Folded Bifurcation in Coupled Asymmetric Logistic Maps

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A system of two coupled logistic maps, one periodic and the other chaotic, is studied. It is found that with the variation of the coupling strength, the system displays several curious features such as the appearance of quadrupling of period, occurrence of isolated period three attractor and the coexistence of the Hopf and pitchfork bifurcations. Possible applications and extensions are discussed.

Chaos in higher dimensional systems is one of the focal subjects of physics today. Along with the approach starting from modeling physical systems with many degrees of freedom, there emerged a new approach, developed by Kaneko, to couple many one-dimensional maps\(^1\) to study the behavior of the system as a whole. However, this model can only be applied to study the dynamics of a single medium such as pattern formation in a fluid. What happens if two media border each other? One may naturally be lead to a model of coupled logistic maps with different strength parameters. Thus it is appropriate to inquire whether loosening the condition of strict identity might bring any new feature while keeping both maps logistic to hold the redundant controlling parameters minimal. Even two logistic maps coupled to each other may serve as a dynamical model of driven coupled oscillators. It has been found that two coupled identical maps possess several characteristic features which are typical of higher dimensional chaos\(^1,4\).

In this paper, we report the results of a numerical investigation of a system of two logistic maps with different strength parameters such that one map lies in a period one stable attractor or a bifurcation point and the other in chaotic region when decoupled. Several new features previously unobserved are found. Most notable among them is the appearance of a period four cycle straight from the stable period one cycle. The other peculiar feature is the almost simultaneous occurrence of periods four, eight and sixteen right after the Hopf bifurcation. This results in a very intriguing metamorphose of the attractor when one changes the coupling parameter.

The system we study is two linearly coupled maps

\[
\begin{align*}
x_{n+1} &= (1 - \epsilon)f(\mu, x_n) + \epsilon f(\nu, y_n), \\
y_{n+1} &= \epsilon f(\mu, x_n) + (1 - \epsilon)f(\nu, y_n),
\end{align*}
\]

where the map \(f\) is taken to be the logistic map\(^5\) with strength parameters \(\mu\):

\[
f(\mu, x) = \mu x (1 - x).
\]
In the case $\mu = \nu$, two maps soon become synchronized no matter what the initial conditions may be, i.e., coupled maps are identical to a single logistic map. The interesting case is that in which $\mu \neq \nu$. In the following we fix the strength parameters above and below the critical value $\mu^* = 3.56994\cdots$ for $\mu$ and $\nu$ respectively. We choose the strength parameters $\mu$ and $\nu$ and regard the coupling parameter $\epsilon$ as the controlling parameter.

In Fig. 1, the attractors of the coupled-map are displayed as functions of the coupling $\epsilon$. Figure 1 (a) shows the result of $\mu = 4$ and $\nu = 3$, and (b) that of $\mu = 4$ and $\nu = 2$. These are two typical examples of various values of $\mu$ and $\nu$. One immediately notices several interesting features. The fact that there are two chaotic regions in both the $\epsilon = 0$ and $\epsilon = 1$ ends seems odd at first sight, but after some reflection, one realizes that a very weak $\epsilon$ means very strong $(1 - \epsilon)$, which brings chaos first to the variable $x$, and then to $y$ however weak the coupling term may be. The most salient feature is the appearance of a stable period four cycle right after the period one cycle around $\epsilon = 0.77$ in Fig. 1 (a). Another, found in both cases (a) and (b) is the sudden filling of the $x$ and $y$ space around $\epsilon = 0.85$ and above. The broad window-like region with period four around $\epsilon = 0.9$ in case (b) is also noteworthy.

Identifying stable and unstable periodic points often gives a skeleton view of the dynamics of the system. With the vector notation of the variables $z = (x, y)$ and the operator notation of the map

$$z_{n+1} = F(z_n),$$

one can define the periodic points of cycle $N$ as the solution of the equation

$$z_N = F^N(z_N), \quad (N \text{ times})$$

where $F^N(z_N)$ is defined by $F^N(z_N) = F(F(F(\cdots F(z_N)\cdots)))$. Bifurcations occur at the parameter value of $\epsilon$ where the periodic point satisfies
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Fig. 2. The "skelton" of Fig. 1, showing the position of the periodic cycles of the maps. (a) is with $\mu=4.0$, $\nu=3.0$, and (b) with $\mu=4.0$, $\nu=2.0$ as before. The caption number indicates the period of the cycle.

\[
E_{\text{max}} \left[ \frac{dF_N}{dz_N} (z_N) \right] = 1,
\]

in addition to Eq. (4). Here $E_{\text{max}}[G]$ is the maximum eigenvalue of the matrix $G$. We numerically solved Eq. (4) for $N=3, 4$ and $8$ for the entire range of $\epsilon = 0 \sim 1$. The results are presented in Fig. 2, where the unstable orbits are shown as well as the stable ones. One can clearly see from the $N=4$ and $N=8$ skeletons that the period four occurring at $\epsilon = 0.77$ is the result of the folded period doubling bifurcation sequence whose period two starts at $\epsilon = 0.75$ after the quadrupling of the period when one moves from period one at larger $\epsilon$ to higher periodic orbits at smaller $\epsilon$. While this type of inversion of the sequence is nothing of unimaginable nature, in principle, it has never been observed in simple maps to our knowledge. One recognizes easily that it is rather hard to obtain in a usual one-dimensional map since it requires special tuning of the functional dependence of the map on the controlling parameter to ensure that the condition (4) is satisfied for $N=4$ before $N=2$. We therefore think it significant that it emerges from nothing but linear coupling of two logistic maps of different strength parameters. One might presume that it should be observed rather frequently in higher dimensional coupled lattice map models once the requirement of strict uniformity of the elements is lifted. In fact, preliminary investigation of three coupled logistic maps supports this view.

There are two notable facts in the periodic cycle diagram above $\epsilon = 0.85$. One is that no periodic point exists in the region $\epsilon < 0.86$. The examination of the $x$-$y$ profile shows that the onset of the "filling area" at approximately $\epsilon = 0.85$ is the result of the
Fig. 3. Spatial portraits of the attractors for the coupled logistic map with $\mu=4.0$ and $\nu=3.0$ at $\epsilon=0.852$ (a), $\epsilon=0.868$ (b), $\epsilon=0.8757$ (c), $\epsilon=0.877$ (d), $\epsilon=0.885$ (e) and $\epsilon=0.900$ (f).

Hopf bifurcation. The second is that around $\epsilon=0.88$, period four and period eight start (as well as other higher periods not shown here), appearing almost simultaneously. While these occurrences fit into the generic transition scenario of “cycle one $\rightarrow$ Hopf $\rightarrow$ cycles $\rightarrow$ $\cdots$ $\rightarrow$ chaos” found by Kaneko in an early study, the second fact, the crowded onset of many different periodic cycles, makes this transition to chaos a very intricate one. These points are visually displayed in the phase portrait of the attractor at several values at and above $\epsilon=0.85$, shown in Fig. 3. The first two figures, (a) and (b), clearly show the Hopf bifurcation around $\epsilon=0.85$. Subsequent distortion and transformation to an aurora-like strange attractor observed in Figs. 3 (c) to (f) certainly match those found in far more complicated systems in aesthetic appeal. One curious feature found both in Figs. 2 and 3 (b) is the early appearance of a period 3 cycle around $\epsilon=0.87$. At present, we are unable to recognize any direct role of this period to the shape and stability of the attractors around this region.

In summary, we have constructed a system consisting of chaotic and periodic maps coupled together. While the dynamics of the system does not go beyond the known bifurcation schemes in two-dimensional dissipative systems (which is certainly not to be expected), it is found that various bifurcations occur in such a combination to give the system several intriguing features.

Finally, a few words on possible applications and extensions are in order. It should not be too difficult to construct a circuit (electronic, for example) to materialize the system proposed here. Actual applicational usage of quadrupling of stable states might be envisioned. Also, replacing the logistic map with other maps, cycle maps or quadratic maps for example, might be interesting to see the generality and/or the new aspects of the findings here. Another application is possible by increasing the number of maps coupled with each other. The coupled lattice map having an “impurity”, or one element with different strength parameter from all the rest might exhibit non-conventional features. A lattice with alternating strength might also be
an interesting system. In a word, loosening the condition of strict uniformity of the elements might bring some new feature to the coupled lattice map.

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References