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Discussion

Discussion: "Shear Coefficients for Timoshenko Beam Theory" (Hutchinson, J. R., 2001, ASME J. Appl. Mech., 68, pp. 87–92)

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In a recent article, Hutchinson [1] employed the Hellinger-Reissner variational principle to construct a beam theory of Timoshenko type, together with a new expression for the inherent shear coefficient κ, as

$$\kappa=\frac{-2(1+\nu)}{\frac{A}{I_y^2}C_4+\nu\left(1-\frac{I_x}{I_y}\right)} \tag{1}$$

where

$$C_4=-\int\int\{\nu(x^2-y^2)f_1+2\nu x y f_2+2(1+\nu)(f_1^2+f_2^2)\}dx\,dy \tag{2a}$$

$$f_1=\frac{-1}{2(1+\nu)}\left(\frac{\partial\chi}{\partial x}+\frac{\nu x^2}{2}+\left(\frac{2-\nu}{2}\right)y^2\right) \tag{2b}$$

$$f_2=\frac{-1}{2(1+\nu)}\left(\frac{\partial\chi}{\partial y}+(2+\nu)xy\right). \tag{2c}$$

In the above, the notation is largely the same as that of Hutchinson except that Love's notation ([2]) has been employed for the beam cross-sectional coordinates, by replacing y by x, and z by y, in [1]; z is then the beam axial coordinate. The motivation of Hutchinson appears to be the construction of a theory in which the shear coefficient takes on the "best" value; for beams of circular and thin rectangular cross section, these are widely accepted to be $\kappa=6(1+\nu)^2/(7+12\nu+4\nu^2)$, and $\kappa=5(1+\nu)/(6+5\nu)$, respectively. The evidence to suggest that these values are "best" comes from comparison with available "exact" elastodynamic analyses and, to a lesser degree, from experiment, and is discussed in [1,3,4].

Expression (1) derived by Hutchinson is exactly equivalent to one derived by the present author and Prof. Mark Levinson [3,4] some two decades ago, which is (Ref. [3], Eq. (20))

$$\kappa=\frac{-4(1+\nu)^2 I_y^2}{2(1+\nu)A\int\int x(\chi+xy^2)dx\,dy+2\nu(1+\nu)I_y(I_y-I_x)+\nu A\int\int\left\{\left(\frac{x^2-y^2}{2}\right)\left(\frac{\partial\chi}{\partial x}+\frac{\nu x^2}{2}+\left(\frac{2-\nu}{2}\right)y^2\right)+xy\left(\frac{\partial\chi}{\partial y}+(2+\nu)xy\right)\right\}dx\,dy}. \tag{3}$$

It is remarkable that three quite different approaches should lead to the same expression for the coefficient. Hutchinson's use of Hellinger-Reissner overcomes the compromises inevitable in a beam theory, allowing "best" choices for both stress and displacement fields, which may be incompatible. In [4], Stephen and Levinson adapted the procedure of Cowper [5], but argued that the stress distribution within a beam performing long wavelength flexural vibration would be approximated better by gravity force body loading (see Love [2], Chapter 16), rather than tip loading of a cantilevered beam, as assumed in [5]. The former has shearing force varying linearly with axial coordinate, while for the latter shearing force is constant. In [3], the coefficient was obtained from the curvature correction during bending, again due to gravity loading (again see Chapter 16 of Love).

Demonstration of the equivalence of the two formulas is somewhat lengthy, and is based upon usage of Green's formula

$$\int\int\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dx\,dy=\oint(f\,dx+g\,dy), \tag{4}$$

and a knowledge of the normal derivative of the (harmonic) flexure function χ on the boundary of the cross section, that is

$$\frac{d\chi}{dn}=-\left(\frac{\nu x^2}{2}+\left(\frac{2-\nu}{2}\right)y^2\right)\cos(x,n)-(2+\nu)xy\cos(y,n). \tag{5}$$

The key step is recognition that the term within Hutchinson's coefficient C_4 involving the area integral of the sum of the squares of the terms f_1 and f_2, in turn involves the area integral of the sum $(\partial\chi/\partial x)^2+(\partial\chi/\partial y)^2$. Transformation of such terms within potential theory is well documented; see, for example, Sokolnikoff [6]. For the present problem an outline of the procedure is as follows:

construct

$$\frac{\partial}{\partial x}\left[(\chi+xy^2)\frac{\partial\chi}{\partial x}\right]=(\chi+xy^2)\frac{\partial^2\chi}{\partial x^2}+\left(\frac{\partial\chi}{\partial x}+y^2\right)\frac{\partial\chi}{\partial x} \quad (6a)$$

$$\frac{\partial}{\partial y}\left[(\chi+xy^2)\frac{\partial\chi}{\partial y}\right]=(\chi+xy^2)\frac{\partial^2\chi}{\partial y^2}+\left(\frac{\partial\chi}{\partial y}+2xy\right)\frac{\partial\chi}{\partial y} \quad (6b)$$

and add, noting that χ is harmonic, to give

$$\frac{\partial}{\partial x}\left[(\chi+xy^2)\frac{\partial\chi}{\partial x}\right]+\frac{\partial}{\partial y}\left[(\chi+xy^2)\frac{\partial\chi}{\partial y}\right]=\left(\frac{\partial\chi}{\partial x}\right)^2+\left(\frac{\partial\chi}{\partial y}\right)^2+y^2\frac{\partial\chi}{\partial x}+2xy\frac{\partial\chi}{\partial y}. \quad (7)$$

Integrate over the cross section, and transform the left-hand side (LHS) of the above using Green's formula, to give

$$\mathrm{LHS}=\oint(\chi+xy^2)\frac{d\chi}{dn}ds \quad (8)$$

where direction cosines $\cos(x,n)=dx/dn=dy/ds$, and $\cos(y,n)=dy/dn=-dx/ds$ have been employed.

Substitute for the normal derivative of χ according to (5), and convert back to an area integral to give

$$\begin{aligned}\int\int\left\{\left(\frac{\partial\chi}{\partial x}\right)^2+\left(\frac{\partial\chi}{\partial y}\right)^2\right\}dx\,dy+\int\int\left(y^2\frac{\partial\chi}{\partial x}+2xy\frac{\partial\chi}{\partial y}\right)dx\,dy\\=-\int\int 2(1+\nu)x(\chi+xy^2)dx\,dy\\-\int\int(2+\nu)xy\frac{\partial\chi}{\partial y}dx\,dy\\-\int\int\left(\frac{\nu x^2}{2}+\left(\frac{2-\nu}{2}\right)y^2\right)\frac{\partial\chi}{\partial x}dx\,dy\\-\int\int\left(\left(4+\frac{5\nu}{2}\right)x^2y^2+\left(\frac{2-\nu}{2}\right)y^4\right)dx\,dy.\end{aligned} \quad (9)$$

Next, expand Hutchinson's expression for coefficient C_4, and substitute the above, when one finds

$$\begin{aligned}C_4=\int\int x(\chi+xy^2)dx\,dy\\+\int\int\frac{\nu(x^2-y^2)}{4(1+\nu)}\left(\frac{\partial\chi}{\partial x}+\frac{\nu x^2}{2}+\left(\frac{2-\nu}{2}\right)y^2\right)dx\,dy\\+\int\int\frac{\nu xy}{2(1+\nu)}\left(\frac{\partial\chi}{\partial y}+(2+\nu)xy\right)dx\,dy.\end{aligned} \quad (10)$$

Lastly substitute the above into Eq. (1) to give expression (3). Not surprisingly, the values of the coefficient for the circular cross section, both solid, hollow and thin-walled, and for the elliptic cross section calculated in [1], are identical to those given in [3,4]. Similarly, Hutchinson's expression for the rectangular cross section reduces to the ''best'' value of $\kappa=5(1+\nu)/(6+5\nu)$ as one approaches plane stress conditions.

A further very interesting feature of [1], Figs. 3 and 4, is the possibility of the shear coefficient taking a negative value for the combination of large width to depth ratio, and for large Poisson's ratio. The effect of a negative coefficient would be to stiffen the structure, leading to a natural frequency higher than that predicted by Euler-Bernoulli theory. However, as one would not normally employ Timoshenko theory for a beam having a large width to depth ratio, this result may turn out to be of little importance. Nevertheless, the physical implication of a possible negative coefficient requires further consideration.

Finally, it is noted that while the above values for the coefficient may be widely accepted as the best, paradoxically Cowper's values appear to be the more widely used; it is to be hoped that investigators will in future make greater use of these ''best'' values.

References

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Closure to ''On Shear Coefficients for Timoshenko Beam Theory'' (2001, ASME J. Appl. Mech., 68, p. 959)

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In his most important point Professor Stephen is completely correct. That is, his shear coefficient and mine are identical. I was aware of his work when it first came out and discounted it because it was derived by solving a static problem for a specific type of loading. It implied that the shear coefficient for the static problem was a function of the type of loading and, further, that by choosing the right type of static loading one could find the best value of the shear coefficient for dynamic loads. As it turns out both of these implications are correct.

Professor Stephen states, ''The motivation of Hutchinson appears to be the construction of a theory in which the shear coefficient takes on the 'best' value.'' I did not mean to convey that impression. My motivation was simply to construct a simple, consistent, dynamic theory which did not require guessing a shear coefficient. This simple consistent theory allowed me to find an expression for the shear coefficient in the Timoshenko beam theory. That this shear coefficient agreed with the ''best'' values simply validated my approach.

Professor Stephen states, ''the values of the coefficient for the circular cross section, both solid, hollow and thin-walled, and for the elliptic cross section calculated in [1], are identical to those given in [3,4].'' What he doesn't note is that the expressions for the rectangular cross section in both his and my paper also produce identical results. He further comments on the fact that the shear coefficient goes to zero for the rectangular cross section. Actually, as shown in my Fig. 4, it is the reciprocal of the shear coefficient that goes to zero which means the shear coefficient would have a pole at that point. Professor Stephen's conclusion that the beam is stiffened with an increase in width is correct. I do not understand, however, his remark that, ''one would not normally employ Timoshenko theory for a beam having large width to depth ratio.'' For a Poisson's ratio of 0.3 the reciprocal of the shear coefficient goes to zero at a width to depth ratio of about 3. This is definitely within the range I would expect Timoshenko theory to be applied. Also the lowest natural frequency would probably occur (depending on boundary conditions) about a neutral axis in the width direction. I should probably also note that the shear coefficient is very sensitive to the assumed shear stress distribution. If one were to assume a simple parabolic shear stress distribution through the thickness of the rectangular beam one would get a shear stress of $5(1+\nu)/6+5\nu)$ independent of the aspect ratio.

In my paper I also had a small section on static problems. In that section I found expressions for a static shear coefficient found by comparing solution of my equations with the elementary beam solution including shear deformation. The comparison was done for a tip loaded cantilever. Since Professor Stephen's work indicates that the coefficient depends on loading I decided to investigate a beam loaded under it's own weight. To accomplish this I inserted a new term γv into the integral in my Eq. (28) and dropped all the time dependent terms. The term γ is the specific weight of the beam and v is the displacement in the y-direction. Proceeding in the same way as in the paper, I came up with the following set of equations,

$$\psi''' + \frac{\gamma A}{EI_z} = 0 \tag{1}$$

$$\psi' - \varphi'' + \frac{C_4}{I_z}\psi''' - \frac{\gamma\nu}{2EI_z}(I_z - I_y) = 0. \tag{2}$$

These equations are solved by integrating the first three times and inserting the result into the second and integrating two more times. The boundary conditions for a cantilever fixed at $x=0$ and free at $x=L$ are $\psi(0)=0$, $\varphi(0)=0$, $\psi'(L)=0$ and $\psi''(L)=0$. Since there are five constants of integration another condition is needed. That condition is, for the axial location at which the shear is zero, the slope ψ equals the slope of the center line deflection φ'. Thus $\psi(L)=\varphi'(L)$. The expression for the center line displacement is then

$$\varphi = \frac{\gamma A}{EI_z}\left(-\frac{x^4}{24} + \frac{Lx^3}{6} - \frac{L^2x^2}{4}\right) + \left[\frac{\gamma A}{EI_z^2}c_4 + \frac{\gamma\nu}{2EI_z}(I_z - I_y)\right] \times \left(Lx - \frac{x^2}{2}\right). \tag{3}$$

The first group of terms in Eq. (3) corresponds to the Euler-Bernoulli beam solution and the second group of terms corresponds to the shear deformation solution. This is the same expression that would result from simple beam theory with shear deformation if the shear coefficient were the expression shown in my paper as Eq. (57). If one uses the integrated average displacement, as was done by Professor Stephen, instead of the center line displacement then one gets the shear coefficient which he found, that is, my Eq. (41) and his comment Eqs. (1) and (3). Thus, even in my approach the shear coefficient is a function of the loading, and for a gravity load the resulting shear coefficient is the same as the dynamic coefficient. Similar equations to (1) and (2) above could be developed for any type of loading on a beam and thus eliminate the need for a shear coefficient entirely.

I recently presented a paper entitled "Shear Coefficients for Thin-Walled Timoshenko Beams" at the Third International Symposium on the Vibrations of Continuous Systems, July 23–27, 2001 at Jackson Lake Lodge, WY. In that paper I considered all the thin-walled cases treated by Cowper plus two additional cases. As in all other cases the "best" shear coefficient agreed with Cowper's values only for Poisson's ratio equal zero.

As to Professor Stephen's final remark, "it is noted that while the above values for the coefficient may be widely accepted as best, paradoxically Cowper's values appear to be more widely used; it is hoped the investigators will in the future make greater use of these 'best' values," I fully agree.