(x,t,\xi) = \lim_{k\to\infty} \sum_{i\equiv 1}^{k} \sum_{i\equiv 1}^{k} \prod_{i\equiv 1}^{k-1} a_{i1} \exp[-a_{i1}(t,\xi)] \qquad (59)$$

Since $k \to \infty$, a_{i1} tends to a fixed value a_1 and Nash condition is obtained. Thus the internal summation is equal to

$$\frac{1}{(k-1)!} \, (at)^{k-1} \, e^{-a,t}$$

Taking now

$$\kappa_\infty(x,t,\xi) = \lim_{k\to\infty} \sum_{i=1}^{k} \frac{1}{(k-1)!} (a_1 t)^{k-1} \, e^{-a,t}$$

$$= \int_0^\infty \frac{1}{(k-1)!} (a_1 t)^{k-1} \, e^{-a_1 t} \, d(a_1 t) \tag{60}$$

$\kappa_\infty(x,t,\xi)$ is found to be constant everywhere and equal to Heaviside function

$$\kappa_\infty(x,t,\xi) \equiv u(x,t,\xi) \tag{61}$$

Hence the linear component is

$$H(x,t) = \int_0^t u(x,t-\xi)q(x,\xi)d\xi \tag{62}$$

The resulting discharge has the following form

$$Q(x,t) = \sum_{j=1}^{m} \int_0^t \ldots \int_0^t \bar{\kappa}_j(x,t,\xi) \prod_{\ell=1}^{j} q(x,\xi)d\xi_1 \ldots d\xi_j \tag{63}$$

Where

$$\bar{\kappa}_j(x,t,\xi) = \int_0^t j a_j \prod_{\ell=1}^{j+1} u(x,t-\xi_j)d\xi_j$$

Now derive the solution from the distributed equation

$$\frac{\partial H}{\partial t} + \frac{\partial Q}{\partial x} = q(x,t) \tag{64}$$

$$Q = \sum_{j=1}^{m} a_j H^j \tag{65}$$

The total differential of $H(x,t)$ is

$$\frac{dH}{dt} \equiv \frac{\partial H}{\partial t} + U \frac{\partial H}{\partial x} \tag{66}$$

By comparing with the continuity Eq. (64) yields

$$\frac{dH}{dt} = q(x,t) \tag{67}$$

$$U = \sum_{j=1}^{m} j \, a_j H^{j-1} \tag{68}$$

Solving Eq. (67) for H, yields Eq. (62), and the resulting discharge can be easily deduced in the form of Eq. (63).

Thus it is proved that when $n \rightarrow \infty$, the generalised lumped solution tends toward the distributed solution of the kinematic wave theory.

Conclusion

Considerable effort has been expended on the theoretical aspects of applicability of spatially varied unsteady flow equation to overland flow. In this paper, a generalized friction factor has been introduced to describe resistance to storm water flow over a natural watershed. This theoretical relationship has been proved to be valid for smooth surface flow, rough surface flow and flow through porous medium. A kinematic relation which accounts for the roughness of the watershed surface has been proposed. The similarity between this kinematic relation and the friction factor has been underlined. This kinematic relationship determines the applicability of the kinematic flow theory and the approximation of the friction slope. When kinematic approximation applies, an analytical solution can be obtained in the form of integral series. The lumped kinematic solution has been proved to tend towards the solution of the distributed approximation when the dimension n of the fundamental matrix tends to infinity.

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