New Method for Exact Calculation of Green Functions in Scalar Field Theory

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We present a new method for calculating the Green functions for a lattice scalar field theory in $D$ dimensions with arbitrary potential $V(\phi)$. The method for non-perturbative evaluation of Green functions for $D=1$ is generalized to higher dimensions. We define "hole functions" $A^i_i$ ($i=0, 1, 2, \cdots, N-1$) from which one can construct $N$-point Green functions. We derive characteristic equations of $A_i^i$ that form a finite closed set of coupled local equations. It is shown that the Green functions constructed from the solutions to the characteristic equations satisfy the Dyson-Schwinger equations. To fix the boundary conditions of $A_i^i$, a prescription is given for selecting the vacuum state at the boundaries.

§ 1. Introduction

In quantum field theory, various physical quantities are calculated from the Green functions (correlation functions). The principal methods for calculating Green functions in a theory for space-time dimensions $D \geq 3$ have been either to resort to the perturbative expansion or to estimate their behavior in Euclidean region by numerical simulation based on a lattice field theory. It is desirable, however, that the Green functions can be calculated other than in perturbative expansion or without large computer calculation.

It is well known that the Green functions obey the Dyson-Schwinger equations, which constitute an infinite hierarchy of equations. For a general interacting field theory, there is no finite closed subset of these equations, which makes it difficult to solve the equations exactly. Meanwhile, for a scalar field theory defined on a lattice (with arbitrary potential $V(\phi)$), we find there exist equations equivalent to the Dyson-Schwinger equations that have finite closed subsets. From the closed equations one may calculate $N$-point Green functions, which can be used as a new technique to evaluate Green functions.

The idea for finding the equations is to generalize the method for non-perturbative evaluation of Green functions in 1 dimension. Consider the 1-dimensional (continuum) scalar field theory given by the action

$$ S[\phi] = \int_0^T dt \left\{ \frac{1}{2} (\partial_t \phi)^2 - V(\phi) \right\}, \quad (1.1) $$

or equivalently, the quantum mechanical system given by the Hamiltonian

$$ \hat{H}_\phi = -\frac{1}{2} \frac{\partial^2}{\partial \phi^2} + V(\phi). \quad (1.2) $$

Suppose we calculate the one-point function defined by
for some initial state $|i\rangle$ and final state $|f\rangle$. For this purpose, first define a function

$$A(\varphi, \varphi'; t) = \langle f | e^{-iH(t-t')} | \varphi' \rangle \langle \varphi | e^{-iH} | i \rangle,$$

(1.4)

where $\tilde{\varphi} | \varphi'^{\prime} \rangle = \varphi | \varphi'^{\prime} \rangle$. Once $A(\varphi, \varphi'; t)$ is known, the one-point function can be calculated as

$$G(t) = \int d\varphi A(\varphi, \varphi; t).$$

(1.5)

$A(\varphi, \varphi'; t)$ satisfies a characteristic equation

$$i \frac{\partial}{\partial t} A(\varphi, \varphi'; t) = (\tilde{H}_\varphi - \tilde{H}_\psi) A(\varphi, \varphi'; t).$$

(1.6)

Note that Eq. (1.6) is satisfied by $A(\varphi, \varphi'; t)$ corresponding to any initial and final states, $|i\rangle$ and $|f\rangle$. As we will see later, there is a prescription to select a particular solution to Eq. (1.6) for which both $|i\rangle$ and $|f\rangle$ are the ground state $|0\rangle$ of the Hamiltonian (1.2). So, in fact we may evaluate the one-point function $\langle 0 | \tilde{\varphi} | 0 \rangle$ by solving Eq. (1.6).

The simplest Dyson-Schwinger equation related to $G(t)$ is given by

$$\frac{d^2}{dt^2} G(t) = - \int d\varphi V(\varphi) A(\varphi, \varphi; t),$$

(1.7)

which is just the Ehrenfest formula when $|i\rangle = |f\rangle$. Using fundamental techniques of quantum mechanics, it is easy to prove Eq. (1.7) for any $A(\varphi, \varphi'; t)$ satisfying Eq. (1.6) and for $G(t)$ constructed via Eq. (1.5).

It is Eqs. (1.4) and (1.6) that we are going to generalize for scalar field theory regularized on a lattice in higher dimensions. We will define a "hole function" $A_n$ associated with a lattice site $n$, and see that it obeys local equations that characterize local property of the field theory in a non-trivial way.

Most of the discussion given in this paper holds in parallel both for Minkowski and Euclidean space-time metrics. For notational convenience, we will adopt the Euclidean theory in the following, and we list Minkowski versions of some important equations in Appendix A.

In § 2 we review the Dyson-Schwinger equations of a lattice scalar field theory. In § 3 we will define a hole function, from which Green functions can be calculated, and derive local equations satisfied by the hole function. We show that these local equations are equivalent to the Dyson-Schwinger equations in § 4. Then we discuss in § 5 how to extract Green functions satisfying a specific boundary conditions, namely those given by the vacuum expectation values of field operator products. Some basic properties of the local equations are briefly summarized in § 6. Section 7 is devoted to a summary and discussion.

We list some equations for Minkowski space-time in Appendix A. Proof of formulas used in § 4 is given in Appendix B.
§ 2. Dyson-Schwinger equations

We start by reviewing the Dyson-Schwinger equations for a Euclidean scalar field theory defined on a $D$-dimensional lattice $A^D$ with a lattice spacing $\epsilon$. The $N$-point Green function for the lattice scalar field theory is given by

$$
\langle \phi_{i_1} \cdots \phi_{i_N} \rangle = \int \prod_{i \in A^D} d\phi_i \phi_{i_1} \cdots \phi_{i_N} e^{-S[\phi]} / \int \prod_{i \in A^D} d\phi e^{-S[\phi]},
$$

(2.1)

where the action is defined as the discretized version of Eq. (1.1) for $D$-dimensional Euclidean space-time:

$$
S[\phi] = \sum_{i \in A^D} \epsilon^D \left\{ \sum_{\bar{\mu}=1}^D \frac{1}{2} \left( \frac{\phi_{i+\bar{\mu}} - \phi_i}{\epsilon} \right)^2 + V(\phi_i) \right\}.
$$

(2.2)

Here and hereafter, $\bar{\mu}$ denotes a unit vector in one of $2D$ directions ($\pm \hat{x}_1, \cdots, \pm \hat{x}_D$). For notational simplicity, the following rules for the sum of directions are understood throughout the paper.

$$
\sum_{\bar{\mu}=1}^D: \text{ sum over } \bar{\mu} = +\hat{x}_1, \cdots, +\hat{x}_D,
$$

$$
\sum_{\bar{\mu}=1}^{2D}: \text{ sum over } \bar{\mu} = \pm \hat{x}_1, \cdots, \pm \hat{x}_D.
$$

(2.3)

The expression for the Green function has a simple form of taking the expectation value of field product $\phi_{i_1} \cdots \phi_{i_N}$ with the weight factor $e^{-S[\phi]}$. In the following, we derive the relationship among the Green functions starting from defining equations of this weight factor $w = e^{-S[\phi]}$:

$$
\frac{\partial}{\partial \phi_n} (e^{S[\phi]} w) = 0 \quad \forall \ n,
$$

(2.4)

or

$$
\left[ \frac{1}{\epsilon^D} \frac{\partial}{\partial \phi_n} - \Box_n \phi_n + V'(\phi_n) \right] w = 0,
$$

(2.5)

where

$$
\Box_n X_n = \sum_{\bar{\mu}=1}^D \left( \frac{X_{n+\bar{\mu}} - 2X_n + X_{n-\bar{\mu}}}{\epsilon^2} \right) = \sum_{\bar{\mu}=1}^{2D} \left( \frac{X_{n+\bar{\mu}} - X_n}{\epsilon^2} \right).
$$

(2.6)

From the first equation (2.4), it is clear that the equation defines $w$ uniquely up to a physically unimportant ($\phi$-independent) coefficient.

Next, we Fourier transform the defining equation (2.5) with respect to all $\phi_i$ as

$$
0 = \int \prod \prod \prod \prod d\phi_i \exp(i\epsilon^D \sum \phi_i) \left[ \frac{1}{\epsilon^D} \frac{\partial}{\partial \phi_n} - \Box_n \phi_n + V'(\phi_n) \right] e^{-S[\phi]}.
$$

(2.7)

We will not pay attention to the boundary conditions of the field configurations until § 5. For definiteness, one may assume periodic boundary conditions for discussion in §§ 2~4.
Replacing $\phi_n$ by $\partial / \partial J_n$, we obtain the following coupled partial differential equations for the partition function, that is, Dyson-Schwinger equations:

$$\left\{ \Box_n \left( \frac{1}{i e^\beta} \frac{\partial}{\partial J_n} \right) - V' \left( \frac{1}{i e^\beta} \frac{\partial}{\partial J_n} \right) + i J_n \right\} Z[J] = 0,$$

$$Z[J] = \int \prod d\phi e^{-S[\phi] + i e^{\beta} \sum_i \phi_i}.$$ (2·9)

Perhaps the physical meaning of the equations is most transparent when we regard them as the expectation values of the equations of motion in the presence of source $J$:

$$\langle \Box_n \phi_n - V'(\phi_n) + i J_n \rangle = 0.$$ (2·10)

Expanding the partition function in Taylor series in $J$ as

$$Z[J] = Z[0] \times \sum_{N=0}^{\infty} \frac{(i e^\beta)^N}{N!} \sum_{i_1 \in A^1} \sum_{i_2 \in A^2} \cdots \sum_{i_N \in A^N} J_{i_1} \cdots J_{i_N} \langle \phi_{i_1} \cdots \phi_{i_N} \rangle,$$ (2·11)

and substituting to Eq. (2·9), we obtain coupled equations among the Green functions:

$$\Box_n \langle \phi_n \phi_{i_1} \cdots \phi_{i_N} \rangle - \langle V'(\phi_n) \phi_{i_1} \cdots \phi_{i_N} \rangle = - \sum_{n=1}^{N} \frac{1}{e^\beta} \delta_{n,n} \langle \phi_{i_1} \cdots \phi_{i_N} \rangle.$$ (2·12)

Since Eq. (2·9) is nothing but Fourier transform of the defining equations for the weight factor $e^{-S}$, Green functions obtained by solving the Dyson-Schwinger equations (2·13) are (almost) equivalent to the Green functions defined by the integral Eq. (2·1).\(^*)\) In this sense, the Dyson-Schwinger equations contain full information on the lattice field theory.

Another important feature of the Dyson-Schwinger equations is that if one tries to solve the equations to obtain the $N$-point Green function for some $N$, one in fact needs to solve all the coupled equations. Namely, the coupled equations never close with some finite subset of the equations, so one needs to deal with infinite-dimensional coupled equations.

\(^*)\) Note that Eq. (2·9) holds for any contour in the complex $\phi$-plane of $\phi$-integral in Eq. (2·10). Therefore, there is additional degree of freedom for solutions to the Dyson-Schwinger equations compared with the original definition (2·1). The degree of freedom corresponds to the number of independent contours in the complex $\phi$-plane for each $\phi$-integration.
§ 3. Hole function and local equations

We consider the lattice scalar field theory as defined in the previous section. For later convenience, we make a slight change in the integral measure of the partition function

\[ Z[J] = \int \prod_{i \in A^D} [d\phi_i] \exp \left[ -S[\phi] + i\epsilon \sum_I \phi_I \right], \tag{3.1} \]

where \([d\phi_i] = d\phi_i / \sqrt{2\pi\epsilon} \). (This change does not alter the Green functions (2.1).)

The action is given by Eq. (2.2).

Let us take one site \( n \in \Lambda^D \), and define the “hole function of \( n \)” as a function of 2D link variables \( u_1, \ldots, u_{2D} \) surrounding the site \( n \) (Fig. 1) by

\[ A_n(u_1, \ldots, u_{2D}; J) \]

\[ = \int \prod_{i \in n} [d\phi_i] \exp \left[ -S_n[\phi] + i\epsilon \sum_{i \in n} J_i \phi_i - \epsilon^D \sum_{\mu=1}^{2D} \frac{1}{2} \left( \frac{\phi_{n+\mu} - \phi_n}{\epsilon} \right)^2 \right], \tag{3.2} \]

where \( S_n[\phi] \) denotes the part of the action \( S[\phi] \) that remains after subtraction of terms depending on \( \phi_n \):

\[ S_n[\phi] = S[\phi] - \epsilon^D \left\{ \sum_{\mu=1}^{2D} \frac{1}{2} \left( \frac{\phi_{n+\mu} - \phi_n}{\epsilon} \right)^2 + V(\phi_n) \right\}. \tag{3.3} \]

Compared to \( Z[J] \), we not only leave \( \phi_n \) unintegrated but also subtracted all \( \phi_n \)-dependent terms in the exponent in Eq. (3.2) first and then re-added the kinetic term after replacing \( \phi_n \) by the link variables \( u_1, \ldots, u_{2D} \). Using this hole function, expectation value of a local operator can be expressed as

**Fig. 1.** The hole function \( A_n(u_1, \ldots, u_{2D}; J) \) is defined as a function of link variables surrounding the site \( n \).

**Fig. 2.** The function \( F_{n, \tilde{n}} \) is defined as a function of link variables surrounding two adjacent sites \( n \) and \( n + \tilde{\mu} \); \( n \) is surrounded by \( u_1, \ldots, u_{2D} \) (except \( u_\mu \)), and \( n + \tilde{\mu} \) is surrounded by \( \tilde{u}_1, \ldots, \tilde{u}_{2D} \) (except \( \tilde{u}_\mu \)). Note that the link connecting \( n \) and \( n + \tilde{\mu} \) is missing.
Also, we can construct the Green functions from the hole function as
\[
\langle \phi_{n} \phi_{i} \cdots \phi_{\omega} \rangle = \frac{\int [d\phi] u^{*} A_{n}(u, \cdots, u; J) e^{-\epsilon^{D} V(u) + ie^{D}_{\mu} J_{\mu}}}{\int [d\phi] A_{n}(u, \cdots, u; J) e^{-\epsilon^{D} V(u) + ie^{D}_{\mu} J_{\mu}}}. \tag{3.5}
\]

Note that from the definition (3.2) the hole function has a property
\[
A_{n}(u_{1}, \cdots, u_{2D}; J) = \text{independent of } J_{n}, \tag{3.6}
\]
although it depends on \( J_{l} \) for \( l \neq n \).

We are going to derive a local equation satisfied by the above hole function \( A_{n} \). For this purpose, we define a function associated with the two adjacent sites \( n \) and \( n + \bar{\mu} \) as follows (see Fig. 2).

\[
F_{n, \bar{\mu}}(u_{1}, \cdots, u_{2D}; \bar{u}_{1}, \cdots, \bar{u}_{2D}; J) = \int \Pi_{l+n,n+\bar{\mu}} [d\phi_{l}] \exp \left[ -S_{n,n+\bar{\mu}}[\phi] + ie^{D}_{\mu} \sum_{l+n,n+\bar{\mu}} J_{\phi_{l}} \right]
\]
\[
- \epsilon^{D}_{\mu} \sum_{\bar{\nu} = \bar{\mu}}^{2D-1} \frac{1}{2} \left( \frac{\phi_{n+\bar{\nu}} - \bar{u}_{\bar{\nu}}}{\epsilon} \right)^{2} - \epsilon^{D}_{\mu} \sum_{\bar{\nu} = \bar{\mu}}^{2D-1} \frac{1}{2} \left( \frac{\phi_{n+\bar{\nu}} + \bar{u}_{\bar{\nu}}}{\epsilon} \right)^{2}. \tag{3.7}
\]

Here, \( S_{n,n+\bar{\mu}}[\phi] \) denotes the part of the action \( S[\phi] \) that remains after subtraction of terms depending on \( \phi_{n} \) and \( \phi_{n+\bar{\mu}} \). The function \( F_{n,\bar{\mu}} \) depends on the variables on the links surrounding the sites \( n \) and \( n + \bar{\mu} \) but the one connecting \( n \) and \( n+\bar{\mu} \). This function is defined such that both \( A_{n} \) and \( A_{n+\bar{\mu}} \) can be constructed from it.

The hole function \( A_{n} \) is obtained from \( F_{n,\bar{\mu}} \) by integrating over the field variable on the site \( n + \bar{\mu} \):
\[
A_{n}(u_{1}, \cdots, u_{2D}; J) = \int [d\bar{u}] \exp \left[ -\epsilon^{D}_{\mu} \left( \frac{1}{2} \left( \frac{\bar{u}_{\bar{\mu}} - \bar{u}}{\epsilon} \right)^{2} + V(\bar{u}) \right) + ie^{D}_{\mu} J_{n+\bar{\mu}} \bar{u} \right]
\]
\[
\times F_{n,\bar{\mu}}(u_{1}, \cdots, u_{2D}; \bar{u}, \cdots, \bar{u}; J) \tag{3.8}
\]
\[
= \exp \left[ \frac{1}{2} \epsilon^{2-D} \frac{\partial^{2}}{\partial u_{\bar{\mu}}^{2}} \right] \exp \left[ -\epsilon^{D} V(u_{\bar{\mu}}) + ie^{D}_{\mu} J_{n+\bar{\mu}} u_{\bar{\mu}} \right]
\]
\[
\times F_{n,\bar{\mu}}(u_{1}, \cdots, u_{2D}; u_{\bar{\mu}}, \cdots, u_{\bar{\mu}}; J). \tag{3.9}
\]
where in the second line we used the identity*)

$$\int \frac{dy}{\sqrt{2\pi a}} e^{-\frac{1}{2} \rho^2} f(y) = e^{\frac{1}{2} \rho^2} f(x). \quad (a > 0) \quad (3.10)$$

Then we can easily invert Eq. (3.9) to find

$$F_{n,\mu}(u_i, \ldots, u_{2D}; u, \ldots, u, \bar{u}; J)$$

$$= \exp\left[ e^D V(u_{\mu}) - ie^D J_{n+\mu} \frac{\partial^2}{\partial u_{\mu}^2} \right] A_n(u, \ldots, u_{2D}; J). \quad (3.11)$$

On the other hand, one may express the hole function $A_{n+\mu}$ in terms of $F_{n,\mu}$:

$$A_{n+\mu}(\bar{u}_1, \ldots, \bar{u}_{2D}; J) = \int [du] \exp\left[ - e^D \left\{ \frac{1}{2} \left( \frac{\bar{u}_{-\mu} - u}{\epsilon} \right)^2 + V(u) \right\} + ie^D J_n u \right] \times F_{n,\mu}(u, \ldots, u; \bar{u}_1, \ldots, \bar{u}_{2D}; J). \quad (3.12)$$

Using representation by differential operator as before, we obtain

$$F_{n,\mu}(\bar{u}_{-\mu}, \ldots, \bar{u}_{-\mu}; \bar{u}_1, \ldots, \bar{u}_{2D}; J)$$

$$= \exp\left[ e^D V(\bar{u}_{-\mu}) - ie^D J_{n+\mu} \bar{u}_{-\mu} \right] \exp\left[ - \frac{1}{2} \frac{\partial^2}{\partial \bar{u}_{-\mu}^2} \right] A_{n+\mu}(\bar{u}_1, \ldots, \bar{u}_{2D}; J). \quad (3.13)$$

Comparing Eqs. (3.11) and (3.13), we find

$$F_{n,\mu}(u, \ldots, u; \bar{u}, \ldots, \bar{u}; J)$$

$$= \exp\left[ e^D V(\bar{u}) - ie^D J_{n+\mu} \bar{u} \right] \exp\left[ - \frac{1}{2} \frac{\partial^2}{\partial \bar{u}^2} \right] A_{n+\mu}(u, \ldots, u, \bar{u}, u, \ldots, u; J) \quad (3.14)$$

Or, equivalently,

*) To derive the integral form (left-hand side) from the differential form (right-hand side), substitute

$$f(x) = \int dy \delta(x-y) f(y) = \int \frac{dy}{2\pi} e^{ip(x-y)} f(y)$$

and integrate over $p$ after replacing $d/dx$ by $ip$.

Also, one may show a similar identity for the inverse transformation:

$$\int_{-\sqrt{2\pi a}}^{\infty} \frac{dy}{\sqrt{2\pi a}} e^{-\frac{1}{2} (y+ia)^2} f(iy) = e^{-\frac{1}{2} \rho^2} f(x). \quad (a > 0)$$
\[
\exp\left[\frac{1}{2} \varepsilon^{2-D} \frac{\partial^2}{\partial u^2}\right]\exp[-\epsilon^0 V(u) + ie^0 J_n u] A_n(u, \ldots, u, \bar{u}, u, \ldots, u; \mu) \\
= \exp\left[\frac{1}{2} \varepsilon^{2-D} \frac{\partial^2}{\partial u^2}\right]\exp[-\epsilon^0 V(\bar{u}) + ie^0 J_{n+\bar{\mu}} \bar{u}] A_{n+\bar{\mu}}(\bar{u}, \ldots, \bar{u}, u, \bar{u}, \ldots, \bar{u}; \mu) \\
\] 
(3.15)

These are the local equations satisfied by the hole function we set out to derive. Using Eq. (3.10), we obtain coupled linear integral equations.

We will show in the next section that the local equations (3.15) together with the condition (3.6) is equivalent to the Dyson-Schwinger equations. In marked contrast to the Dyson-Schwinger equations, however, we obtain sets of closed equations among the hole functions when we expand Eqs. (3.15) in Taylor series in \( J \). For example, if we define

\[
A_n^{(0)} \equiv A_n|_{J=0}, \\
A_n^{(1)} k = \frac{1}{ie^0} \frac{\partial}{\partial J_k} A_n|_{J=0},
\]
(3.16)
(3.17)

then the zeroth order and the first order equations, respectively, become as follows.

\[
\exp\left[\frac{1}{2} \varepsilon^{2-D} \frac{\partial^2}{\partial u^2}\right]\exp[-\epsilon^0 V(u)] A_n^{(0)}(u, \ldots, u, \bar{u}, u, \ldots, u) \\
= \exp\left[\frac{1}{2} \varepsilon^{2-D} \frac{\partial^2}{\partial u^2}\right]\exp[-\epsilon^0 V(\bar{u})] A_{n+\bar{\mu}}^{(0)}(\bar{u}, \ldots, \bar{u}, u, \bar{u}, \ldots, \bar{u}) \\
\] 
(3.18)

and

\[
\exp\left[\frac{1}{2} \varepsilon^{2-D} \frac{\partial^2}{\partial u^2}\right]\exp[-\epsilon^0 V(u)] A_n^{(1)}(u, \ldots, u, \bar{u}, u, \ldots, u) \\
- \exp\left[\frac{1}{2} \varepsilon^{2-D} \frac{\partial^2}{\partial u^2}\right]\exp[-\epsilon^0 V(\bar{u})] A_{n+\bar{\mu}}^{(1)}(\bar{u}, \ldots, \bar{u}, u, \bar{u}, \ldots, \bar{u}) \\
\]

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\[
\begin{align*}
&= \delta_{\mu, \nu, \lambda} \exp \left[ \frac{1}{2} \varepsilon^{2-\delta} \frac{\partial^2}{\partial \mu^2} \right] \exp \left[ -\varepsilon^0 V(\bar{u}) \right] \bar{u} A_{\mu, \nu, \lambda}(\bar{u}, \cdots, \bar{u}, u, \bar{u}, \cdots, \bar{u}) \\
&\quad - \mu \\
&= - \delta_{\mu, \nu, \lambda} \exp \left[ \frac{1}{2} \varepsilon^{2-\delta} \frac{\partial^2}{\partial \mu^2} \right] \exp \left[ -\varepsilon^0 V(u) \right] u A_{\mu, \nu, \lambda}(u, \cdots, u, \bar{u}, \cdots, \bar{u}, u) .
\end{align*}
\]

Also, a condition follows from Eq. (3.6):

\[ A_{\mu, \nu, \lambda}(u, \cdots, u, \bar{u}) = 0 . \] (3.20)

So, if we solve Eqs. (3.18)～(3.20), we can calculate one-point and two-point Green functions, respectively, using Eq. (3.5).

\[ \S 4. \text{ Equivalence to the Dyson-Schwinger equations} \]

We have seen in the previous section that the hole function defined in Eq. (3.2) obeys the local equations (3.15). Conversely, one may define a hole function to be the solution to the local equations (3.15) satisfying condition (3.6). Now we are going to show that the hole function defined in this way generates the partition function that obeys the Dyson-Schwinger equations.

Let us begin by showing that \( Z_n \) defined as

\[ Z_n = \int [du] \exp \left[ -\varepsilon^0 V(u) + i\varepsilon^0 J_n u \right] A_n(u, \cdots, u; J) \] (4.1)

is independent of \( n \). This property would have been trivial if we had adopted the definition (3.2) since \( Z_n \) is just the partition function \( Z[J] \). Using the local equations (3.15), we see

\[ Z_{n+\bar{u}} = \int [du] \exp \left[ -\varepsilon^0 V(u) + i\varepsilon^0 J_n u \right] A_{n+\bar{u}}(u, \cdots, u; J) \]

\[ = \int [du] \left\{ \exp \left[ -\varepsilon^0 V(\bar{u}) + i\varepsilon^0 J_{n+\bar{u}} \bar{u} \right] A_{n+\bar{u}}(\bar{u}, \cdots, \bar{u}, u, \bar{u}, \cdots, \bar{u}, J) \right\} \bar{u} \rightarrow u \]

\[ = \int [du] \left\{ \exp \left[ -\varepsilon^0 V(u) + i\varepsilon^0 J_n u \right] A_n(u, \cdots, u, \bar{u}, \cdots, \bar{u}, u; J) \right\} \bar{u} \rightarrow u \]

\[ = \int [du] \exp \int \left[ -\varepsilon^0 V(u) + i\varepsilon^0 J_n u \right] A_n(u, \cdots, u; J) = Z_n, \] (4.2)
where in the last line we used the identity

$$\int dx \left\{ \exp \left[ -\frac{1}{2} \epsilon^2 \frac{\partial^2}{\partial y^2} + \frac{1}{2} \epsilon^2 \frac{\partial^2}{\partial x^2} \right] f(x, y) \right\}_{y-x} = \int dx f(x, x)$$

(see Appendix B for the proof). Thus, $Z_n$ is independent of $n$, so we will denote $Z_n = Z[J]$ in the following.

We can derive yet another local property of $Z[J]$ in a similar manner. Again using Eq. (3·15), we find

$$\int [du] u \exp[-\epsilon^d V(u) + i\epsilon^d J_{n+\mu} u] A_n(u, \cdots, u; J)$$

$$= \int [du] u \left\{ \exp[-\epsilon^d V(\tilde{u}) + i\epsilon^d J_{n+\tilde{\mu}} \tilde{u}] A_n(\tilde{u}, \cdots, \tilde{u}, u, \tilde{u}, \cdots, \tilde{u}; J) \right\}_{\tilde{u} \to u}$$

$$= \int [du] u \left\{ \exp[-\frac{1}{2} \epsilon^2 \frac{\partial^2}{\partial \tilde{u}^2} + \frac{1}{2} \epsilon^2 \frac{\partial^2}{\partial u^2}] \right\}_{\tilde{u} \to u}$$

$$\times \exp[-\epsilon^d V(u) + i\epsilon^d J_n u] A_n(u, \cdots, u, \tilde{u}, u, \cdots, u; J) \right\}_{\tilde{u} \to u}$$

$$= \int [du] \left\{ u \exp[-\epsilon^d V(u) + i\epsilon^d J_n u] A_n(u, \cdots, u) \right\}_{\tilde{u} \to u}$$

$$+ \epsilon^2 \exp[-\epsilon^d V(u) + i\epsilon^d J_n u] \left\{ \frac{\partial}{\partial \tilde{u}} A_n(u, \cdots, u, \tilde{u}, u, \cdots, u; J) \right\}_{\tilde{u} \to u} \right\}_{\tilde{u} \to u}$$

(4·4)

where in the last line we used the identity

$$\int dx x \left\{ \exp \left[ -\frac{1}{2} \epsilon^2 \frac{\partial^2}{\partial y^2} + \frac{1}{2} \epsilon^2 \frac{\partial^2}{\partial x^2} \right] f(x, y) \right\}_{y-x}$$

$$= \int dx \left( xf(x, x) + \epsilon^2 \frac{\partial}{\partial y} f(x, y) \right)_{y-x}$$

(4·5)

(see Appendix B for the proof). Thus, we have shown the relation

$$\int [du] u e^{-\epsilon^d V(u)} e^{-2[A_n+\tilde{\mu} u]e^{i\epsilon^d J_{n+\tilde{\mu}} u} - A_n(u, \cdots, u; J)e^{i\epsilon^d J_{\tilde{\mu}} u}]$$

$$= e^{-\epsilon^d} \int [du] e^{-\epsilon^d V(u) + i\epsilon^d J_{n+\mu}} \left\{ \frac{\partial}{\partial \tilde{u}} A_n(u, \cdots, u, \tilde{u}, u, \cdots, u; J) \right\}_{\tilde{u} \to u}$$

(4·6)
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Taking sum over $\tilde{\mu}$ in 2D directions, we find

$$
\int [du] e^{-\epsilon v(u)} \sum_{\tilde{\mu}} e^{-2\left[A_{n+\tilde{\mu}}(u, \ldots, u; J) e^{i e p_{n+\tilde{\mu}} u} - A_n(u, \ldots, u; J) e^{i e p_{n} u}\right]} = e^{-D} \int [du] e^{-\epsilon v(u) + i e p_{n} u} \frac{\partial}{\partial u} A(u, \ldots, u; J) .
$$

(4.7)

Then one may convert this equality to the equation for the partition function $Z[J]$ using Eqs. (3.6) and (4.2):

$$(\text{l.h.s.}) = \sum_{\tilde{\mu} = 1}^{2D} e^{-2} \left\{ \frac{1}{i e p} \frac{\partial}{\partial f_{n+\tilde{\mu}}} - \frac{1}{i e p} \frac{\partial}{\partial f_{n}} \right\} Z[J],$$

$$(4.8)$$

$$(\text{r.h.s.}) = - \int [du] \left[ - V(u) + i f_n \right] e^{-\epsilon v(u) + i e p_{n} u} A_n(u, \ldots, u; J)$$

$$= \left\{ V\left(\frac{1}{i e p} \frac{\partial}{\partial f_{n}}\right) - i f_n \right\} Z[J].$$

(4.9)

Hence, the partition function satisfies the Dyson-Schwinger equations (2.9). This shows that the Green functions constructed from the hole function (3.5) satisfy the Dyson-Schwinger equation (2.13).

§ 5. Extracting the vacuum state at the boundaries

So far we have discussed method for calculating Green functions that obey the Dyson-Schwinger equations, while we have left aside the issue of their boundary conditions. In fact, Green functions satisfy the Dyson-Schwinger equations for arbitrary boundary conditions. In quantum field theory, however, we usually want to find the Green functions given by the vacuum expectation values of time-ordered field operator products. We will see that such Green functions can be obtained by solving the zeroth and first order local equations for the hole functions, Eqs. (3.18) and (3.19), with appropriate conditions.

The Hamiltonian $\tilde{H}$ for the lattice field theory (3.1) is defined from the transfer matrix of this theory:

$$e^{-\tilde{\theta}} = \exp \left[ -\frac{1}{2} e^D \sum_{l \in S} \left( \sum_{\tilde{\nu} = 1}^{2D} \frac{1}{2} \left( \phi_{l+\tilde{\nu}} - \phi_l \right)^2 + V(\phi_l) \right) \right] \times \exp \left[ \frac{1}{2} e^{2-D} \sum_{l \in S} \frac{\partial^2}{\partial \phi_l^2} \right] \exp \left[ -\frac{1}{2} e^D \sum_{l \in S} \left( \sum_{\tilde{\nu} = 1}^{2D} \frac{1}{2} \left( \phi_{l+\tilde{\nu}} - \phi_l \right)^2 + V(\phi_l) \right) \right],$$

(5.1)

where $\phi_l$ are the fields on a $(D-1)$-dimensional lattice hyperplane $S$ corresponding to some fixed time. $\tilde{H}$ is hermitian, and we denote the energy eigenstates as

$$\tilde{H} |a\rangle = E_a |a\rangle .$$

(5.2)

We can define Heisenberg operator $\tilde{\Phi}_a$ just as in the continuum theory. Then the Green functions that obey the Dyson-Schwinger equations (2.13) are given by
for arbitrary \( C_{ab} \).

Suppose we look for translationally invariant solutions to the zeroth order equations (3·18):

\[ A_n^{(0)} = \text{independent of } n. \]

Then any one-point function calculated from \( A_n^{(0)} \) will also be translationally invariant

\[ \langle \phi_n \rangle = \frac{\int [du] u^m A_n^{(0)}(u, \ldots, u) e^{-\epsilon V(u)}}{\int [du] A_n^{(0)}(u, \ldots, u) e^{-\epsilon V(u)}} = \text{independent of } n. \]

It means we have selected a particular class of boundary conditions in (5·3). Since

\[ \sum_{a, b} \langle a | \hat{\Phi}_n^m | \beta \rangle = \sum_{a, b} C_{ab} e^{-\tau(E_b - E_a)} \langle a | \hat{\Phi}_{(n, 0)} | \beta \rangle, \]

where we regard the \( D \)-th component of \( n \) as the Euclidean time, \( n=(n, n_0) \) and \( \tau = \epsilon n_0 \), we see solutions (5·4) correspond to the boundary conditions

\[ C_{ab} = 0 \quad \text{for} \quad E_a \neq E_b. \]

Next, consider the solution \( A_{n,k}^{(1)} \) to the first order equations (3·19), from which one may calculate the two-point Green function \( \langle \phi_n \phi_k \rangle \) subject to the same boundary conditions as that for the one-point function, Eq. (5·7). For \( n_D > k_D \),

\[ \langle \phi_n \phi_k \rangle = \sum_{a, a'} C_{aa'} e^{-\tau(E_a - E_{a'})} \langle a | \hat{\Phi}_{(n, 0)} | \beta \rangle \langle \beta | \hat{\Phi}_{(k, 0)} | a' \rangle, \]

where \( k=(k, k_0) \) and \( \tau' = \epsilon k_0 \). Therefore, if we demand the two-point function to be well-behaved as \( |n_D - k_D| \to \infty \), \( E_a \geq E_b \), so that the vacuum state will be selected at the boundaries. It is the condition that should be imposed on \( A_{n,k}^{(1)} \).

To calculate higher Green functions satisfying the right boundary conditions, solve successively higher order local equations for higher order hole functions using thus obtained \( A_n^{(0)} \) and \( A_{n,k}^{(1)} \).

\section*{§ 6. Basic properties of the local equations}

\subsection*{6.a. \( D=1 \) Case}

Let us examine the local equation (3·15) for the 1-dimensional lattice with \( N+1 \) sites \((n=0, 1, \ldots, N)\):

\[ e^{(1/2)e(\partial^2/\partial u^2)} e^{-\epsilon V(u) + i\epsilon J_n u} A_n(u, \tilde{u}; J) = e^{(1/2)e(\partial^2/\partial \tilde{u}^2)} e^{-\epsilon V(\tilde{u}) + i\epsilon J_{n+1} \tilde{u}} A_{n+1}(u, \tilde{u}; J). \]

Noting that \( A_n(u, \tilde{u}; J) \) is independent of \( J_n \), the solution is found to be

\[ A_n(u, \tilde{u}; J) = \sum \left[ e^{(1/2)e(\partial^2/\partial u^2)} e^{-\epsilon V(u) + i\epsilon J_{n-1} u} \ldots e^{(1/2)e(\partial^2/\partial u^2)} e^{-\epsilon V(u) + i\epsilon J_0 f(u)} \right] \]
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\[ \times e^{-iV(\bar{u}) + i\epsilon N \cdot \hat{u} \cdot e^{(1/2)\epsilon (\hat{u}^{\mu} / \alpha \hat{u}^{\mu})} \cdots e^{-iV(\bar{u}) + i\epsilon N \cdot \hat{u} \cdot e^{(1/2)\epsilon (\hat{u}^{\mu} / \alpha \hat{u}^{\mu})} g_i(\bar{u})} \]  

(6·2)

for \( \forall f_i(u), g_i(\bar{u}) \).

Now we take the continuum limit \( \epsilon \to 0 \) fixing \( T = \epsilon N \) and \( \tau = \epsilon n \). Equation (6·1) reduces to

\[ \frac{\partial}{\partial \tau} A(\tau; u, \bar{u} ; J) = \left[ (\hat{H}_u - iJ(\tau)\bar{u}) - (\hat{H}_{\bar{u}} - iJ(\tau)u) \right] A(\tau; u, \bar{u} ; J) , \]

(6·3)

where

\[ \hat{H}_u = -\frac{1}{2} \frac{\partial^2}{\partial u^2} + V(u) , \quad \hat{H}_{\bar{u}} = -\frac{1}{2} \frac{\partial^2}{\partial \bar{u}^2} + V(\bar{u}) , \]

(6·4)

and the hole function (6·2) to

\[ A(\tau; u, \bar{u} ; J) = \sum_i \langle g_i | T e^{-\int_0^{\tau} dr (\bar{u} - \bar{u}(r) \hat{\phi})} \bar{u} \rangle \langle u | T e^{-\int_0^{\tau} dr (\bar{u} - \bar{u}(r) \hat{\phi})} f_i \rangle , \]

(6·5)

which are the slightly generalized forms of the equations discussed in § 1. Then one obtains the correct partition function of the theory from the hole function

\[ Z[J(\tau)] = \lim_{\epsilon \to 0} \int du e^{-\epsilon V(u) + i\epsilon n u} A_n(u, u ; J) \]

(6·6)

\[ = \sum_i \langle g_i | T e^{-\int_0^{\tau} dr (\bar{u} - \bar{u}(r) \hat{\phi})} f_i \rangle . \]

(6·7)

The Green functions are obtained from \( Z[J] \) (or from \( A(\tau; u, \bar{u} ; J) \)), and it is easy to check that the procedure indicated in the previous section picks up the vacuum \( |f_i> \), \( |g_i> \to |0> \).

6.b. Formal solutions

We present formal solutions to the local equations (3·15).

First consider the hole function (3·2) in the case of free field theory \( V(\phi) = (1/2)m^2 \phi^2 \). Since the integrand is Gaussian, one can explicitly perform the integration. We find

\[ A_n^{(\text{free})}(u_1, \cdots, u_{2D}; J) = \exp \left[ -\frac{1}{2} (u + J)^T N^{-1} (u + J) - \frac{1}{2} \epsilon^{0-2} \sum_{\mu=1}^{2D} u_\mu^2 \right] , \]

(6·8)

where

\[ (J)_i = e^\rho f_i , \]

\[ (u)_i = \sum_{\mu=1}^{2D} e^{\rho-2} u_\mu \delta_{\nu + \mu, i} , \]

(6·9)

\[ (N^{-1})_{\nu \kappa} = G(l - k) - \frac{1}{G(0)} G(l - n) G(n - k) \]

and

\[ G(l) = \frac{1}{\epsilon^{0}} \int d^D_\rho \frac{ \rho_{\nu} d_\nu}{(2\pi)^D} \frac{e^{i\rho \cdot l}}{m^2 + 2\epsilon^{0-2} \sum_{\mu=1}^{2D} [1 - \cos \rho_\mu]} , \quad B = (-\pi, \pi)^D . \]

(6·10)
It is straightforward to verify that Eq. (6·8) satisfies the local equation (3·15).

For a general potential $V(\phi)$, we write

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + L_{\text{int}}(\phi).$$

(6·11)

Then from the definition (3·2), the formal solution can be written as

$$A_n(u_1, \ldots, u_{2D}; J) = \exp \left[ - e^D \sum_{i \in A_D} \left( \frac{1}{i} \frac{\partial}{\partial a_i} \right) \right] A_{n}^{(\text{free})}(u_1, \ldots, u_{2D}; J).$$

(6·12)

Again, it is straightforward to show that this formal solution satisfies the local equations (3·15).

6.c. Reduction of variables

The closed equations (3·18) and (3·19) are for the hole functions of 2D link variables. We shall turn the problem of solving these equations into that of solving coupled linear equations for a set of functions of two variables. When the system in question is translationally invariant as well as invariant under rotation by 90°, we define

$$A(u, \bar{u}) \equiv A^{(0)}_n(u, \ldots, u, \bar{u}, u, \ldots, u); \quad \text{independent of } n, \bar{\mu},$$

(6·13)

$$B_{n-k; \bar{\mu}}(u, \bar{u}) \equiv A^{(1)}_{n-k}(u, \ldots, u, \bar{u}, u, \ldots, u).$$

(6·14)

Then Eqs. (3·18)~(3·20) are rewritten as

$$e^{(1/2) e^{2-0} (\partial/\partial u)} e^{-e^{0} (u)} A(u, \bar{u}) = e^{(1/2) e^{2-0} (\partial/\partial \bar{u})} e^{-e^{0} (\bar{u})} A(\bar{u}, u),$$

(6·15)

$$e^{(1/2) e^{2-0} (\partial/\partial u)} e^{-e^{0} (u)} B_{1; \bar{\mu}}(u, \bar{u}) - e^{(1/2) e^{2-0} (\partial/\partial \bar{u})} e^{-e^{0} (\bar{u})} B_{1+\bar{\mu}, -\bar{\mu}}(\bar{u}, u)$$

$$= \delta_{1+\bar{\mu}, \bar{\mu}} e^{(1/2) e^{2-0} (\partial/\partial \bar{u})} e^{-e^{0} (\bar{u})} \bar{u} A(\bar{u}, u) - \delta_{1, \bar{\mu}} e^{(1/2) e^{2-0} (\partial/\partial u)} e^{-e^{0} (u)} u A(u, \bar{u}),$$

(6·16)

$$B_{0; \bar{\mu}}(u, \bar{u}) = 0.$$

(6·17)

In addition, the following conditions should be satisfied according to the definitions (6·13) and (6·14).

$$\frac{d}{du} A(u, \bar{u}) = 2D \left[ \frac{\partial}{\partial \bar{u}} A(u, \bar{u}) \right]_{\bar{u} = u},$$

(6·18)

$$B_{1; \bar{\mu}}(u, \bar{u}) = B_{1}(u) ; \quad \text{independent of } \bar{\mu},$$

(6·19)

$$\frac{d}{du} B_{1}(u) = \sum_{\bar{\mu} = 1}^{2} \left[ \frac{\partial}{\partial \bar{u}} B_{1; \bar{\mu}}(u, \bar{u}) \right]_{\bar{u} = u}.$$
Following the discussion in § 4, one can verify that when \( A(u, \bar{u}) \) and \( B_{i; \vec{\nu}}(u, \bar{u}) \) satisfy Eqs. (6·15)~(6·20), the one-point and two-point Green functions constructed from these hole functions obey the Dyson-Schwinger equations (2·13). Thus, it suffices to solve these sets of coupled linear equation for \( A(u, \bar{u}) \) and \( B_{i; \vec{\nu}}(u, \bar{u}) \) to obtain the one-point and two-point Green functions.

§ 7. Summary and discussion

In § 2 we review the Dyson-Schwinger equations for a Euclidean lattice scalar field theory. We argue that they carry full information of the theory since the equations are identified with the Fourier transform of the defining equations for the weight factor \( e^{-s|\phi|} \).

In § 3 we define the “hole function”, \( A_n \), associated with a lattice site \( n \). It is defined so that any \( N \)-point Green function can be constructed from it. We see that the hole function obeys a set of local equations, which are shown to be equivalent to the Dyson-Schwinger equations (§ 4). The remarkable feature is that the local equations, when expanded in terms of source \( J \), reduce to sets of closed equations for the hole functions. It is in contrast to the Dyson-Schwinger equations which are the infinite series of coupled equations.

To obtain the Green functions satisfying the right boundary conditions; find translationally invariant solutions \( A_n^{(0)} \) to the zeroth order equation (3·18). Then solve the first order equation (3·19) for \( A_{n; \vec{\nu}}^{(1)} \), and demand that the two-point function calculated from \( A_{n; \vec{\nu}}^{(1)} \) to behave well as \( |n-k| \to \infty \) (§ 5).

Section 6 summarizes some basic properties of the local equations. We have seen that the equation correctly reproduces the known results from quantum mechanics when \( D=1 \) (§ 6.a). The existence of formal solution to the local equations is anticipated from the ordinary perturbation theory, and the explicit form is given (§ 6.b). The closed sets of local equations for the hole functions can be reduced to the coupled linear equations for functions of two variables (§ 6.c).

It would be straightforward to solve numerically the local equations given in the form of § 6.c. If we expand the functions \( A(u, \bar{u}) \) and \( B_{i; \vec{\nu}}(u, \bar{u}) \) in terms of some appropriate functional bases, task to solve the equations would be reduced to matrix calculations.

Finally, it may be instructive to see which part of the information on the Dyson-Schwinger equations is contained in each set of closed equations. Consider the solution \( A_n^{(0)} \) to the simplest equations (3·18). Since we may obtain \( F_{n; \vec{\nu}}^{(1)} \) from \( A_n^{(0)} \) via Eq. (3·14), not only the one-point functions \( \langle \phi_n^m \rangle \) but also two-point functions of the nearest neighbors \( \langle \phi_n^m \phi_{\vec{\nu}}^m \rangle \) can be constructed from \( A_n^{(0)} \).* In fact, one can show using the zeroth order equations (3·18) alone the following particular part of the Dyson-Schwinger equations

\[
\left( \Box_n \phi_n \right) \phi_n^m - V'(\phi_n) \phi_n^m + \frac{m}{\epsilon} \phi_n^{m-1} = 0, \quad (m=0, 1, 2, \ldots) \quad (7·1)
\]

*) Since we defined the hole function to be a function of links surrounding \( n \) rather than of the field on \( n \), it carries some information on the nearest neighbors.
Similarly, from the solutions to Eqs. (3·18) and (3·19), one can show
\[
\left\langle \left( \Box_{\kappa} \phi_{n} \right) \phi_{n}^{\mu} \phi_{k} - V(\phi_{n}) \phi_{n}^{\mu} \phi_{k} + \frac{m}{\epsilon \partial} \phi_{n}^{\mu-1} \phi_{k} + \frac{1}{\epsilon \partial} \delta_{n.k} \phi_{n}^{\mu} \right\rangle = 0. \quad (m=0, 1, 2, \ldots)
\] (7·2)

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Appendix A

--- Minkowski Versions ---

We present here the local equations for Minkowski space-time.

Define
\[
\eta_{\mu} = \begin{cases} 
-1 & \text{for } \mu = \pm \vec{x}_{i}, \ldots, \pm \vec{x}_{D-1} \\
+1 & \text{for } \mu = \pm \vec{x}_{D} 
\end{cases} \quad \text{(space direction)}
\] (A·1)

The action is given by
\[
S[\phi] = \sum_{J} e^{\partial} \left\{ \frac{1}{2} \eta_{\mu} \frac{\left( \phi_{t+\mu} - \phi_{t} \right)^{2}}{\epsilon} - V(\phi_{t}) \right\},
\] (A·2)

and the hole function is defined by
\[
A_{n}(u_{1}, \ldots, u_{2D}; J) = \int \prod_{i=n} \left[ d\phi_{i} \right] \exp \left[ iS_{n}[\phi] + i e^{\partial} \sum_{i=n} j_{i} \phi_{i} + i e^{\partial} \frac{1}{2} \sum_{i=1}^{2D} \eta_{\mu} \left( \frac{\phi_{n+\mu} - u_{\mu}}{\epsilon} \right)^{2} \right] \] (A·3)

with
\[
S_{n}[\phi] = S[\phi] - e^{\partial} \left\{ \frac{1}{2} \sum_{i=1}^{2D} \eta_{\mu} \left( \frac{\phi_{n+\mu} - \phi_{n}}{\epsilon} \right)^{2} - V(\phi_{n}) \right\}. \] (A·4)

The integral measure is defined as \([d\phi_{i}] = d\phi_{i}/\sqrt{2\pi\epsilon}.

The local equations satisfied by the hole function is
\[
\exp \left[ i e^{\partial} \eta_{\mu} \frac{\partial^{2}}{\partial u_{\mu}^{2}} \right] \exp \left[ - i e^{\partial} V(u) + i e^{\partial} j_{i} u_{i} A_{n}(u, \ldots, u, \vec{u}, u, \ldots, u; J) \right] \]

\[\uparrow_{\mu}
\]
\[\uparrow_{\vec{\mu}}
\]

\[= \exp \left[ i e^{\partial} \eta_{\mu} \frac{\partial^{2}}{\partial \vec{u}_{\mu}^{2}} \right] \exp \left[ - i e^{\partial} V(\vec{u}) + i e^{\partial} j_{n+\mu} \vec{u}_{i} A_{n+\mu}(\vec{u}, \ldots, \vec{u}, \vec{u}, u, \ldots, \vec{u}; J) \right] \]

\[\downarrow_{\mu}
\]

\[\downarrow_{\vec{\mu}}
\]

(A·5)

where a condition that follows from the definition Eq. (A·3) is
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\[ A_n(u_1, \ldots, u_{2\alpha}; f) = \text{independent of } f. \quad (A \cdot 6) \]

An identity corresponding to Eq. (3.10) is given by

\[ \int \frac{dy}{\sqrt{2\pi a}} e^{i(x-y)^2/2a} f(y) = e^{(1/2)ia(x^2 + x^2)} f(x). \quad (A \cdot 7) \]

When we expand the local equation (A.5) in \( f \), the zeroth-order and first-order equations are, respectively,

\[ \exp \left[ \frac{1}{2} i e^{2p} \eta \frac{\partial^2}{\partial u^2} \right] \exp \left[ -i e^{p} V(u) \right] A_0^{(0)}(u, \ldots, u, \tilde{u}, u, \ldots, u) \]

\[ \uparrow \bar{\mu} \]

\[ = \exp \left[ \frac{1}{2} i e^{2p} \eta \frac{\partial^2}{\partial u^2} \right] \exp \left[ -i e^{p} V(\tilde{u}) \right] A_0^{(0)}(\tilde{u}, \ldots, \tilde{u}, u, \tilde{u}, \ldots, \tilde{u}) \quad (A \cdot 8) \]

and

\[ \exp \left[ \frac{1}{2} i e^{2p} \eta \frac{\partial^2}{\partial u^2} \right] \exp \left[ -i e^{p} V(u) \right] A_1^{(1)}(u, \ldots, u, \tilde{u}, u, \ldots, u) \]

\[ \uparrow \bar{\mu} \]

\[ - \exp \left[ \frac{1}{2} i e^{2p} \eta \frac{\partial^2}{\partial u^2} \right] \exp \left[ -i e^{p} V(\tilde{u}) \right] A_1^{(1)}(\tilde{u}, \ldots, \tilde{u}, u, \tilde{u}, \ldots, \tilde{u}) \]

\[ \uparrow -\bar{\mu} \]

\[ = \delta_{\kappa + \mu, \lambda} \exp \left[ \frac{1}{2} i e^{2p} \eta \frac{\partial^2}{\partial u^2} \right] \exp \left[ -i e^{p} V(\tilde{u}) \right] \tilde{A}_1^{(1)}(\tilde{u}, \ldots, \tilde{u}, u, \tilde{u}, \ldots, \tilde{u}) \]

\[ \uparrow -\bar{\mu} \]

\[ - \delta_{\kappa, \lambda} \exp \left[ \frac{1}{2} i e^{2p} \eta \frac{\partial^2}{\partial u^2} \right] \exp \left[ -i e^{p} V(u) \right] u A_1^{(0)}(u, \ldots, u, \tilde{u}, u, \ldots, u) \]

\[ \uparrow \bar{\mu} \quad (A \cdot 9) \]

where

\[ A_0^{(0)} = A_n|_{f=0}, \quad (A \cdot 10) \]

\[ A_1^{(1)} = \frac{1}{i e^p} \frac{\partial}{\partial f_k} A_n|_{f=0} \quad (A \cdot 11) \]

and
Appendix B

In this appendix, we prove Eqs. (4·3) and (4·5):

\[
\int dx \left\{ \exp \left[ -\frac{1}{2} e^{2-D} \frac{\partial^2}{\partial y^2} + \frac{1}{2} e^{2-D} \frac{\partial^2}{\partial x^2} \right] f(x, y) \right\}_{y=x} = \int dx f(x, x) \tag{B·1}
\]

and

\[
\int dx x \left\{ \exp \left[ -\frac{1}{2} e^{2-D} \frac{\partial^2}{\partial y^2} + \frac{1}{2} e^{2-D} \frac{\partial^2}{\partial x^2} \right] f(x, y) \right\}_{y=x} \\
= \int dx \left( xf(x, x) + e^{2-D} \left( \frac{\partial}{\partial y} f(x, y) \right) \right)_{y=x} \tag{B·2}
\]

Here, we assume that the function \( f(x, x) \) swiftly vanishes as \(|x| \to \infty\), as well as

\[
e^{-(1/2)e^{2-D}p^2}f(x, y) \text{ swiftly vanishes as } |x|, |y| \to \infty.
\]

We write \( f(x, y) = \langle x | \tilde{f} | y \rangle \). Then, we see

\[
\int dx \left\{ \exp \left[ -\frac{1}{2} e^{2-D} \frac{\partial^2}{\partial y^2} + \frac{1}{2} e^{2-D} \frac{\partial^2}{\partial x^2} \right] f(x, y) \right\}_{y=x} \\
= \int dx \langle x | e^{-(1/2)e^{2-D}p^2} \tilde{f} e^{(1/2)e^{2-D}p^2} | x \rangle \\
= \text{Tr} \left[ e^{-(1/2)e^{2-D}p^2} \tilde{f} e^{(1/2)e^{2-D}p^2} \right] = \text{Tr} [ \tilde{f} ] \\
= \int dx f(x, x) \tag{B·3}
\]

Also,

\[
\int dx x \left\{ \exp \left[ -\frac{1}{2} e^{2-D} \frac{\partial^2}{\partial y^2} + \frac{1}{2} e^{2-D} \frac{\partial^2}{\partial x^2} \right] f(x, y) \right\}_{y=x} \\
= \int dx \langle x | e^{-(1/2)e^{2-D}p^2} \tilde{f} e^{(1/2)e^{2-D}p^2} \tilde{x} | x \rangle \\
= \text{Tr} \left[ e^{-(1/2)e^{2-D}p^2} \tilde{f} e^{(1/2)e^{2-D}p^2} \tilde{x} \right] \\
= \text{Tr} [ \tilde{f} (e^{(1/2)e^{2-D}p^2} \tilde{x} e^{-(1/2)e^{2-D}p^2}) ] = \text{Tr} [ \tilde{f} (\tilde{x} - ie^{2-D}p) ] \\
= \int dx \left( xf(x, x) + e^{2-D} \left( \frac{\partial}{\partial y} f(x, y) \right) \right)_{y=x} \tag{B·4}
\]
References

1) F. Dyson, Phys. Rev. 75 (1949), 1736.