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THREE-DIMENSIONAL STABILITY ANALYSIS OF OPEN CHANNEL FLOW OVER AN ERODIBLE BED

FRANK ENGELUND and JØRGEN FREDSØE

Hydraulic Laboratory, Technical University of Denmark,
Copenhagen, Denmark

The formation of ripples and dunes (lower range bed waves) is assumed to be related to the transport of sediment as bed load. From the present theory it is concluded that the formation of the upper range bed configurations (standing waves, antidunes) may be explained on the assumption that the predominant part of the sediment transport is in suspension. The paper presents a mathematical model of the formation of double-periodic antidunes, first-order potential flow theory being applied. It differs from previous models in taking account of the non-uniform distribution of the suspended load. The theory predicts regions of stability and instability. Results are compared with measurements made by different observers.

The formation of sandwave patterns in erodible channels has been discussed by several authors in the recent years. For a thorough review, readers are referred to a recent monograph by Allen (1968).

The two-dimensional periodic instability was treated by Kennedy (1961) and Reynolds (1965) on the basis of potential flow theory. The formation of three-dimensional (double-periodic) bed waves has attracted less attention, but the analysis by Reynolds also comprises a discussion of the problem. The present investigations differ from those performed by Reynolds in using a definite physical model of the sediment transport mechanism.

The formation of ripples and dunes (lower range bed waves) starts at very low sediment transport rates, where the sand particles move essentially as bed load. Upper range bed waves (standing waves and antidunes), on the other hand, occur when the sediment transport is vigorous, so that it is natural to assume that sediment transport in suspension will be a significant feature of this range.

This paper presents a mathematical model of flow over a double-periodic, movable bed. The sediment is assumed to be transported in suspension only. Consequently, the formation of lower range bed waves is not accounted for by this model.

In this non-uniform flow the sediment concentration will be non-uniform, too. In order to account for this in a reasonably simple way, the undisturbed concentration distribution is replaced by an exponential distribution. This corresponds to assuming a constant value of the diffusivity ε over the whole depth. This is known to be a rather poor approximation close to the bed. Hence, we have to introduce a nominal value c_b of the sediment concentration at the bed, different from the one actually occurring.

The non-uniform sediment distribution may then be accounted for by an equation describing the equilibrium between convection, settling, and diffusion. For the sake of simplicity the diffusivity is assumed equal to the eddy viscosity ε. Then, a linear stability analysis may be carried out in the conventional way, the originally plane bed being stable only if the flow conditions are such that the amplitude of the sinusoidal bed waves is attenuated.

The Basic Equations of First-order Theory

The flow is considered a small double-periodic perturbation of a uniform flow with the velocity V. The coordinates are x_1 in the direction of the undisturbed flow, x_2 vertical, and x_3 horizontal in the undisturbed bed. The wave lengths of the perturbations are correspondingly L_1 and L_3, and the cyclic wave numbers are

$$k_1 \equiv \frac{2\pi}{L_1} \text{ and } k_3 \equiv \frac{2\pi}{L_3} \tag{1}$$

The migration velocity a of the bed form is generally treated as a complex quantity $a \equiv a_r + \iota a_i$, where $\iota \equiv \sqrt{-1}$.

The velocity potential φ for the flow of a frictionless fluid over the double-periodic bed is given by

$$\varphi = -Vx_1 - [c_1 \exp(k_2x_2) + c_2 \exp(-k_2x_2)] \exp\{\iota k_1 (x_1 - at)\} \times \exp(\iota k_3 x_3) \tag{2}$$

where c_1 and c_2 are arbitrary. It is easily verified that Laplace's equation is fulfilled if

$$k_2^2 = k_1^2 + k_3^2 \tag{3}$$

These expressions are now made non-dimensional by the introduction of $\varphi = VD\varphi'$, $\xi_i \equiv x_i/D$, $t' \equiv Vt/D$ and $a' \equiv a/V$, (4)

where D is the undisturbed depth:

$$\varphi' = -\xi_1 - [c_1 \exp(k_2 D\xi_2) + c_2 \exp(-k_2 D\xi_2)] \exp\{\iota k_1 D(\xi_1 - a't')\} \times \exp(\iota k_3 D\xi_3) \tag{5}$$

Next, we consider the continuity of the sediment load, assuming suspension to be the actual transport mechanism.

The distribution of the suspended load in the non-uniform flow is expressed mathematically as an equilibrium between settling, diffusion, and convection.

If c denotes the volume concentration of sediment and $\mathbf{w}$ the settling velocity, the sediment flux due to settling is equal to $c\mathbf{w}$.

If ε is the diffusivity, the sediment flux caused by diffusion equals ε grad c.

Hence, for a unit volume of the water-sediment mixture, we obtain the following equation of continuity

$$\frac{dc}{dt} \equiv \text{div}(-c\mathbf{w} + \varepsilon \,\text{grad}\, c) \tag{6}$$

or

$$\frac{dc}{dt} = w\frac{\delta c}{\delta x_2} + \varepsilon \nabla^2 c \tag{7}$$

in which x_2 is the vertical coordinate.

In the case of steady uniform flow, Eq. (7) is reduced to a simple ordinary differential equation with the solution

$$c_o \equiv c_{bo} \exp(-wx_2/\varepsilon) \tag{8}$$

in which c_{bo} is a nominal concentration at the bed, a quantity to be discussed later in connection with the boundary conditions. When the uniform flow is perturbed, the concentration c is given by

$$c \equiv c_o + \tilde{c} \tag{9}$$

in which c is supposed to be small as compared with c_o.

Now, the total derivative of the concentration is considered

$$\frac{dc}{dt} = \frac{\delta\tilde{c}}{\delta t} + v_1\frac{\delta\tilde{c}}{\delta x_1} + v_2\frac{\delta}{\delta x_2}(c_o + \tilde{c}) + v_3\frac{\delta\tilde{c}}{\delta x_3} \tag{10}$$

in which v_1, v_2, v_3 are the velocity components. By linearization we obtain

$$\frac{dc}{dt} = \frac{\delta\tilde{c}}{\delta t} + V\frac{\delta\tilde{c}}{\delta x_1} - \frac{\delta\varphi}{\delta x_2}\frac{dc_o}{dx_2} \tag{11}$$

Introducing the non-dimensional variables the complete continuity equation becomes

$$\frac{\delta \tilde{c}}{\delta t'} + \frac{\delta \tilde{c}}{\delta \xi_1} - \frac{\delta \varphi'}{\delta \xi_2}\frac{dc_o}{d\xi_2} = \frac{w}{V}\frac{\delta \tilde{c}}{\delta \xi_2} + \frac{\varepsilon}{VD}\nabla^2 \tilde{c} \tag{12}$$

The assumption of periodicity is expressed by

$$\tilde{c} = f(\xi_2) \exp\left\{\iota k_1 D (\xi_1 - a't')\right\} \exp(\iota k_3 D \xi_3) \tag{13}$$

in which f is an unknown function. When this is inserted in Eq. (12), we obtain the following ordinary differential equation of the second order:

$$f'' + \frac{wD}{\varepsilon} f' - \iota k_1 D \frac{VD}{\varepsilon} \left[(1-a') - \iota \frac{(k_2 D)^2}{k_1 D}\frac{\varepsilon}{VD}\right] f \equiv$$

$$\frac{VD}{\varepsilon}\frac{dc_o}{d\xi_2} k_2 D \left(c_1 \exp(k_2 D \xi_2) - c_2 \exp(-k_2 D \xi_2)\right) \tag{14}$$

When Eq. (8) is introduced, the general solution of this equation is easily obtained analytically.

The Boundary Conditions

(1) The perturbation of the sand bed is described by the local height h of the bed above the x_1, x_3-plane and is given by the expression

$$h \equiv h_o \exp\left\{\iota k_1 D (\xi_1 - a't')\right\} \exp(\iota k_3 D \xi_3) \tag{15}$$

The kinematical condition for the sand bed is

$$v_2 \equiv \frac{dh}{dt} \simeq \frac{\delta h}{\delta t} + \frac{\delta h}{\delta x_1} \cdot V \equiv \left(-\frac{\delta \varphi}{\delta x_2}\right)_{\xi_2 \equiv 0}$$

If Eqs. (5) and (15) are substituted, we get

$$c_1 - c_2 \equiv \iota \frac{k_1 D}{k_2 D}\frac{h_o}{D}(1-a') \tag{16}$$

(2) The kinematical boundary condition for the water surface is formulated exactly analogous to the condition for the sand bed. The water surface deviates from the unperturbed level by

$$\eta \equiv \eta_o \exp\left\{\iota k_1 D (\xi_1 - a't')\right\} \exp(\iota k_3 D \xi_3) \tag{17}$$

h_o as well as η_o are real.

The boundary condition is

$$v_2 = \frac{\delta\eta}{\delta t} + \frac{\delta\eta}{\delta x_1} V = \left(-\frac{\delta\varphi}{\delta x_2} \right)_{\xi_2 = 1}$$

and again with Eqs. (5) and (17)

$$c_1 \exp(k_2 D) - c_2 \exp(-k_2 D) \equiv \iota \frac{k_1 D}{k_2 D} \frac{\eta_o}{D} (1 - a') \tag{18}$$

(3) In order to eliminate η_o from Eq. (18), we use Bernoulli's equation on the surface

$$\frac{dv_1}{dt} = -g \frac{\delta\eta}{\delta x_1}, \; \xi_2 = 1$$

In this we substitute the velocity v_1 obtained from the potential Eq. (5):

$$ik_1 V^2 (1 - a') (c_1 \exp(k_2 D) + c_2 \exp(-k_2 D)) \equiv -\frac{g\eta_o}{D} \tag{19}$$

From Eqs. (18) and (19), η_o is eliminated. Introducing the Froude number $\mathsf{F} \equiv V/\sqrt{gD}$ we get the definitive expressions for c_1 and c_2 from Eqs. (16), (18), and (19):

$$c_1 = \frac{\iota \frac{k_1 D}{k_2 D} (1 - a') \exp(-k_2 D) \left[+ \mathsf{F}^2 (1 - a')^2 + \frac{k_2 D}{(k_1 D)^2} \right]}{2\mathsf{F}^2 (1 - a')^2 \cosh(k_2 D) - 2 \frac{k_2 D}{(k_1 D)^2} \sinh(k_2 D)} \frac{h_o}{D}$$

$$c_2 = \frac{\iota \frac{k_1 D}{k_2 D} (1 - a') \exp(k_2 D) \left[- \mathsf{F}^2 (1 - a')^2 + \frac{k_2 D}{(k_1 D)^2} \right]}{2\mathsf{F}^2 (1 - a')^2 \cosh(k_2 D) - 2 \frac{k_2 D}{(k_1 D)^2} \sinh(k_2 D)} \frac{h_o}{D} \tag{20}$$

(4) The vertical sediment flux must vanish at the water surface

$$\varepsilon \frac{\delta \tilde{c}}{\delta x_2} + w\tilde{c} = 0, \; \xi_2 = 1$$

By substitution of Eq. (13), this gives the condition

$$f'(1) + \frac{wD}{\varepsilon} f(1) \equiv 0 \tag{21}$$

(5) The last boundary condition is based on the relation between the bed concentration c_b and the friction velocity.

In order to find c_b, the reasonable accurate empirical formula for the average concentration of the sediment in suspension is used (Engelund 1970):

$$\frac{q_{so}}{q} = \alpha \left(\frac{U_f}{w}\right)^4 \tag{22}$$

q_{so} is the sediment transport rate per unit width, q the corresponding water discharge, $q \equiv VD$, w is the fall velocity of the sediment grains, U_f the friction velocity, and α is a constant. From experiments in which the suspended load was a dominating part of the total sediment transport it is found that α has the value 0.00056. Eq. (22) is not assumed to be more accurate than most other transport formulae, but has been applied because of its convenience.

Eq. (22) relates the sediment transport to the bed shear stress. With the exponential distribution c_o of the suspended sediment (Eq. (8)), the relation between the sediment transport and the bed concentration is

$$q_{so} \equiv \int_0^D v_1\, c_o\, dx_2 = V c_{bo} \frac{\varepsilon}{w}$$

The diffusivity is assumed equal to the eddy viscosity ε. From experiments and dimension considerations it has been found to be

$$\varepsilon = 0.077\, U_f D \tag{23}$$

From the three equations, we obtain the following expression for the bed concentration:

$$c_{bo} \equiv 13\alpha \left(\frac{U_f}{w}\right)^3 \tag{24}$$

This result is immediately seen to make physical sense. The square of the friction velocity equals the kinematical shear stress. The larger the shear, the more violent is the tendency for fluid turbulence to pick up sediment particles from the bed. Similarly, the larger the settling velocity w, the more pronounced is the tendency for sediment particles near the bottom to leave the state of suspension by settling.

In the case of gradually varied flow it is a reasonable assumption that the bottom concentration c_b still varies according to Eq. (24) when the local value of U_f is substituted, so that the process of establishing the equilibrium between pick-up and settling takes place without any spatial lag.

In the present theory the friction velocity does not appear directly, since we use the potential theory. We introduced it through the definition of the diffusivity. Like in ordinary channel hydraulics it is assumed that U_f is proportional to the relative velocity between fluid and bed form:

$$U_f \equiv (U_b - a)/K \tag{25}$$

in which U_b is the flow velocity along the bed and K is a constant.

The bottom concentration is now written as

$$c_b \equiv \frac{13\alpha}{K^3}\left(\frac{V}{w}\right)^3\left(\frac{U_b}{V} - a'\right)^3 \equiv \mathrm{H}\left(\frac{U_b}{V} - a'\right)^3 \tag{26}$$

From the Eqs. (8), (9), and (13) we obtain

$$c_b = c_{bo} + \frac{dc_o}{dx_2}\,\mathrm{h} + f\,(o)\exp\left\{\iota k_1 D\,(\xi_1 - a't')\right\}\exp\,(\iota k_3 D\xi_3)$$

or

$$c_b = c_{bo} + \left(-\,c_{bo}\frac{wD}{\varepsilon}\,\frac{h_o}{D} + f\,(o)\right)\exp\left\{\iota k_1 D\,(\xi_1 - a't')\right\}\exp\,(\iota k_3 D\xi_3)$$

By the application of Eq. (26), using $U_b = \left(-\frac{\delta\varphi}{\delta x_1}\right)_{\xi_2 = 0}$ we get the alternative expression

$$c_b \equiv \mathrm{H}\left\{1 - a' + (c_1 + c_2)\,D\iota k_1\exp\left\{\iota k_1 D\,(\xi_1 - a't')\right\}\exp\,(\iota k_3 D\xi_3)\right\}^3$$

or, after linearization,

$$c_b = \mathrm{H}\,(1 - a')^3 + 3\mathrm{H}\,(1 - a')\;(c_1 + c_2)\,D\iota k_1\exp\left\{\iota k_1 D\,(\xi_1 - a't')\right\}\exp\,(\iota k_3 D\xi_3)$$

By comparison with the previous expression we find

$$\mathrm{H}\,(1 - a')^3 \equiv c_{bo} \text{ and } 3\mathrm{H}\,(1 - a')\,(c_1 + c_2)\,D\iota k_1 \equiv -\,c_{bo}\frac{wW}{\varepsilon}\,\frac{h_o}{D} + f\,(o)$$

or, eliminating H

$$3\,\frac{(c_1 + c_2)}{(1 - a')^2}\,D\iota k_1\,c_{bo} = -\,c_{bo}\frac{wD}{\varepsilon}\,\frac{h_o}{D} + f\,(o) \tag{27}$$

With the two boundary conditions Eqs. (21) and (27), the differential equation (14) can be solved directly. The solution is

$$f = A_1\exp\,(r_1\xi_2) + A_2\exp\,(r_2\xi_2) + A_3\exp\,(r_3\xi_2) + A_4\exp\,(r_4\xi_2) \tag{28}$$

in which

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} \equiv \pm\,k_2 D - \frac{0.077\,U_f}{w}$$

$$\left.\begin{matrix} r_3 \\ r_4 \end{matrix}\right\} = -\frac{wD}{2\varepsilon} \pm \sqrt{\left(\frac{wD}{2\varepsilon}\right)^2 - c}\,,\; c = \frac{-\,\iota k_1 D}{0.077}\,\frac{V}{U_f}\left[(1 - a') - \iota\frac{(k_2 D)^2}{k_1 D}\,\frac{U_f}{V}\,0.077\right]$$

$$A_1 \equiv \frac{-13\alpha}{(0.077)^2} \frac{V}{U_f} \left(\frac{U_f}{w}\right)^2 \frac{k_2 D c_1}{\left(r_1^2 + \frac{wD}{\varepsilon} r_1 + c\right)}$$

$$A_2 \equiv \frac{+13\alpha}{(0.077)^2} \frac{V}{U_f} \left(\frac{U_f}{w}\right)^2 \frac{k_2 D c_2}{\left(r_2^2 - \frac{wD}{\varepsilon} r_2 + c\right)}$$

$$A_3 = \left[A_1 \left(-k_2 D \exp(r_1) + \left(r_4 + \frac{wD}{\varepsilon}\right) \exp(r_4)\right) + A_2 \left(k_2 D \exp(-r_2) + \left(r_4 + \frac{wD}{\varepsilon}\right) \exp(r_4)\right) - 13\alpha \left(\frac{U_f}{w}\right)^3 \left(\frac{3(c_1 + c_2)}{(1-a')^2} iDk_1 + \frac{wD}{\varepsilon} \frac{h_o}{D}\right)\left(r_4 + \frac{wD}{\varepsilon}\right) \times \exp(r_4)\right] \Big/ \left[\left(r_3 + \frac{wD}{\varepsilon}\right) \exp(r_3) - \left(r_4 + \frac{wD}{\varepsilon}\right) \exp(r_4)\right]$$

$$A_4 \equiv 13\alpha \left(\frac{U_f}{w}\right)^3 \left[3\iota \frac{k_1 D}{(1-a')^2} (c_1 + c_2) + \frac{wD}{\varepsilon} \frac{h_o}{D}\right] - A_1 - A_2 - A_3$$

The Stability Analysis

Having obtained the function f, it is easy now to carry out a stability analysis, that is, to predict the conditions under which the initial bed wave amplitude h_o will decrease or increase.

The total rate of suspended sediment transport in the x_1 direction is found from

$$q_{s1} \equiv \int_h^{D+\eta} v_1 c \, dx_2$$

$$q_{s1} \equiv \int_{h/D}^{1} VD\left\{1 + (c_1 \exp(k_2 D \xi_2) + c_2 \exp(-k_2 D \xi_2))\, \iota k_1 D \exp(\iota k_1 D (\xi_1 - a't')) \times \exp(\iota k_3 D \xi_3)\right\}\left\{c_o + f(\xi_2) \exp\left\{\iota k_1 D (\xi_1 - a't')\right\} \exp(\iota k_3 D \xi_3)\right\} d\xi_2$$

After substituting f and c_o from Eqs. (28) and (8) the expression can be integrated. Introducing the sediment transport rate q_{so} for uniform flow the integrated expression can be written as

$$\frac{q_{s1} - q_{so}}{VD} \equiv -c_{bo} \frac{h_o}{D} + \exp\left\{\iota k_1 D (\xi_1 - a't')\right\} \exp(\iota k_3 D \xi_3) \times \mathrm{E} \tag{29}$$

in which

$$E \equiv \frac{(\exp(r_1)-1)}{r_1}\left[c_1 \cdot 13\alpha\left(\frac{U_f}{w}\right)^3 \iota k_1 D + A_1\right] - \frac{(\exp(-r_2)-1)}{r_2}\left[c_2 \cdot 13\alpha \times \left(\frac{U_f}{w}\right)^3 \iota k_1 D + A_2\right] + \frac{A_3}{r_3}[\exp(r_3)-1] + \frac{A_4}{r_4}[\exp(r_4)-1]$$

The total rate of suspended sediment transport in the transverse direction is

$$q_{s3} \equiv \int_h^{D+\eta} v_3 c dx_2$$

$$\equiv \int_{h/D}^{1} VD\left\{c_1 \exp(k_2 D\xi_2) + c_2 \exp(-k_2 D\xi_2)\right\} \cdot \iota k_3 D \times \exp(\iota k_1 D(\xi_1 - a't')) \times \exp(\iota k_3 D\xi_3)\Big\}\Big\{c_o + f(\xi_2)\exp\left\{\iota k_1 D(\xi_1 - a't')\right\}\exp(\iota k_3 D\xi_3)\Big\} d\xi_2$$

Substituting c_o from Eq. (8), we obtain

$$\frac{q_{s3}}{VD} = \exp\left\{\iota k_1 D(\xi_1 - a't')\right\}\exp(\iota k_3 D\xi_3) \times G$$

in which

$$G \equiv \frac{(\exp(r_1)-1)}{r_1}\left[13\alpha \cdot c_1\left(\frac{U_f}{w}\right)^3 \iota k_3 D\right] - \frac{(\exp(-r_2)-1)}{r_2}\left[13\alpha \cdot c_2\left(\frac{U_f}{w}\right)^3 \iota k_3 D\right]$$

For an unsteady situation we have the following continuity equation for the sediment:

$$\frac{\delta q_{s1}}{\delta x_1} + \frac{\delta q_{s3}}{\delta x_3} \equiv -(1-n)\frac{\delta h}{\delta t} \tag{30}$$

n is the porosity of the sand bed, and the total transport rate q_t is assumed equal to the suspended load transport rate. Substituting Eqs. (15) and (29) in this continuity equation, we get

$$(1-n)\,a' \equiv \left[-13\alpha\left(\frac{U_f}{w}\right) + E\frac{D}{h_o} + G\frac{k_3 D}{k_1 D}\frac{D}{h_o}\right] \tag{31}$$

From Eq. (31), a' is calculated through an iteration process: putting a'_n to the right, we obtain a'_{n+1} on the left. Three or four iterations are sufficient. The number, $a' \equiv a'_r + a'_i$, depends on the non-dimensional parameters

F, k_1D, k_3D, U_f/w, and V/U_f

From the expression $h = [h_o \exp(k_1 D a'_i t')] \exp\left\{\iota k_1 D(\xi_1 - a'_r t')\right\}\exp(\iota k_3 D\xi_3)$ it is seen that the perturbation will increase (indicating instability) if the purely imaginary part a'_i is positive. If a'_i is negative, the perturbation will be attenuated, corresponding to a stable, plane bed. The neutral perturbation corresponds to $a'_i = 0$, in which case the amplitude of the bed waves will keep its original magnitude.

In Fig. 1, a'_i is plotted against the Froude number **F** for some reasonable, arbitrary values of the parameters. The example is typical of the unstable situation. There is a singularity for a'_i for a Froude number $\mathbf{F}_1$ near the critical value and a zero for a larger Froude number F_2. The zeroes indicate points of neutral stability, and for Froude numbers in the interval $\mathbf{F}_1 < \mathbf{F} < \mathbf{F}_2$ we find po-

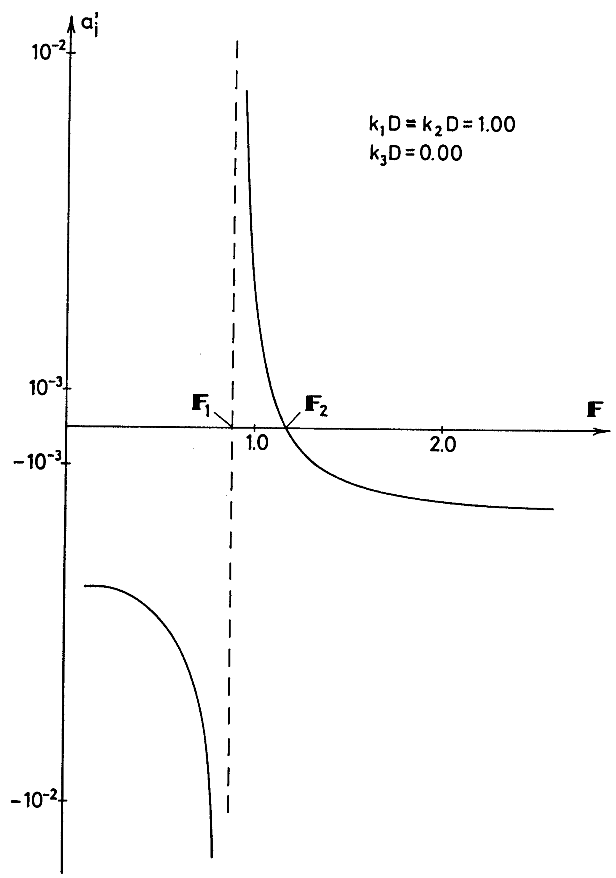

Fig. 1.
Variation of the imaginary part of the bed wave celerity as a function of the Froude number **F**.

sitive values of a'_i, corresponding to instability. For values of the Froude number outside this interval the bed waves will be attenuated.

When a sufficient number of such calculations have been carried out with new values of k_1D, keeping k_3D constant (that is, $k_2D \equiv (k_1^2D^2 + k_3^2D^2)^{1/2}$), we are able to work out stability diagrams as presented in Figs. 2–6. The abscissa is still a Froude number, while the ordinate is the parameter k_1D. The curves of neutral stability form a parametric family, the parameter being k_3D. The diagrams assume a constant value of the parameters U_f/w and V/U_f. However, the diagram is not sensitive to changes in V/U_f, nor do the curves depend in any

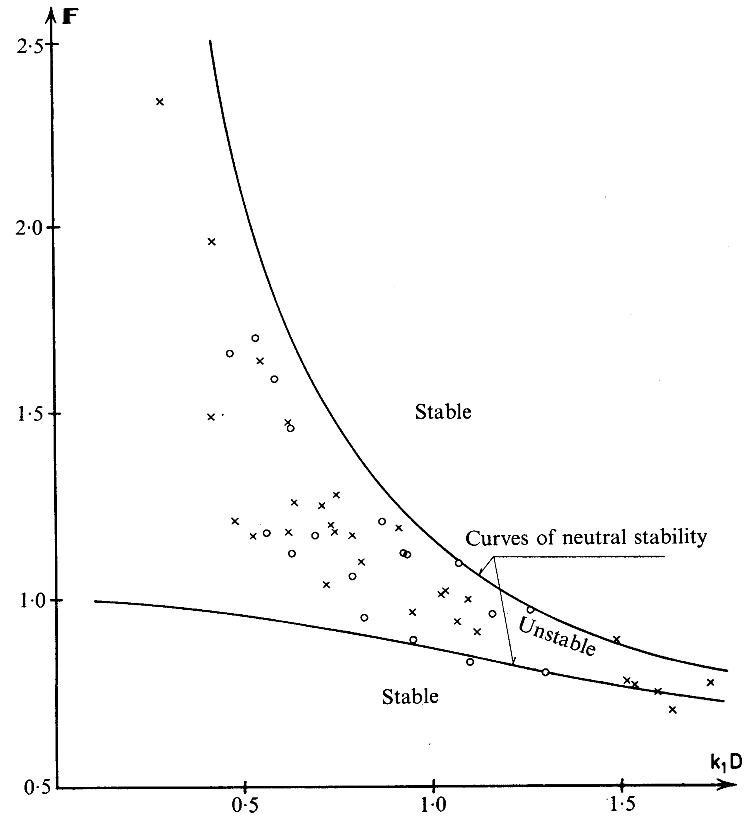

Fig. 2.
Stability diagram for two-dimensional flow, $k_3D \equiv 0$.
× Data from Kennedy (1961).
○ Data from Guy et al. (1966).
△ Data from Navntoft (1968).

essential degree on the parameter U_f/w, as this has been varied between the limits 1 and 4.

The unstable area is bounded by two limiting curves, the upper one being the stability boundary predicted by Reynolds (1965), given by the equation

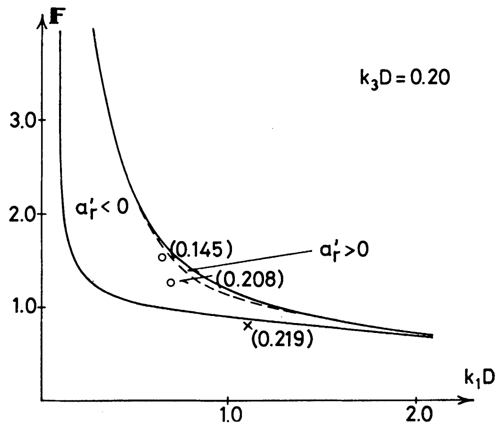

Fig. 3.

Stability diagram for three-dimensional flow, $k_3D \equiv 0.2$. For explanation of symbols, see legend to Fig. 2.

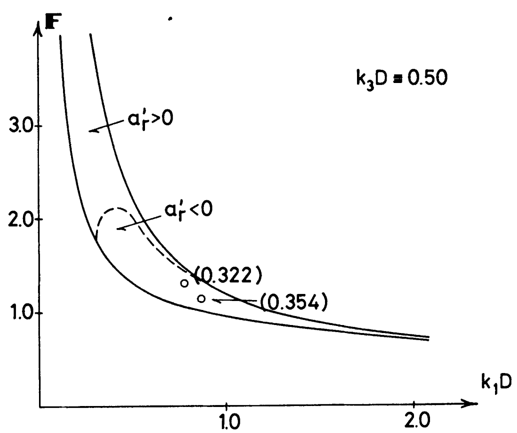

Fig. 4.

Stability diagram for three-dimensional flow, $k_3D \equiv 0.5$. For explanation of symbols, see legend to Fig. 2.

$$F^2 = \frac{[(k_1D)^2 + (k_3D)^2]^{1/2} \tanh [(k_1D)^2 + (k_3D)^2]}{(k_1D)^2} = \frac{k_2D \tanh (k_2D)}{(k_1D)^2}$$

In the special case where the flow is plane corresponding to $k_3D = 0$, the expression becomes

$$F^2 \equiv \frac{\coth (k_1D)}{k_1D}$$

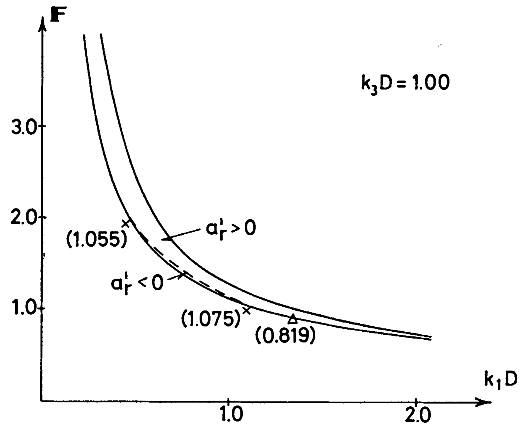

Fig. 5.
Stability diagram for three-dimensional flow, $k_3D \equiv 1$. For explanation of symbols, see legend to Fig. 2.

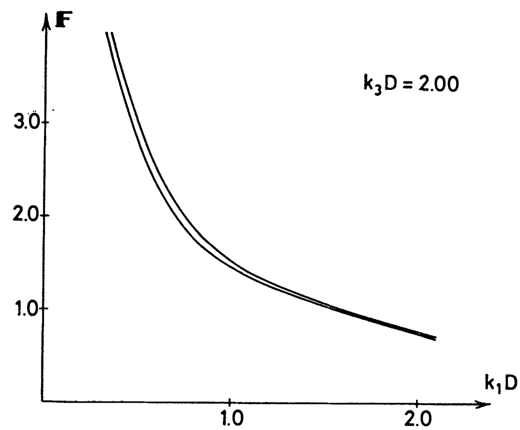

Fig. 6.
Stability diagram for three-dimensional flow, $k_3D \equiv 2$.

The lower limiting curve corresponds to the condition of three-dimensional critical flow (Engelund & Munch-Petersen 1953):

$$\mathsf{F}^2 = \frac{[(k_1D)^2 + (k_3D)^2]^{1/2} \coth [(k_1D)^2 + (k_3D)^2]}{(k_1D)^2} = \frac{k_2D \coth (k_2D)}{(k_1D)^2}$$

In the plane case, $k_3D = 0$ and $k_1D = k_2D$, we obtain

$$\mathsf{F}^2 = \frac{\tanh (k_1D)}{k_1D}$$

These latter curves are stability boundaries in the present model, while in previous potential analyses they mark the transition from antidunes to dunes.

As will be seen from Figs. 2–6, the unstable area becomes smaller for increasing k_3D and, in practice, is disappearing for $k_3D \sim 2.0$. Theoretically, the area will never disappear, because the upper and lower limiting curves approach each other only asymptotically.

In the plane case, antidunes are always observed moving upstream, which explains the name of antidunes. In a flow of a three-dimensional character, they are occasionally observed moving downstream, and it appears that the present simplified model is actually able to describe this phenomenon. The real part of a', a'_r, is the dimensionless migration velocity of the antidunes. For $a'_r < 0$, the antidunes are moving upstream, and for $a'_r > 0$ the antidunes are moving downstream. In the plane case, the model gives only negative values of a'_r, but for $k_3D > 0$, a'_r becomes positive in some regions as shown in Figs. 3, 4, and 5. For small values of k_3D, a'_r is positive only for great values of k_3D, but for increasing k_3D, a'_r becomes positive in a still greater part of the unstable region.

The two-dimensional case is presented in Fig. 2, where the experimental points correspond to the flow configuration characterized as plane standing waves or antidunes.

For the three-dimensional case a similar comparison is rather difficult, because a regular double-periodic flow pattern will seldom occur. If the flow has a significant three-dimensional character, it will often be a superposition of several sinusoidal waves with different wave numbers, so that k_3 is not very well defined. The existence of "rooster tails" is an extreme example. How the superposition of two waves may produce a water surface approaching the rooster tail conditions is illustrated in Fig. 7.

In Figs. 3, 4, and 5 some flume data are plotted for the three-dimensional case. For k_3 is substituted the value corresponding to the dominant wave component. This has been estimated from descriptions or from photos. There is a large amount of uncertainty in this comparison.

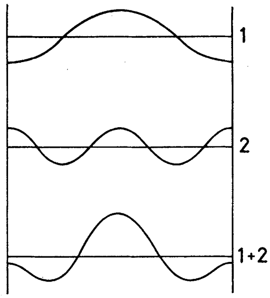

Fig. 7.
Superposition of first- and second-order waves.

CONCLUSION

The result of the analysis is that the conclusions obtained by A. J. Reynolds are essentially confirmed, apart from the fact that the introduction of a definite transport mechanism changes the transition curve between dunes and antidunes in the previous models to a stability boundary in the present. For fine sediment this is in accordance with experimental findings.

NOTATION

$a \equiv a_r + \iota a_i$	: complex migration velocity of sand waves
c	: volume concentration of sediment
D	: water depth
F	: Froude number
h	: height
k_1, k_2, k_3	: wave numbers
q	: water discharge per unit width
q_s	: sediment discharge per unit width
t	: time
$U_f = \sqrt{\tau_0/\varrho}$	: friction (shear) velocity
V	: mean velocity
v_1, v_2, v_3	: velocity components
w	: settling velocity of sediment particles
x_1, x_2, x_3	: space coordinates
η	: water surface perturbation
ξ_1, ξ_2, ξ_3	: dimensionless coordinates
ε	: diffusivity
$\iota \equiv \sqrt{-1}$	: imaginary unit

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Address:
Hydraulic Laboratory, Technical University of Denmark,
Building 115, 2800 Lyngby, Denmark

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