Soliton-Typed Solutions to the Generalized Cylindrical Kadomtsev-Petviashvili Equation with Variable Coefficients

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The variable-coefficient generalizations of nonlinear evolution equations are able to realistically model various physical situations. In this paper, we make use of the truncated Painlevé expansion and symbolic computation to obtain a new class of soliton-typed solutions to the generalized cylindrical Kadomtsev-Petviashvili equation with variable coefficients.

The variable-coefficient generalizations of nonlinear evolution equations are a currently exciting subject in physical sciences. The reason is that, for example, the physical situations in which the celebrated Kadomtsev-Petviashvili (KP) equation arises tend to be highly idealized, owing to the assumptions of the constant coefficients. Correspondingly, there has recently been remarkable interest in the investigation of its variable-coefficient generalizations, which are able to provide us with more realistic models in various physical situations. See, e.g., Refs. 1)~4).

The development of symbolic computation and the truncated Painlevé expansion enables us to obtain new solutions to the generalized cylindrical KP equation with variable coefficients (GCKP) as follows:

\[ \left\{ u_t + 6u u_x + u_{xx} + \frac{u}{2t} + [H(t) + yG(t)]u_x + F(t)u_y \right\}_x + \frac{a}{4t^2} u_{yy} = 0, \]

where \( G(t), H(t) \) and \( F(t) \) are arbitrary functions of \( t \) and \( a = \pm 1 \) are for media with positive or negative dispersions. For \( F = G = H = 0 \), the GCKP reduces to the well-known cylindrical KP equation, which originates from the investigation of propagation of ion-acoustic waves in a collisional plasma with scales of variation and cylindrical symmetry imposed, and of nonlinear cylindrical waves in shallow water with a weak azimuthal \( y \) dependence.\(^5\)~\(^7\) The Kac-Moody-Virasoro algebra and the relation to the KP equation are also discussed for the GCKP by a Lie point transformation.\(^8\) Of a hierarchy of nonlinear equations in (2+1) dimensions with variable coefficients, the GCKP is the first member.\(^8\)

It is known that the sufficient condition for a partial differential equation (PDE) to be completely integrable is that it possesses the Painlevé property, i.e., as addressed in Refs. 9) and 10), that the solutions to the PDE, written as

\[ u(x, y, t) = \phi^{-1}(x, y, t) \sum_{l=0}^{\infty} u_l(x, y, t) \phi^l(x, y, t), \]

\(^1\) Correspondence to: Yi-Tian Gao.
are single-valued in the neighbourhood of a non-characteristic, movable singularity manifold

\[ M = \{(x, y, t) \mid \phi(x, y, t) = 0\}, \]  

where \( J \) is a natural number to be determined; \( u_i(x, y, t) \) and \( \phi(x, y, t) \) are analytic functions with \( u_0(x, y, t) \neq 0 \).

However, it is not necessary, hereby, to investigate the system's complete integrability and/or Painleve property. Instead, we aim at deriving certain special solutions for the GCKP by virtue of the truncation of the Painlevé expansion (Eq. (2)) at the constant level term, i.e.,

\[ u(x, y, t) = \phi^{-1}(x, y, t) \sum_{l=0}^{J} u_l(x, y, t) \phi^l(x, y, t), \]  

as well as at obtaining the constraints on the coefficient functions, if any, for the aforementioned solutions to exist.

Then, the leading-order analysis gives that \( J = 2 \), or

\[ u(x, y, t) = u_0(x, y, t) \phi^{-2}(x, y, t) + u_1(x, y, t) \phi^{-1}(x, y, t) + u_2(x, y, t). \]  

When substituting Eq. (5) into the GCKP with Mathematica, we make the coefficients of like powers of \( \phi \) to vanish, so as to get the set of Painlevé-Bäcklund (PB) equations as follows:

\[ \phi^{-6}: u_0 = -2 {\phi_x}^2, \]  
\[ \phi^{-5}: u_1 = 2 {\phi_{xx}}, \]  
\[ \phi^{-4}: \mathcal{Z} = 3 {\phi_x}^2 - \frac{\alpha}{4t^2} {\phi_y}^2 - F {\phi_x} {\phi_y} - (Gy + H) {\phi_x}^2 - {\phi_{xx}} - 6 {\phi_x}^2 u_2 - 4 {\phi_x} {\phi_{xxx}} = 0, \]  
\[ \phi^{-3}: {\phi_x}^2 \mathcal{Q} - 2 {\phi_x} \mathcal{Z} - {\phi_{xx}} \mathcal{Z} = 0, \]  
\[ \phi^{-2}: \mathcal{Z}_{xx} - {\phi_{xx}} \mathcal{Q} - 2 {\phi_x} \mathcal{Q}_x = 0, \]  
\[ \phi^{-1}: \mathcal{Q}_{xx} = 0, \]  
\[ \phi^0: \left[ u_{2,t} + 6 u_2 u_{2,x} + u_{2,xxx} + \frac{u_2}{2t} \right] + (H + yG) u_{2,x} + F u_{2,y} \bigg|_{x} + \frac{\alpha}{4t^2} u_{2,yy} = 0, \]  

where \( \phi_x \neq 0 \), and

\[ \mathcal{Q} = \frac{\alpha}{4t^2} {\phi_{yy}} + \frac{\phi_x}{2t} + {\phi_{xx}} + F {\phi_x} y + (Gy + H) {\phi_{xx}} + {\phi_{xxx}} + 6 {\phi_{xx}} u_2 = 0. \]  

Next we consider a trial solution generalized from that in Ref. 11,

\[ \phi(x, y, t) = 1 + e^{i q(x, y, t)}, \]  

where \( q(x, y, t) \) could be complex. Substituting Eq. (14) into Eqs. (8) and (13), we try to separate the real terms from the coefficients of the imaginary terms, and to make the coefficients of like powers of \( x \) to vanish. Having used Mathematica to deal with the complicated calculations, we end up with
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\[ q_{xx} = 0 \quad \Rightarrow \quad q(x, y, t) = \phi(x, y, t) + \lambda(y, t), \quad (15) \]

\[ \phi_{yy} = 0 \quad \Rightarrow \quad \phi(x, y, t) = \rho(t) y + \theta(t), \quad (16) \]

\[ u_2(x, y, t) = \sum_{m=0}^{2} u_{2m}(y, t)x^m, \quad (17) \]

\[ \frac{\rho y + \theta}{2t} + (\rho y + \theta t) + F \rho + \frac{\alpha}{4t^2} \lambda_{yy} = 0, \quad (18) \]

\[
(Gy + H)(\rho y + \theta)^2 - (\rho y + \theta)^4 + 6(\rho y + \theta)^2u_{20} + (\rho y + \theta)\lambda_{t} + F(\rho y + \theta)\lambda_{y} + \frac{\alpha}{4t^2} \lambda_{y}^2 = 0, \quad (19)
\]

\[ 6(\rho y + \theta)^2u_{21} + (\rho y + \theta)(\rho y + \theta t) + F(\rho y + \theta)\rho + \frac{\alpha}{2t^2} \lambda_{y} \rho = 0, \quad (20) \]

\[ 6(\rho y + \theta)^2u_{22} + \frac{\alpha}{4t^2} \rho^2 = 0, \quad (21) \]

where \( \phi(x, y, t), \lambda(y, t), \rho(t), \) and the \( u_{2m}(y, t) \)'s are differentiable functions.

Twice of integrations of Eq. (18) over \( y \) gives

\[
\lambda(y, t) = -\frac{2t^2}{3\alpha} \left[ \frac{\rho(t)}{2t} + \rho_{t}(t) \right] y^3 - \frac{2t^2}{\alpha} \left[ \frac{\theta(t)}{2t} + \theta_{t}(t) + F(t) \rho(t) \right] y^2 + \mu(t) y + \delta(t), \quad (22)
\]

where \( \mu(t) \) and \( \delta(t) \) are also differentiable functions. From Eqs. (19)~(22), we find the expressions for \( u_{22}(y, t), u_{21}(y, t), \) and \( u_{20}(y, t), \) with which Eq. (12) is satisfied, i.e.,

\[ u_2(x, y, t) = \sum_{m=0}^{2} u_{2m}(y, t)x^m \]

now satisfies Eq. (1).

To make the solutions physically meaningful, we choose \( q(x, y, t) \) to be pure imaginary so that \( \phi(x, y, t) \) become real. Correspondingly,

\[
\rho(t) = i\eta(t), \quad \theta(t) = i\zeta(t), \quad \mu(t) = ir(t) \text{ and } \delta(t) = i\gamma(t), \quad (23)
\]

where \( \eta(t) \neq 0, \zeta(t), \) \( r(t) \) and \( \gamma(t) \) are all real.

Having applied the truncated Painlevé expansion and symbolic computation, we are able to complete the paper with a new class of the soliton-typed solutions to the GCKP, i.e., Eq. (1), as follows:

\[ u(x, y, t) \]

\[
= u_0(x, y, t) - \phi^{-1}(x, y, t) + u_1(x, y, t) - \phi^{-1}(x, y, t) + u_2(x, y, t)
\]

\[
= \left[ \begin{array}{l}
\frac{\eta(t) y + \zeta(t)^2}{2}
\end{array} \right] \text{sech} \left[ \begin{array}{l}
\frac{\eta(t) y + \zeta(t)^2}{2}
\end{array} \right] x - \frac{t^2}{3\alpha} \left[ \begin{array}{l}
\frac{\eta(t)}{2t} + \eta_{t}(t)
\end{array} \right] y^2 + \frac{\tau(t) y + \gamma(t)}{2}
\]

\[
- \frac{t^2}{\alpha} \left[ \frac{\zeta(t)}{2t} + \zeta_{t}(t) + F(t) \eta(t) \right] y^2 - \frac{ax^2 \eta(t)}{24t^2 \left[ \eta(t) y + \zeta(t) \right]^2}
\]

\[
- \frac{x \left[ \eta(t) y + \zeta(t) + \eta(t) F(t) \right]}{6 \left[ \eta(t) y + \zeta(t) \right]} - \frac{ax \gamma(t)}{12t^2 \left[ \eta(t) y + \zeta(t) \right]^2} \left[ \frac{t \eta(t)^2}{\alpha} \right]
\]

\[
- \frac{2 \eta(t) y + \zeta(t) + \eta(t) F(t)}{6 \left[ \eta(t) y + \zeta(t) \right]} - \frac{ax \gamma(t)}{12t^2 \left[ \eta(t) y + \zeta(t) \right]^2} \left[ \frac{t \eta(t)^2}{\alpha} \right]
\]

\[
- \frac{2 \eta(t) y + \zeta(t) + \eta(t) F(t)}{6 \left[ \eta(t) y + \zeta(t) \right]} - \frac{ax \gamma(t)}{12t^2 \left[ \eta(t) y + \zeta(t) \right]^2} \left[ \frac{t \eta(t)^2}{\alpha} \right]
\]
\[ + \frac{2\nu}{a} \left[ 2t\eta(t)F(t) + \xi(t) + 2t\xi(t) \right] - \tau(t) \]
\[ + \frac{\nu^2}{a} \left[ \eta(t) + 2t\eta(t) \right] - \frac{2\nu}{a} \left[ 2t\eta(t)F(t) + \xi(t) + 2t\xi(t) \right] \]
\[ + \frac{1}{6[\eta(t)\gamma + \xi(t)]} \left\{ \frac{\nu^3}{3a} \left[ \eta(t) + 5t\eta(t) + 2t^2\eta(t) \right] + \frac{\nu^2}{a} \left[ 5t\eta(t)F(t) + 4t^2\eta(t)F(t) + 5t\xi(t) + 2t^2\xi(t) \right] \]
\[ + \frac{2\nu}{a} \left[ 2t\eta(t)F(t) + \xi(t) + 2t\xi(t) \right] - \gamma(t) \}
\]
\( (24) \)

where the real, differentiable functions \( \eta(t), \xi(t), \tau(t) \) and \( \gamma(t) \) are all arbitrary except that \( \eta(t) \neq 0 \).

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