tangent-gas approximation, this equation can be reduced to the Laplace equation

$$\psi_{\omega\omega} + \psi_{\theta\theta} = 0,$$

where

$$\omega = \ln \frac{\dfrac{w}{w_m} M_m/(1 - M_m{}^2)^{1/2}}{1 + \left[1 + \dfrac{w^2}{w_m{}^2} M_m{}^2/(1 - M_m{}^2)\right]^{1/2}},$$

and subscript m refers to the match state which is assigned experimentally, as discussed below. A similar procedure (with the added assumption of irrotational flow) can be followed to show that the velocity potential also satisfies the Laplace equation. The solution for the stream function and potential function which can be found by the method of singularities is, therefore,

$$\Omega(\omega - i\theta) = \mu \ln \frac{\cosh \dfrac{\pi}{\alpha} (\omega - i\theta - \omega_\infty) - \cosh \dfrac{\pi}{\alpha} (\omega_\infty - \omega_a)}{\cosh \dfrac{\pi}{\alpha} (\omega - i\theta - \omega_\infty) - 1}$$

The solution is then transformed numerically back to the physical plane and lines of constant flow angle, called isoclines, are determined.

Assuming that the nozzle is operating at a given supercritical pressure ratio, the solution for the jet can be obtained with the use of the characteristic equations of axisymmetric flow which can be written as

$$\left.\frac{dy}{dx}\right|_{\mathrm{I,II}} = \tan (\theta \mp \alpha),$$

$$\left.\frac{1}{w}\frac{dw}{d\theta}\right|_{\mathrm{I,II}} = \mp \tan \alpha + \frac{\tan^2 \alpha \tan \theta}{\tan \theta \mp \tan \alpha} \frac{1}{y}\frac{dy}{d\theta}.$$

Using standard numerical procedures these equations can be combined with the solution for the local flow angle in the throat to provide the position of the sonic line. The match Mach number is then selected in such a way as to produce the best possible agreement between the theoretical and experimental sonic lines at the choked pressure ratio. The solution is then completed by calculating the discharge and thrust coefficients with the use of the following relationships

$$C_D = \frac{2}{R^2} \int_S y(\sin \theta dx - \cos \theta dy),$$

and $C_T =$

$$\frac{\dfrac{P^*}{P_T}\left[1 + \dfrac{2k}{R^2}\displaystyle\int_S y \cos \theta(\sin \theta dx - \cos \theta dy)\right] - \dfrac{P_\infty}{P_T}}{{}_D \dfrac{2k}{k-1}\left(\dfrac{P^*}{P_T}\right)^{1/k}\left\{\left[1 - \left(\dfrac{P^*}{P_T}\right)^{(k-1)/k}\right]\left[1 - \left(\dfrac{P_\infty}{P_T}\right)^{(k-1)/k}\right]\right\}^{1/2}},$$

where the subscript S indicates that the integration is to be carried out along the theoretical sonic line, from the nozzle lip to the center line.

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DISCUSSION

J. E. Postlewaite[2]

To the design and test engineers the results of this work provide useful data for selecting and calibrating propulsion nozzles currently in use. The theoretical solution to the mixed flow problem, even though it is not an exact solution from the viewpoint of the mathematician, is none the less extremely useful. First it provides a clear understanding of the flow field through the nozzle throat and second it provides the capability to generate the parametric design data presently required by industry.

An important point in this analysis is the insight to explain why the flow coefficient is dependent upon the local static pressure above critical pressure ratio. We have encountered this phenomenon in wind tunnel tests of scale model airplanes with blowing nozzles where the discharge coefficient varied with both the external velocity and the airplane angle of attack. The engine thrust is a function of the local static pressure above critical pressure ratio until the nozzle flow is choked. This was quite disturbing when first discovered because performance guarantees did not recognize that the jet engines considered operate between the critical pressure ratio and the choked pressure ratio at design point conditions and thus could not provide the required airflow. As a result the production nozzles had to be modified.

I suggest that other areas in which a fundamental understanding of the flow field is desireable and requires further research are:

1 In axisymmetric and two-dimensional plug type nozzles for a single flow with a uniform total pressure profile, and

2 in coaxial flows through conical convergent and plug nozzles with the two flows having different total pressures.

H. S. Hillbrath[3]

The discusser would like to compliment the authors on an excellent paper which, together with another publication by one of the authors[4] has considerably clarified the performance of an important class of constrictions.

Perhaps the most interesting point is that choking does occur for these devices, which corrects the generally held opinion. Although the limiting case of 90 deg half angle (which is the familiar thin plate orifice) is not considered in the present paper, this behavior has been shown to apply in this case.[4] Although the difference in performance is small, compared to previous estimates, the difference in concept is important.

It is also interesting, in an obviously carefully controlled study, to find a disagreement of $1^1/_2$ percent between two methods of flow rate determination. This is not vital to the present study, but one wonders how often even greater errors go undetected because no cross check is made.

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[4] Brown, E. F., and Chow, W. L., "Supercritical Flow Through Convergent, Conical Nozzles," Paper No. 1-3-20; Presented at the First Symposium on Flow—Its Measurement and Control in Science and Industry, Pittsburgh, Pa., May 10-14, 1972.

The authors' study was directed toward a propulsion application, however, similar devices are often used as flow meters. There is considerable interest in development and codification of such devices having the smallest possible uncertainty in flow rate. Intuitively, it would seem that this would result when the actual flow was as close as possible to ideal one-dimensional flow. The authors seem to agree in selecting a device similar to the Smith-Matz circular arc venturi as a comparison standard. Do the authors have any further observations related to measurement applications?

The results shown in Fig. 2 are, at first examination, surprising. Can the authors elaborate on why the thrust coefficients should increase as the velocity profile becomes more peaked?

J. F. Runkel[5]

The authors of the paper provide a valuable contribution for understanding the performance of convergent-conical nozzles. This simple type thrust nozzle has been used with gas turbine engines for decades, yet the lack of reliable data showing the effect of shape variations on the exhaust flow field has caused considerable confusion for exhaust system designers. A continuing problem in this work is the inherent difficulty of achieving high accuracies in the measurement of mass flow and thrust. In addition, the definitions of discharge coefficient and thrust coefficient for convergent nozzles may differ between various authors. For example, the compressible flow discharge coefficient has been variously assumed to include such items as boundary layer effects (area contraction coefficient), jet inclination effects (velocity coefficient), sonic line curvature effects, centrifugal force effects, virial effects, internal flow distortion and swirl effects, and whether or not the discharge coefficient (or effective area ratio) applies to only the momentum force term or also is included in the pressure-area force term in the convergent nozzle thrust equation (references [5 and 6] and references [9] through [12][6]). In the present paper, Thornock and Brown have chosen to examine only the inviscid effects of sonic line curvature.

In the "Testing Procedure" section the authors indicate a flow measurement problem with the curved internal surface reference nozzle. Although the authors do not include a statement on the variation of Reynold's number with jet pressure ratio for the nozzles, the results of reference [10] would indicate that the actual discharge coefficient for the reference nozzle was probably higher than the value of 0.993 selected from reference [5] of the paper. Fig. 1 shows the variation of discharge coefficient with pressure ratio where the theoretical line becomes constant above choked conditions, a trend not exactly verified by the extent of the experimental data. A gradually rising mass flow ratio curve is sometimes observed for convergent nozzles, particularly those having short cylindrical throat sections, which is somewhat similar to the effect on discharge coefficient of increasing Reynolds number. The experimental data scatter of Fig. 1 make it difficult to evaluate how "good" the agreement is compared to the choked condition theoretical line. The faired lines through the experimental data shown in Thornock's reference [1] are as much as $^1/_2$ percent lower than the theoretical line for the 15 deg convergence nozzle and the point at which the curve becomes linear is subject to interpretation.

With Fig. 2 depicting the variation of experimental and theoretical thrust coefficient with jet pressure ratio, the results indicate the thrust coefficient increases as the nozzle convergence angle increases. Thus the nozzle with the lowest discharge coefficient has the highest thrust ratio. For this definition of thrust ratio the actual mass flow is used both for the measured (actual) thrust and for the isentropic thrust. Thornock's experimental measurements are also used in reference [12] but the theoretical thrust variation with pressure ratio shows an opposite trend from that of the present paper because of a different definition of thrust coefficient. The denominator of the thrust ratio in reference [12] contains the ideal one-dimensional mass flow in the isentropic thrust term. This causes an almost direct reduction in thrust ratio, C_F, with reduction in discharge coefficient over the pressure ratio range in reference [12]. In contrast, the theoretical variation in thrust ratio, C_T, of the present paper predicts an increase above the reference nozzle values with increasing discharge coefficient (internal loss) and this divergence between the nozzles with different convergence angles increases also with pressure ratio. If one considers the thrust ratio to be a measure of nozzle thrust efficiency it is difficult to see how it is possible to exceed the performance of a nozzle most closely approaching the one-dimensional ideal case. Some physical explanation of this anomaly should be given by the authors if the statement that "the thrust coefficient increases with increasing nozzle angle" remains in the Conclusions. The authors may find the work of Milford and Migdal (reference [13]) useful for examining this problem.

Although these comments may seem somewhat critical, all propulsion specialists are aware of the difficulty in obtaining highly accurate thrust and flow measurements and the problems of incorporating all loss parameters in theoretical treatments of nozzle flow. Further work along the lines proposed by the authors would be very useful. It would be interesting to examine some additional parameters such as conical nozzles with successive convergence angles, concave versus convex curvature, cylindrical throat extensions, and the effect of rounding the sharp internal lip (reference [14] for example).

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Authors' Closure

The authors would like to express their appreciation for the comments of Mr. Postlewaite, Mr. Hillbrath, and Mr. Runkel.

Those problems suggested by Mr. Postlewaite for further study can be treated by a time dependent solution [12]. While this method requires large amounts of computer time, another solution has been developed[7] which may reduce the time required. It is notable that neither of these solutions requires an experimentally determined parameter for the analysis, something which was necessary in the analysis described in this paper.

The authors agree with Mr. Hillbrath regarding the need for improved accuracy in flow measuring systems. It is felt that the work of Stratford [10] provides a good method of selecting a high

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[6] Numbers [9–14] in brackets designate Additional References at end of discussion.

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accuracy flow measuring nozzle. The nozzle should have neither completely uniform flow at the throat, with the resulting large boundary layer nor a contour to minimize boundary layer growth, in which case the deficit due to potential flow effects is large. Stratford [10] has suggested a wall curvature radius to throat radius of 4.

As indicated by Mr. Runkel, the value of discharge coefficient chosen for our reference nozzle was lower than the value given by Stratford [10]. The value was chosen from the curve of Smith and Matz [5] because of their detailed experimental verification of the analysis. The amount of uncertainity quoted by Stratford (0.15 percent) allows it to be said that his analysis and the value from Smith and Matz are in agreement.

The authors agree with Mr. Runkel's statement that it is difficult to determine the actual choked pressure ratio from the given experimental data. When more precise determinations, are made of nozzle discharge coefficient, a gradual increase with increasing pressure ratio (Reynold's number) is noted, even past the "choked" limit. As applied to experimental data, the term choked can only refer to the pressure ratio at which the change of discharge coefficient with pressure ratio becomes small. In conical convergent nozzles of moderate half angle, the mass flux deficit due to potential flow curvature is larger than that due to the boundary layer and thus when the potential flow field becomes independent of nozzle pressure ratio, the nozzle can be said to be effectively choked. It is to be expected that the data of reference [1] should be somewhat below the theoretical curves due to the effect of the boundary layer which was not accounted for in the analysis. This difference should decrease with increasing half angle.

The definition of thrust coefficient chosen for this paper is the one most often used in aircraft propulsion, though others have been and continue to be used. The reason for this definition is that, when evaluating candidate nozzle systems for a particular engine, the comparison must be made between nozzles operating at the same pressure ratio and passing the same mass flow. In nozzle testing this is done by using, as a normalizing quantity, the thrust which would ideally be produced by the mass flow actually leaving the nozzle. The effect of nozzle half angle on the thrust coefficient appears at first to be anomalous. This behavior is the equivalent of a convergent conical nozzle producing more thrust than an "ideal, one dimensional" convergent nozzle having the same mass flow. It is a direct result of the definition of thrust coefficient—an ideal, one dimensional convergent nozzle passing the same mass flow as the convergent conical nozzle has a smaller throat area. When components of the thrust produced by these two nozzles are compared, the smaller momentum flux of the conical convergent nozzle is more than offset by a larger (pressure) × (area) force, resulting in a larger overall thrust at supercritical pressure ratios.