Binding Energy for Three-Nucleon System

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The theoretical triton bound state energies for several two-nucleon potentials are calculated from the Faddeev equation by analytic continuation method. The calculation has been performed from 5 to 42 channels. The spectator momentum dependence is discussed on the Argonne $V_{14}$ potential.

In a previous paper,\(^1\) we have shown that the three-nucleon binding energy can be obtained from the Faddeev equation for the $n$-$d$ scattering by analytic continuation. In this paper, we show that the same equation can be derived from the original Faddeev $T$-matrix equation\(^2\) and show the theoretical triton binding energies for several potentials. The potentials used are the Argonne $V_{14}$ ($A$ $V_{14}$),\(^3\) the Paris,\(^4\) the modified Paris,\(^1\) the Bonn A and B (nonrelativistic versions)\(^5\) and the Funabashi\(^6\) potentials.

Following Ref. 1) for the meaning of notations, we have the $i$-th Faddeev component of the wave function,

$$|\Psi^{(i)}(z)\rangle = K^{(i)}(z)(1 - K^{(i)}(z))^{-1}|\Phi^{(i)}\rangle,$$

where

$$K^{(i)}(z) = -G_0(z) 2 \tilde{t}^{(i)}(z) P_\delta .$$

This equation was derived from the Faddeev equation for the initial state in the $n$-$d$ system. However, the Faddeev equation for the initial state in the free state is different from the one for the initial state in the $n$-$d$ system. So we must have the same type of equation from the former for the bound state. To do so, we start with the original Faddeev $T$ matrix,\(^2\)

$$
\begin{bmatrix}
T^{(1)}(z) \\
T^{(2)}(z) \\
T^{(3)}(z)
\end{bmatrix} =
\begin{bmatrix}
t_\alpha(z) \\
t_\beta(z) \\
t_\gamma(z)
\end{bmatrix} +
\begin{bmatrix}
0 & t_\alpha(z) & t_\alpha(z) \\
t_\beta(z) & 0 & t_\beta(z) \\
t_\gamma(z) & t_\gamma(z) & 0
\end{bmatrix}G_0(z)
\begin{bmatrix}
T^{(1)}(z) \\
T^{(2)}(z) \\
T^{(3)}(z)
\end{bmatrix}. \tag{3}
$$

The first term on the right-hand side (rhs) of the above equation turns out an unconnected term which gives rise to a $\delta$ function singularity and does not contribute to making a bound state. So that we must subtract the term from $T^{(i)}(z)$. Omitting $z$, defining $U^{(i)} = T^{(i)} - t_\alpha$ etc. and multiplying $G_0$, we have the equation,

$$
\begin{bmatrix}
U^{(1)} \\
U^{(2)} \\
U^{(3)}
\end{bmatrix} =
\begin{bmatrix}
0 & t_\alpha & t_\alpha \\
t_\beta & 0 & t_\beta \\
t_\gamma & t_\gamma & 0
\end{bmatrix}G_0
\begin{bmatrix}
U^{(1)} + t_\alpha \\
U^{(2)} + t_\beta \\
U^{(3)} + t_\gamma
\end{bmatrix}. \tag{4}
$$

This equation is symmetric as to the exchange of any pairs.
Now, to make an antisymmetrized wave function, we define the free state $|\phi^{(i)}\rangle = |\phi(i,j,k)\rangle$. The Roman characters $i, j$ and $k$ are one of cyclic permutations of 1, 2 and 3. $\phi^{(i)}$ means that all particles are in the free state, but the $j$ and $k$-th particles are in antisymmetric states. These states have the properties of Eq. (1) in Ref. 1,

$$P_{ij}|\phi^{(k)}\rangle = -|\phi^{(k)}\rangle \quad (k \neq i, j), \quad P_{ij}|\phi^{(i)}\rangle = -|\phi^{(i)}\rangle.$$  

We can make the totally antisymmetric free state vector $|\Phi\rangle = |\phi^{(1)} + \phi^{(2)} + \phi^{(3)}\rangle$. Operating Eq. (4) to the free state $|\Phi\rangle$ from the left, we have the state vector,

$$\begin{pmatrix}
|\Psi^{(1)}\rangle \\
|\Psi^{(2)}\rangle \\
|\Psi^{(3)}\rangle
\end{pmatrix} = G_{ij}\begin{pmatrix}
0 & t_a & t_a \\
t_a & 0 & t_a \\
t_a & t_a & 0
\end{pmatrix}\begin{pmatrix}
|\Psi^{(1)}\rangle + |\Phi^{(1)}\rangle \\
|\Psi^{(2)}\rangle + |\Phi^{(2)}\rangle \\
|\Psi^{(3)}\rangle + |\Phi^{(3)}\rangle
\end{pmatrix},$$

where

$$|\Phi^{(1)}\rangle = G_0 t_a |\Phi\rangle,$$

$$|\Phi^{(2)}\rangle = G_0 t_a |\Phi\rangle,$$

$$|\Phi^{(3)}\rangle = G_0 t_a |\Phi\rangle,$$

$$|\Psi^{(n)}\rangle = G_0 U^{(n)} |\Phi\rangle. \quad (n=1, 2 \text{ and } 3)$$

These vectors also satisfy the properties of Eq. (5). Then we have the equation for the $i$-th component,

$$|\Psi^{(i)}\rangle = -2G_0 \tilde{\tau}^{(i)} P_{ij} |\Phi^{(j)} + \Psi^{(j)}\rangle,$$

where we have made use of the relations,

$$2 \tilde{\tau}^{(i)} = t_a (1 - P_{2a}), \quad \text{etc.}$$

This is just Eq. (1). The equation $|\Phi^{(1)}\rangle = G_0 t_a |\Phi\rangle$ can be written as

$$|\Phi^{(1)}\rangle = G_0 t_a (|\phi^{(1)} + \phi^{(2)} + \phi^{(3)}\rangle)$$

$$= G_0 t_a |\phi^{(1)} + G_0 2 \tilde{\tau}^{(1)} |\phi^{(2)}\rangle.$$

In the bound state problem, $|\Phi^{(i)}\rangle$ is arbitrary, because the binding energy and the wave function can be determined only from the singular point of the matrix $K^{(i)} \times (1 - K^{(i)})^{-1}$, for example, the first component of the Faddeev wave functions can be determined from the residue of the expression,

$$\langle p_i, p_a, JM, a | \Psi^{(1)} \rangle = \langle p_i, p_a, JM, a | K^{(1)} (1 + K^{(1)} + (K^{(1)})^2 + \cdots) | \Phi^{(1)} \rangle$$

$$= \frac{W^{(i)}}{Z - E_b} + R(Z),$$

where $E_b$ is the binding energy and $R(Z)$ is an entire function of $Z$ which is unimportant at $Z = E_b$.

For the summation of the rhs, the Padé method has been used. The residue satisfies the homogeneous integral equation $(1 - K^{(1)}) W^{(1)} = 0$ independent of the $|\Phi\rangle$.

Now, all of the potentials used here, except for the $AV_{14}$, include the $p^2$ term,
which comes from the nonrelativistic approximation of the relativistic meson theory. We have exactly taken into account the term through the two-body off-shell t-matrix.\(^7\) Our off-shell t-matrix is represented in terms of only the wave function and its derivative of the off-shell Lippmann-Schwinger equation, and no potential appears directly. So the \(P^2\) term in the potential can be treated exactly. Of course, the Lippmann-Schwinger equation may have the \(P^2\) term in the potential. The method of Ref. 7) can also be applicable to the negative energy state by analytic continuation.

In the practical Faddeev calculation, we have used the maximum spectator momentum \(p_{2\text{max}}=6\text{ fm}^{-1}\), which gave the good description for the continuum state. The maximum of the pair momentum \(p_{\text{max}}\) is constrained to \(1.5 p_{2\text{max}}\). The calculations have been performed for 5, 18, 26, 34 and 42 channels. In the previous preliminary calculation,\(^1\) 30 point Gaussian integration has been used for both the momenta \(p_2\) and \(p_\beta\). This choice was good for \(p_\beta\), but too small for \(p_2\). In the present calculation, 45 point Gaussian integration has been used for \(p_\beta\). As the consequence, we got binding energies greater than the previous ones by about 15 keV.\(^1\) The present results are displayed in Table I.

The results of the Paris potential are consistent with those of Kamada et al.\(^8\) and Friar et al.\(^9\) However, the present results of the Bonn A and B are considerably different from those of Ref. 8), because the relativistic versions have been used in Ref. 8). We have used the nonrelativistic versions. So our results cannot be directly compared with them. The Bonn A and B potentials show a curious convergence, i.e., the binding energies of 5 channel calculation are greater than those of 18 channel. The result of 42 channel calculation for the \(AV_{14}\) potential is consistent with the variational one\(^10\) 7.684 MeV.

The Funabashi potential gives a greater binding energy than the Bonn B, but smaller than the Bonn A.

The three-body binding energy depends not only on the two-body \(d\)-state \(P_\alpha(\%)\) but also on the two-body scattering length \(1^a_{\text{ann}}\text{(fm)}\). So we displayed these quantities in Table I, together with the binding energies.

We have calculated the case \(p_{2\text{max}}=8\text{ fm}^{-1}\) for the \(AV_{14}\) potential. In this case, the binding energy for 5 channel calculation was 7.432 MeV, which is about 5 keV less than the one in Table I. From our numerical experience, for all potentials and all channels, we can expect about 5 keV smaller binding energies than the ones in Table I. However, the higher channel calculation will give the slightly larger binding

<table>
<thead>
<tr>
<th>channel numbers</th>
<th>(AV_{14})</th>
<th>(\text{mod})</th>
<th>(P_\alpha(%))</th>
<th>(1^a_{\text{ann}}(\text{fm}))</th>
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<tr>
<td>5</td>
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<td>7.607</td>
<td>8.200</td>
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<td>7.834</td>
<td>8.197</td>
<td>5.53</td>
</tr>
<tr>
<td>42</td>
<td>7.676</td>
<td>7.839</td>
<td>8.199</td>
<td>5.53</td>
</tr>
<tr>
<td>(P_\alpha)</td>
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<td>5.77</td>
<td>4.75</td>
<td>5.53</td>
</tr>
<tr>
<td>(1^a_{\text{ann}})</td>
<td>-23.70</td>
<td>-23.75</td>
<td>-23.71</td>
<td>-23.70</td>
</tr>
</tbody>
</table>

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energies.

All of the potentials used here, except for the modified Paris potential, are the phase shift equivalent potentials. However, there is considerable difference in the three-body binding energies. Such difference comes from the potential shape, especially the short range part which is phenomenologically determined. From the fact mentioned above and Table I, we can say that the convergence of the Faddeev calculation is slow and the binding energies are not yet settled for the order of several keV. The convergence of the three-body binding energies depends on the potentials used.

A merit of the present method is that the three-body problem can be treated by analytic continuation as well as the two-body one and a spurious or an unphysical solution never appears, as shown in Ref. 1). For the complete three-body Faddeev calculation, we have the problem, how large spectator momentum, how many Gaussian points and how many channels should be taken.

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