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Order Selection of AR Models of Hydrologic Time Series

G. Padmanabhan
North Dakota State University, Fargo ND-58105, USA

A.R. Rao
Purdue University, W. Lafayette, IN-47907, USA

Recently several criteria have been proposed for selection of orders of AR models. One of these, namely the AIC has been used in hydrologic analysis. However, the results given by these criteria when observed hydrologic data are analyzed by using them have not been studied to determine the variability in the results obtained by these criteria.

Some of the more widely discussed of these criteria have been analyzed in this paper by using both hydrologic and nonhydrologic time series. The orders of AR models given by Akaike's information criterion, the final prediction error criterion, the criterion autoregression transfer function, Schwarz's criterion and the posterior probability criterion have been analyzed, and the results are discussed.

Although the results obtained by these criteria are often similar, the posterior probability criterion has been recommended for use because it is possible to estimate the probability of the order being correct by using it.

Introduction

Autoregressive (AR) models are extensively used in modeling Hydrologic Time Series. These models are used for forecasting and generation of synthetic data. Autocorrelation functions and partial autocorrelation functions of the time series are often used to identify the approximate order of AR models. The commonly used present procedure to select the correct order of AR models is to increase the order of models and test the residuals from these models. The model with the smallest number of parameters but which has uncorrelated and nonperiodic residuals is selected as a valid model of the process. Although the procedure is feasible, it is subjective. It is possible to fit several contiguous and noncontiguous AR models with the same number of parameters and the model choice in such cases is not easy. Although it is possible to resort to simulation procedures to select the models better methods which would quantify the error involved in the decisions are needed.

AR models are also used in the Maximum Entropy Spectral (MES) estimation of hydrologic data. This is a high resolution spectral estimation method which is being extensively used by geophysicists. By virtue of its high resolution and computational ease, it can be advantageously used in the analysis of hydrologic data in preference to conventional methods of spectral analysis (Padmanabhan and Rao 1980). However, one of the problems associated with the MES method is the determination of the correct order of AR model to be used. At present, the orders of AR models in MES estimation are subjectively selected.

Recently, several statistical decision rules have been proposed for selection of orders of AR models. Since they are designed to provide correct order estimates, it is desirable that these decision rules posses certain statistical properties such as consistency, optimality, etc. Some of the currently used decision rules do not have these properties. For example, the decision rule based on Akaike's Information Criterion (*AIC*) (Akaike 1970, 1971, 1974) has been widely discussed as a part of hydrologic modeling. However, the *AIC* rule has been proved to be inconsistent (Shaibata 1976, Kashyap 1980). Consequently there is no apriori justification for choosing *AIC* rule over other consistent decision rules. In view of such occurrences there is a need to investigate these criteria which have been proposed for AR model order selection. The results from such investigations, preferably based on real data, would enable us to rank these decision rules and use them with greater confidence in modeling hydrologic time series.

The basic objective of the research reported herein is to evaluate and compare some of the recently proposed decision rules. Some of the decision rules investigated herein are the Akaike's Information Criterion (*AIC*), Final Prediction Error Criterion (*FPE*), Criterion Autoregression Transfer Function ($CAT(p)$), Schwarz's Criterion ($S(p)$) and Posterior Probability Criterion. Hydrologic and Meteorologic time series – several of which are well known – are used in the study. The results of the analysis would indicate the relative merits of these decision rules.

The paper is organized as follows. The decision rules used in the study are discussed first, next details about the data used in the study are presented, and then the results of the data analysis are given. A discussion of results are given in the conclusion.

Criteria for Determination of Order of AR Models

The first approach to the problem of order determination of AR models was posed as a multiple hypothesis testing problem by Anderson (1963). However, this approach has the disadvantage of being based on arbitrary significance levels. Therefore, the method is subjective and hence is not further discussed herin.

Two recently developed criteria for order selection are the criteria based on Final Prediction Error (*FPE*) and on Information theoretic concepts (*AIC*). Both of these criteria are developed by Akaike (1969, 1970, 1971, 1974) and have been used for determining orders of stochastic models. These two criteria are herin referred to as *FPE* and *AIC* criteria. The *FPE* is applicable only for AR models while *AIC* can be used for Autoregressive-Moving Average (ARMA) models also. The *FPE* and *AIC* criterion functions are given below. The model with minimum *FPE* or *AIC* value is selected as the appropriate model.

$$FPE = \frac{n+p+S}{n-p-S} \hat{\sigma}_p^2 \quad (1)$$

$$AIC = n \log \rho_p + 2p \quad (2)$$

In Eqs. (1) and (2), n is the sample size, $\hat{\sigma}_p^2$ is the estimated mean square one-step-ahead prediction error computed by using a p^{th} order model, S is the number of functionally independent parameters used in detrending the series and ϱ_p is the residual variance of the model with p parameters.

Bhansali and Downham (1977) have suggested variants of AIC in which positive integers like 3, 4 and 5, instead of 2, are used in Eq. (2). They have demonstrated by simulation that the power of the decision rule increases with the larger integers in Eq. (2). Later, Akaike (1979) justified the increasing power of the decision rule by using the Bayesian approach. However, based on the asymptotic distribution of *AIC* and simulation studies, Shibata (1976) found that *AIC* is not a consistent estimator of the model order. Kashyap (1980) has analytically derived the lower bound for the probability of error associated with *AIC* and its variants when the number of observations is very large and has shown that there is always a significant probability that the order chosen by the *AIC* would be asymptotically incorrect. In other words, *AIC* rule and its variants are inconsistent. Furthermore, *AIC* is not an optimal decision rule (Kashyap 1977 and Schwarz 1978) in the sense that it does not minimize the probability of error involved in the decision.

A decision rule based on the Criterion Autoregression Transfer function (*CAT(p)*) was suggested by Parzen (1974) to select AR model orders. Schwarz (1978) proposed a decision rule based on another criterion herein called *S(p)* to determine AR model order. Parzen's decision rule is also inconsistent whereas Schwarz's decision rule is consistent under specific conditions only. Fine and Hwang (1979) have proposed a consistent decision rule applicable both for AR and ARMA models. However, they have stated that further simulation studies are required to justify the use of the criterion in their decision so that it can be used as a practical estimator of AR and ARMA model orders. Consequently Fine and Hwang's (1979) decision rule is not considered further herein.

Parzen's *CAT(p)* and Schwarz's *S(p)* criteria are given in Eqs. (3) and (4) respectively

$$CAT(p) = \left(\sum_{j=1}^{p-1} \frac{\hat{\sigma}_j^2}{n-j}\right) - \frac{n-1}{n-p}\,\hat{\sigma}_p^2 \qquad (3)$$

$$S(p) = n \log \hat{\rho}_p + p \log n \qquad (4)$$

In Eqs. (3) and (4), *CAT(p)* and *S(p)* are the values of respective criterion functions for the p^{th} order AR model, $\hat{\sigma}_j^2$ is the one step-ahead prediction mean square error of the j^{th} order AR model, $\hat{\varrho}_p$ is the estimate of residual variance of p^{th} order AR model, and n is the number of observations. The order of the model which gives the minimum value of *CAT(p)* or *S(p)* is selected to be the correct order of the model.

A Bayesian decision rule which possesses optimality and aymptotic consistency has been developed by Kashyap (1977). The principle of parsimony is also quantified by this decision rule. An important property of this rule called the posterior probability rule is that, unlike the decision rules discussed above, it gives the probability of error involved in selecting the order of the model. The criterion function for order selection is given below in terms of the posterior probability $P(C_i|\xi)$ that the model order selected is the correct order.

$$P(C_i|\xi) = K \exp[0.5\, h_i(\xi)] \qquad (5)$$

where

$$h_i(\xi) = -(n-m_i)\log\left[\rho_i + \frac{(\rho-\rho_i)}{n}\right] - \rho_i \log(n) - m_i \log(\rho) + m_i - \frac{1}{\rho}\sum_{j=1}^{m_i}(y_j^2) \qquad (5a)$$

$$K = 1 \Big/ \sum_{1}^{r} \exp[0.5\, h_i(\xi)]$$

$P(C_i|\xi)$ is the posterior probability that the i^{th} model C_i, is the correct one given the observation set $\xi = y(1),\ldots,y(n)$, m_i is the maximum lag used in i^{th} model, p_i is the number of parameters in the model, ϱ_i is the residual variance of the i^{th} model,

ϱ is the variance of the observations $y(1),\ldots,y(n)$. r is the number of models considered.

In this decision rule, the model having the maximum posterior probability is selected to be the correct model. If two or three models with consecutively increasing orders have approximately equal posterior probability followed by models of higher orders with low posterior probability, then one may choose the higher order model. Such a selection is justifiable on the basis that the probability of a model order higher than the one selected being correct is insignificant.

However, if the intent is to select the order of an AR model for spectral estimation by using the maximum entropy method, the criterion function can be suitably modified as given below and the resulting decision rule used for *MES* estimation.

$$\text{Min.}[\{\log P(C_i|\xi) - H\}^2] \qquad (6)$$

Where

$$H = \sum_{i=1}^{r} [P(C_i|\xi) \log P(C_i|\xi)]$$

The details of derivation and proof of properties of this decision rule are found in Kashyap (1977). The use of the decision rule for selecting the order of models which are used for spectral estimation is found in (Padmanabhan and Rao 1980) and is not discussed further herin.

Of the different rules compared in this study, the parsimony is emphasized using separate terms only in *AIC*, *S(p)* and Posterior Probability rule. The penalty per parameter placed on additional parameters is $2p$ in *AIC* and 3, 4 and 5 times p instead of $2p$ in the variants of *AIC* suggested by Bhansali and Downham (1977). In the Posterior Probability rule, and Schwarz's criterion the penalty is $p\log(n)$. Also $p\log(n)$, instead of $2p$, when introduced into the *AIC* has the advantage of making *AIC* consistent (Kashyap 1980). In addition, Posterior Probability rule has several other terms, including one to account for the number of lag terms used in the model. It may be noted that in noncontiguous AR models the lag terms do not appear sequentially.

One of the important assumptions in many of these decision rules is that the candidate models which are considered for selection are equally acceptable *apriori*. This assumption is justified on the premise that usually no *apriori* information would be available about the correct order of models. Therefore it is important that proper candidate models should be included for comparison. The decision rules can only indicate the best model among the models considered.

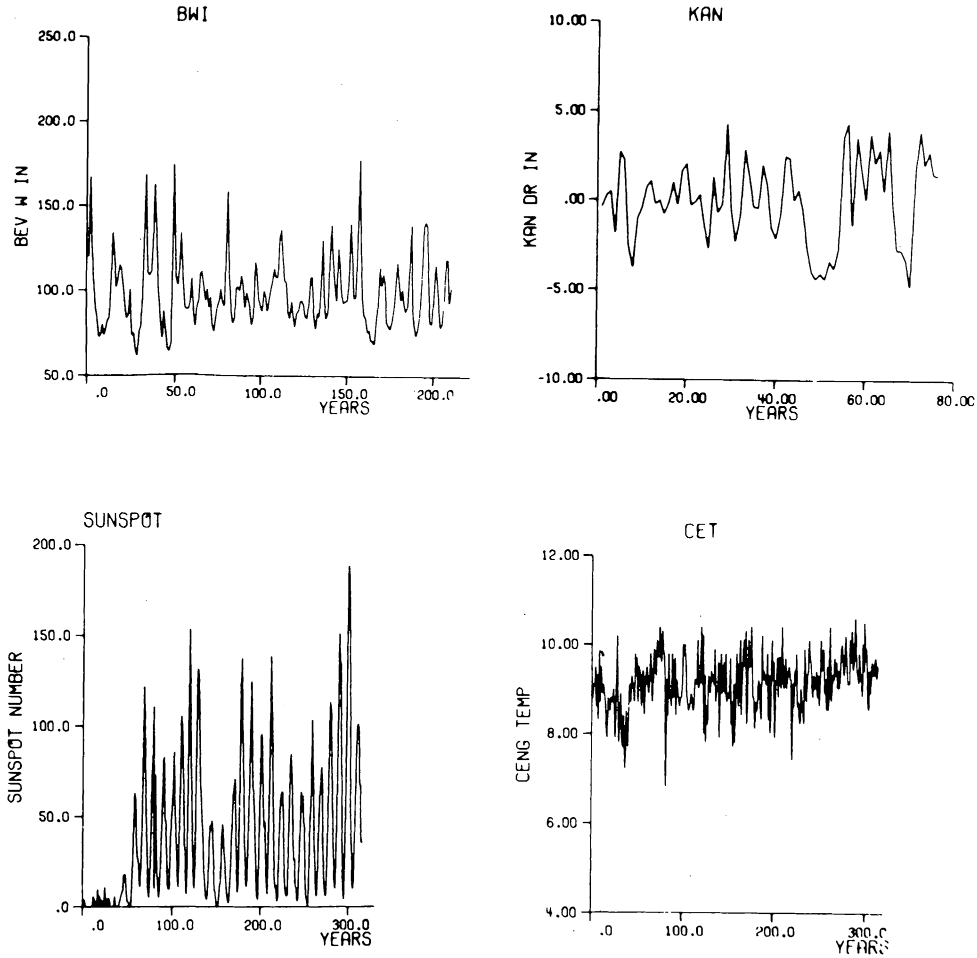

Fig. 1. Some of the Time Series Used in the Study, BWI: Beveridge Wheat Index Series, KAN: Monthly Palmer's Drought Index Series, SUNSPOT: Annual Wolfer's Sunspot Numbers, CET: Annual Central England Temperatures.

Data Used

Some well known time series together with others which have not been extensively analysed but which have interesting characteristics are used in this study (Table 1). Some of these time series are shown in Fig. 1. All the series are annual data except the Palmer's drought index series of Kansas and Iowa which are monthly series. The Sunspot, Beveridge Wheat Index and Central England Tem-

Table 1 – Orders of AR Models for the Data Series Selected by Using Different Criteria

Series	Posterior Probability	*AIC*	*FPE*	*CAT*	*S(p)*	Source* of data
Sunspot	9**	9	9	9	9	29
Beveridge Wheat Index (Y)	2	2	8	2	2	7
Central England Temperatures (Y)	2	4	4	2	2	29
Central England Rainfall (Y)	1	1	1	1	1	23
Kew Rain (Y)	1	1	1	1	1	36
Kalburn Rainfall (Y)	1	1	1	1	1	24
Mangalore Rainfall (Y)	1	1	1	1	1	35
Waltair Rainfall (Y)	1	1	1	1	1	35
Kansas Drought Index (Y)	5	5	4	1	4	26
Iowa Drought Index (Y)	4	4	4	1	4	26
Kansas Drought Index (m)	1	3	3	1	1	26
Iowa Drought Index (m)	1	5	5	1	1	26

(Y) Yearly Series
(m) Monthly Series
* Numbers in this column refer to entries in the List of References
** Non contiguours AR Model with 3 parameters given in Eq. (7)

perature data are well known and have been extensively analysed. The sources of the data series used are indicated in Table 1. Sunspot series is included because of strong 11-year cycle present in the data which would result in higher order AR models. For all the annual rainfall series low order AR models appear to be adequate.

Results and Discussion

The decision rules discussed above were applied to the hydrologic and climatologic time series listed in Table 1. Orders of AR models given by different decisions are listed in Table 1.

For all annual rainfall series, first order models are indicated to be adequate by all the decision rules. For Beveridge's Wheat Index series only the *FPE* criterion indicates an AR(8) model, while others indicate that a AR(2) model is adequate. For Central England temperature data, *AIC* and *FPE* rules indicate AR(4) mo-

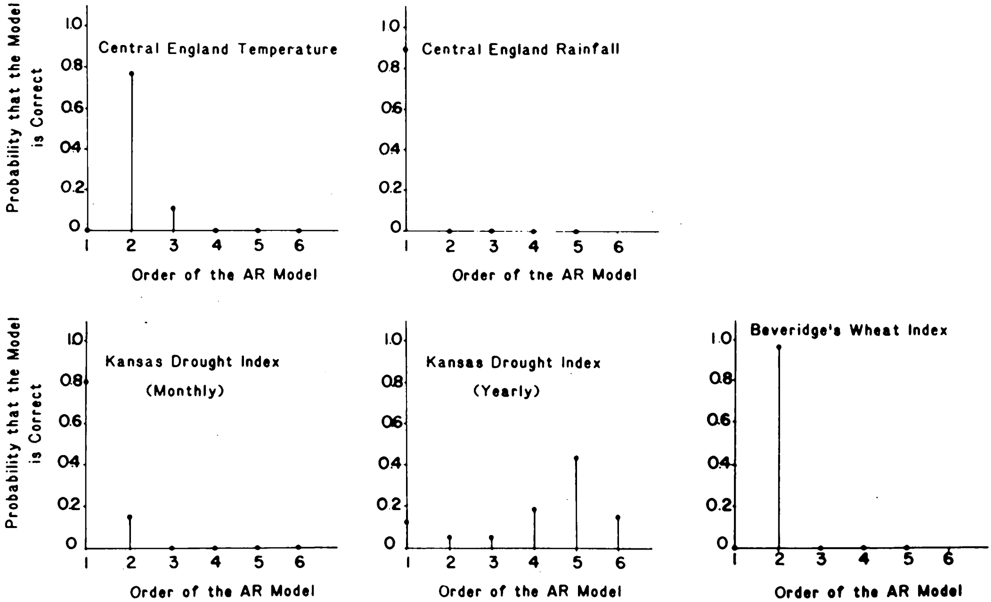

Fig. 2. Posterior probabilities of Orders of AR Models.

dels while others indicate second order models, except *CAT(p)*, which indicates first order models. Higher order models are indicated by *AIC* and *FPE* rules for monthly drought index series.

The orders suggested by various criteria for all the models are not significantly different although in some cases there are variations. In some of the series the number of observations 'n' is not very large. Therefore, the penalty per parameter $\log(n)$ in Posterior Probability and $S(p)$ criteria is not very different from 2 in *AIC*. Consequently the decisions are also not too different. The penalty per parameter could be large in the Posterior Probability and Schwarz's rules if the number of observations were large. However, only the Posterior Probability rule gives estimates of posterior probabilities of the chosen order being correct. These probabilities are shown in Fig. 2. It is interesting to note that the probability of the correct model is clearly evident in the results presented in Fig. 2.

The importance of selecting suitable candidate models and the variation in the posterior probabilities brought about by considering sets of different models are illustrated in Fig. 3 by using Posterior Probability decision rule. In Fig. 3a the posterior probabilities computed for a set of six AR models – AR(1) to AR(6) – fitted to Sunspot data are shown. The posterior probability is maximum for the AR(2) model. In Fig. 3b posterior probabilities computed for a set of 10 AR models – AR(1) to AR(10) – are shown and in this case AR(9) model clearly is to

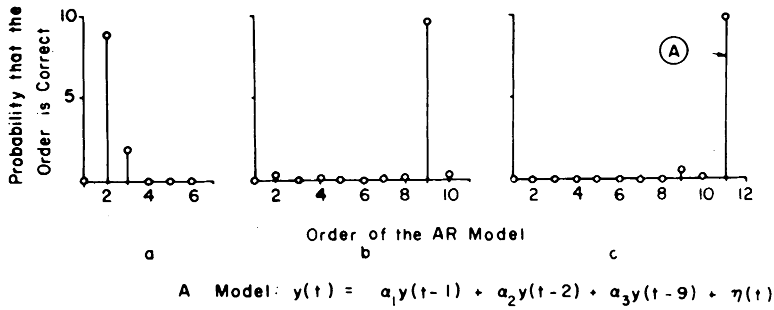

Fig. 3. Variation of Posterior Probabilities of AR Models of Different Orders Being Correct with Different Sets of Candidate Models. (a) Set of Six AR Models from AR(1) to AR(6). (b) Set of Ten AR Models from AR(1) to AR(10). (c) Set of Eleven AR Models Consisting of Models AR(1) to AR(10) and the Noncontiguous Model A Shown in the Figure. All the Models are Fitted to Annual Sunspot Data.

be preferred. Finally in Fig. 3c, posterior probabilities estimated for 11 models consisting of the 10 AR models and a noncontiguous model given in Eq. (7) are shown. In this case the noncontiguous model is clearly preferable thus demonstrating the importance of considering appropriate models.

$$y(t) = \alpha_1 y(t-1) + \alpha_2 Y(t-2) + \alpha_3 y(t-9) + \eta(t) \tag{7}$$

It should be noted that the number of parameters and maximum lag used in noncontiguous models are different whereas they are the same for contiguous models. These two terms contribute differently in prior probability rule whereas the maximum lag used does not play any role in other decision rules.

It should be noted that Gaussian assumption on observations is involved in the derivation of Posterior Probability decision rule. Therefore, the posterior probabilities computed by using this decision rule, may not be correct if the observations are not Gaussian distributed. Although the results of this comparative study have not brought out the power of Posterior Probability decision rule dramatically mainly because of the short length of data series, this decision rule is considerably better than the others investigated in the study because of its superior statistical properties, particularly when long data series are available.

Conclusions

AIC rule is being currently used for the determination of order of AR models of hydrologic time series modeling in spite of the fact that it is statistically inconsistent. There are several other decision rules which have been proposed for selection orders of AR models. One such rule is the posterior probability rule which is statistically consistent and has several other desirable properties such as optimality. The probability of selecting the correct model can also be estimated by using this decision rule. Other decision rules do not have this property. These decision rules have been compared with each other in this paper. Although the order of AR models given by the different decision rules are not very different in many of the time series analysed in the study, Posterior Probability criterion is recommended because of its properties of optimality and asymptotic consistency.

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Address:
G. Padmanabhan,
Department of Civil Engineering,
North Dakota State University,
Fargo, ND-58105,
USA.

A. Ramachandra Rao,
School of Civil Engineering,
Purdue University,
W. Lafayette, IN-47907,
USA.