Scaling Properties of Type-II Intermittency

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It is verified by the numerical simulations of a two-dimensional map that the mean laminar length for type-II intermittency obeys the scaling law $\langle N \rangle \sim \varepsilon^{-1}$, or the Liapunov exponent obeys $\lambda \sim \varepsilon^{1}$. The probability of a laminar length is quite locally distributed, and such behavior of the probability is in sharp contrast with that exhibited by the conventional theory with an exponential decay. These results obtained from the present simulations are consistent with those from experiments. The significant effects of reinjection probability on the mean laminar length are stressed through the properties of the mapping function.

§ 1. Introduction

Many studies for intermittency, which is characterized by the intermittent state from regular motion to chaos seen in various phenomena of natural science, have been presented since the pioneering work by Manneville and Pomeau. They proposed three types of intermittencies I, II and III, although more three types of intermittencies other than these have been classified by the local Poincaré map. Among them, type-I intermittency has been exemplified by a Bénard experiment as a typical example, and type-III intermittency also has been observed in Bénard Convection of the well-known experiment by Dubois et al. Theoretical understandings of type-I and type-III intermittencies have been further advanced by renormalization-group theory by Hu and Rudnick, Procaccia, Schuster, and Kodama et al. etc.

The mean laminar length between neighboring bursts is known as a scaling quantity which obeys the power law $\varepsilon^{-(d-1)/2}$ for type-I intermittency, independent of reinjection probability. However, the mean laminar length for type-II and type-III intermittencies strongly depends on the distribution of reinjection points, as pointed out for type-III intermittency and hence the scaling law for the mean laminar length sensitively depends on it. For the case of a uniform reinjection probability, type-III intermittency obeys the scaling law $\langle N \rangle \sim \varepsilon^{-(d-2)/(d-1)}$. The distribution of reinjection points is considered to significantly affect the mean laminar length for type-II intermittency because of two-dimensional phase space.

Type-II intermittency has the clear-cut features that the Poincaré plot exhibits a spiraling behavior and an inverse Hopf bifurcation occurs, and its prototype map is given by

\begin{align}
    r & \mapsto (1 + \varepsilon) r + r^2 , \\
    \theta & \mapsto \theta + \Omega .
\end{align}
The length of a laminar from an initial point \( r_0 \) to a final point \( R_l \) can be expressed as

\[
N = \epsilon^{-1} z_{-1} \ln \left( \frac{1 + \epsilon r_0^{1-z}}{1 + \epsilon R_l^{1-z}} \right)
\]  

through continuous approximation. The mean laminar length \( \langle N \rangle \) taken over the whole reinjection points \( r_0 \) is approximated by

\[
\langle N \rangle = \int_{R_l}^{R_2} N \tilde{P}(r_0) r_0 \, dr_0
\]

\[
= \int_{N_1}^{N_2} N P(N) \, dN
\]

under the assumption of the rotational symmetry, where \( \tilde{P}(r_0) \) and \( P(N) \) denote the probability of reinjection points and of the laminar length, respectively. These are related by

\[
P(N) = \tilde{P}(r_0) \left| \frac{dr_0}{dN} \right| = \tilde{P}(r_0)(1 + \epsilon R_l^{1-z})e^{(z-1)\epsilon N - 1}z^{(1+z)/(1-z)} e^{(1+z)/(1-z)},
\]

using Eq. (3). Therefore, \( \epsilon \)-dependence of \( \langle N \rangle \) is generally different from that of \( N \).

For the uniform distribution of reinjection points and \( R_l = 0 \) the scaling relation for the mean laminar length is given by

\[
\langle N \rangle \propto \epsilon^{-(z-3)/(z-1)}, \quad (z > 3)
\]

\[
\langle N \rangle \propto \ln(1/\epsilon), \quad (z = 3)
\]

\[
\langle N \rangle \propto \epsilon^0, \quad (1 < z < 3)
\]

for infinitesimal \( \epsilon \). The scaling law \( \langle N \rangle \propto \ln(1/\epsilon) \) is the result obtained by Pomeau and Manneville.\(^2\) This result, however, is in disagreement with the experimental result in the coherence collapse of a semiconductor laser,\(^1\)\(^1\)\(^,\)\(^1\)\(^2\) which exhibits \( \langle N \rangle \propto \epsilon^{-1} \). One of the origins of this disagreement is thought to come from the distribution of reinjection points.

Richetti et al.\(^1\)\(^3\)\(^,\)\(^1\)\(^4\) carried out numerical simulations for the ordinary differential equation (ODE) system which describes a periodically driven non-linear oscillator, and found that type-II intermittent chaos appears and reinjection points are one-dimensionally distributed. On the basis of these results, Argoul et al.\(^1\)\(^4\) found \( \langle N \rangle \propto \epsilon^{-1/2} \) for \( \epsilon \ll R_l^2 \) and \( \langle N \rangle \propto \ln(1/\epsilon) / \epsilon \) for \( \epsilon \gg R_l^2 \). Experimental observation for type-II intermittency is also exemplified for a coupled non-linear oscillator with a \( p-n \) junction diode.\(^1\)\(^5\)

The aim of this paper is to derive the scaling law for the mean laminar length by the numerical simulation for a complex logarithmic map which exhibits type-II intermittency. We use no assumption for a reinjection mechanism or reinjection probability. The reason why the reinjection points one-dimensionally distribute will be also touched upon. In \$ 2 \) the bifurcation diagram of this map is described and the
transitive behaviors of iterated points from chaos to the period-2 region are exemplified as a representative example of type-II intermittency. In § 3 the scaling law $\langle N \rangle \propto \epsilon^{-1}$ is numerically obtained. Numerical simulations to examine the relation between the laminar length and the position of the reinjection points, the approximate return map and the probability distribution of a laminar length are carried out in order to assure this scaling law. The Liapunov exponent $\lambda$ is also evaluated, and the scaling law $\lambda \propto \epsilon^1$ is ascertained near the bifurcation point. Section 4 gives discussion and a brief summary.

§ 2. Complex logarithmic map

We do not know any dynamical systems generating type-II intermittency in which the reinjection process is included such as in a tent map or a logistic map in codimension one system. The prototype map (1) and (2) possesses no reinjection mechanism in the map either. Therefore, for the case of a map like this, the artificial assumption is necessitated for a reinjection mechanism or for the probability distribution of reinjection points in order to obtain the scaling law for the mean laminar length.

We have clarified$^{16}$ that the logarithmic map $x \mapsto \ln(a|x|)$ exhibits the type-I and type-III intermittent chaos near $a=e$ and $a=e^{-1}$, respectively. Furthermore, the complex map defined by

$$Z \mapsto f(Z) = \ln Z + C,$$  \hfill (10)

where $C$ is a complex control parameter, reveals the characteristic properties: Hopf bifurcation, period-adding rule, Devil's staircase, and scaling properties common to complex maps.$^{17,18}$ We have recently found that chaos appears for

$$-\pi/2 \leq \arg(Z) < \pi/2,$$ \hfill (11)

when we define $\ln Z = \ln |Z| + i\arg(Z)$, while there exists no chaos for $\alpha \leq \arg(Z) < \alpha + 2\pi$ with $\alpha$ arbitrary.$^{19}$ This results from the fact that the mapping function is double-valued for the definition with a range $\pi$ of $\arg(Z)$ and then the map becomes non-invertible, while it is single-valued for that with a range $2\pi$. The bifurcation diagram of the complex logarithmic map defined by (10) and (11) is shown in Fig. 1, and chaos and regions with $2k+1$ periodic sequence can be found. The dirty dust indicates the existence of the region with large period. Chaos appears also on the
line segment $-1 < C_x < 1$, which corresponds to the one-dimensional logarithmic map mentioned above, and type-I and type-III intermittencies occur near the onset points $C_x = 1$ and $C_x = -1$, respectively. 16) Inverse Hopf bifurcation (type-II intermittency) occurs at the critical point $P$ with the value $C = (-0.65, -1.179397 \ldots)$, where the bifurcation diagram changes from chaos to period 2 when we decrease $C_y$, keeping $C_x = -0.65$ constant (we will always use this value for $C_x$ hereafter in this paper).

Let us examine how the iterated points behave when we vary $C_y$ along the arrow across the point $P$ in Fig. 1. We can determine the transitive behavior from chaos for $C_y = -1.14$ (a) to systematic periodic region for $C_y = -1.19$ (c) in Fig. 2. Small vacant spaces around the unstable fixed points A and B in Fig. 2(b) are brought about by a repulsive effect due to the instability, and the iterative points occupy the region surrounded by the vacant spaces. Iterative points proceed to trace alternately near A and B as whirling outward.

The burst in the present iteration process is brought about by the points which are swelled out in the region (denoted by a) given by $0 > X$ (where $Z = X + iY$) near A, because $\arctan(Y/X)$ which is an imaginary part of $\ln(Z)$ discontinuously changes by...
π rad at $X=0$. An iterative point is reinjected as a new laminar state somewhere near the center $B$ via the regions $b$, $c$ and $d$. In Fig. 2 (c) the iterated points describe the clear spiraling traces toward the stable periodic points $A$ and $B$.

On taking every fourth step of iteration process for $C_y = -1.178$ the line connecting the iterated points exhibits behavior as shown in Fig. 3, like a trajectory segment on the center manifold for the Lorenz system, although it is the continuous dynamical system of three-dimensional phase space. Every fourth iterated point goes out describing the spiraling trajectory from the neighborhood of the point $A$ or $B$, in contrast with the case $C_y = -1.19$ in Fig. 2(c) converging to the stable fixed points, and such behavior is interrupted at the instant when an iterated point is injected in the region $a$ described in Fig. 2(b).

Next, we show the time-sequence of $|Z_n|$ for $C_y = -1.17852$. This consists of 62 sequences corresponding to 31 quasi-periodic points around the center $A$ and $B$, respectively, as seen from Figs. 3 and 4, and it is interrupted by three burst points. So, for the present parameter $C$ a laminar length $N$ is defined by the iterated number between neighbouring bursts divided by 62.

§ 3. Scaling law for the mean laminar length

A laminar length is known to be scaled by the control parameter $\varepsilon$ which vanishes at the onset point of chaos. In the present map the small parameter $\varepsilon$ is related to the parameter $C_y$ by

$$\varepsilon = |F'(Z^*)| - 1$$
Fig. 5. (a) The mean laminar length \( \langle N \rangle \) and (b) the iteration length \( n \) versus \( \varepsilon \). \( n \) expresses the iterated number which takes to reach the stable periodic point from the position apart by a finite distance. The slope of solid lines in the log-log plot has been determined by the least squares method fittings.

Fig. 6. The distribution of 230 reinjection points for \( \varepsilon = 0.001 \). The iterated points are inevitably first reinjected in the region of \( B \) and then reinjected in that of \( A \) in the following step. It should be noted that the reinjection points are not uniformly distributed but rather are one-dimensionally distributed.

\[
= \frac{1}{|Z_A Z_B|} - 1, \quad (12)
\]

where \( Z^* \) denotes the unstable periodic point \( Z_A \) or \( Z_B \) which is given by the solution of \( Z = F(Z) = f(f(Z)) \). For the present case \( C_x = -0.65 \), we can numerically obtain the approximate expression \( \Delta C_y = C_y + 1.179397 \cdots \approx 0.8 \varepsilon \). The value \( C_y = -1.17852 \) mentioned above corresponds to \( \varepsilon = 0.0010 \). We can see in Fig. 5 that the relation between the mean laminar length \( \langle N \rangle \) and \( \varepsilon \) is numerically given by \( \langle N \rangle \propto \varepsilon^{-1.000} \). This result is in good agreement with the experimental one.\(^{11}\)

On the other hand, the iteration length \( n \) in the period-2 region which takes to reach the fixed point \( A \) or \( B \) also obeys the same power law \( n \propto |\varepsilon|^{-1.000} \), as shown in Fig. 5(b).

The exponent \(-1\) for the scaling law on the mean laminar length is expected to be closely related to the reinjection distribution. The distribution of reinjection points for \( \varepsilon = 0.001 \) is depicted in Fig. 6. Reinjection points near \( B \) are traced after four iterated steps from the region \( a \) in Fig. 2(b), and those near \( A \) follow at the next iteration. Both the reinjection points near \( A \) and \( B \) are confined in the one-dimensional region rather than the two-dimensional region, in agreement with the numerical result for a non-linear oscillator system (see Fig. 4 in Ref. 13)). Such a distribution of reinjection points appears to be more realistic rather than uniform distribution used in most theoretical assumption. The region of the distribution contracts further as \( \varepsilon \) vanishes.

The one-dimensional shape in the distribution of reinjection points is brought about due to the general property of type-II intermittency for which the iterated points
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describe the spiral orbit and due to the reinjection mechanism included in the map. The laminar state which expands outward describing the spiral orbit will be interrupted at an instant by the critical line, which corresponds to the y-axis near the unstable point A in the present map. The shape cutoff by the critical line after many iterations displays a form like a crescent, which corresponds to the region a in Fig. 2(b) in our case. The iterated points which intersect the line will be injected in the new laminar state after several burst steps. Since the Liapunov exponent is nearly zero in the intermittent chaos, the one-dimensional region which includes above-mentioned iterated points will be transferred into the reinjection region, almost maintaining the shape.

Now, let us see three numerical simulations to assure the scaling law \( \langle N \rangle \propto \varepsilon^{-1} \) clarified in Fig. 5(a). The first simulation is executed in order to examine the relation between the reinjection points and the laminar length. The laminar length for \( \varepsilon = 10^{-3}, 10^{-4}, 10^{-5} \), as shown in Fig. 7, is about 10 times apart from one another for the various values of \( r_0 = |Z_0 - Z_\lambda| \) of reinjection points. We can easily find the relation \( N \propto \varepsilon^{-1} \).

Judging from the \( r_0 \)-dependence of the laminar length and the density of reinjection points in Fig. 7 it is plausible that the scaling law \( \langle N \rangle \propto \varepsilon^{-1} \) holds. Here, we may add further that \( Z^* \) in Eq. (12) depends little on \( \varepsilon \). We can also suppose from the distribution on similar line

![Fig. 7. The relation between the laminar length and the distance from the unstable point \( Z_\lambda \) to the reinjection point \( Z_0 \). A relation \( N \propto \varepsilon^{-1} \) can be readily found.](https://academic.oup.com/ptp/article-abstract/96/1/1/1852673/3)

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segments in Fig. 7 that the points are reinjected within a similar contour in the two-dimensional phase space and that the contour contracts as \( \varepsilon \) diminishes.

The second simulation is to examine the approximate behavior of the present map. On turning our attention to one of 62 quasi-periodic sequences mentioned in the preceding section, the behavior of its return map near the point \( Z^* \) will yield interesting information about a laminar length. On inspecting the time-sequence of \( r_n = |Z_n - Z^*| \) for several sequences among them the coefficients of the mapping function are provided as listed in Table I. Numerical results exhibit that any sequence obeys quite accurately a linear map \( r \rightarrow (1 + kr)r \) with a constant \( k \) depending on sequence. Since this holds in much the same way also for the other sequences the present map just above the bifurcation point \( P \) approximately yields the relation \( N \propto \varepsilon^{-1}\ln r_0 \) through the continuous transformation of the differential equation \( dr/dN = ker \).

We have carried out another numerical simulation to support this result.\(^{21}\)

The third simulation is to investigate the probability distribution \( P(N) \) for the laminar length \( N \). We see from Fig. 8 that the probability distribution \( P(N) \) is restricted in a narrow range of the laminar length \( N \) of which the center

![Fig. 8. Probability distribution \( P(N) \) of the laminar length \( N \). Here, \( P(N) \) satisfies the condition \( \sum_i P(N_i) = 1 \), where \( i \)-summation is taken over the numerical data denoted by dots. Numerical analyses have been carried out over the iteration process of \( 9 \times 10^7 \), and the solid lines denote spline curves connecting dots.](image)

![Fig. 9. (a) Liapunov exponent \( \lambda \) versus \( C_y \). (b) Scaling law for the Liapunov exponent \( \lambda \) near the bifurcation point \( P \) in (a). Numerical analyses have been carried out over \( 2 \times 10^7 \) iterations and the slope of a solid line in the log-log plot has been determined by the least squares method fittings.](image)
situates around $N$ proportional to $\varepsilon^{-1}$. Hence, Fig. 8 supports the relation $\langle N \rangle \propto \varepsilon^{-1}$ from Eq. (5). The localization of the probability for smaller values of $\varepsilon$ implies that the iteration is more densely repeated toward $r$-direction as $\varepsilon$ decreases, judging from the fact that the iterated points are confined in a definite range as shown in Fig. 2(b). The probability distribution of the laminar length is quite different from the expression $P(N) \sim \varepsilon^{2N}e^{2\pi N(\varepsilon^{2\pi N}-1)^{-2}}$ which was obtained for the prototype map on the assumption of the uniform distribution of reinjection points. We find from these simulations that the laminar length is not determined by the distance from the unstable periodic point to the reinjection point but almost by $\varepsilon$; that is, the increase of the laminar length due to the decrease of $\varepsilon$ is brought about by the high density population of iterated points in proportion to $\varepsilon^{-1}$ in the $r$-direction.

The Liapunov exponent $\lambda$ for $C_x = -0.65$ was numerically obtained as a function of $C_y$ in Fig. 9(a), and a power law is clearly observed as shown in Fig. 9(b) when the Liapunov exponent just within chaos near the bifurcation point $P$ in (a) is magnified. This result suggest that the scaling law $\lambda \propto \varepsilon^1$ holds, and it indicates that it is consistent with the relation $\lambda \propto \langle N \rangle^{-1}$. On the other hand, the Liapunov exponent just within the period-2 region in Fig. 8(a) reveals analytical results $\lambda = \ln|F'(Z_\lambda)|/2 \sim -\varepsilon/2$.

§ 4. Discussion

We have obtained the scaling law $\langle N \rangle \propto \varepsilon^{-1}$ through the numerical simulation of the complex logarithmic map, and reinjection points distribute one-dimensionally, which is consistent with the numerical results in ODE system by Argoul et al. Although the scaling law is not the same as the one found by them, some experimental results support $\langle N \rangle \propto \varepsilon^{-1}$.

The logarithmic map $Z \mapsto \ln Z + C$ restricted by $-\pi/2 \leq \arg(Z) < \pi/2$ exhibits clear-cut Type-II intermittency for $C_x = -0.65$. For $C_x \neq -0.65$ we will also be able to expect the emergence of a similar type-II intermittency from the bifurcation diagram in Fig. 1. However, we could not find type-II intermittency near the boundary between the fixed point region and chaos. This is due to the fact that the existence of many fine islands with large period, which distribute like a dust in Fig. 1, disturbs the observation of type-II intermittency.

The behavior of iterated points for the present parameter $C$ is not always simple period-2 pattern. When we expand the map $Z \mapsto f(f(Z))$ about the unstable period-2 point $Z^*$ we can easily transform it into the map with the form given by Eq. (1) after taking the average over the angle $2\pi$ (refer to p. 259 of Ref. 22). The transformation like this, however, is not appropriate as far as the branch is restricted by $-\pi/2 \leq \arg(Z) < \pi/2$, because the mapping function $f(Z)$ includes the discontinuous point $X = 0$. The numerical simulation actually indicates that the time-sequence generated from every 62-th iterated point of the map may be well approximated by $r \mapsto (1 + ke)r$, where $r$ almost satisfies $r < 1$. The non-linear term may appear in much higher order of $r$.

The reinjection mechanism is also an important dynamical process. Hence, if the dynamical equation in which it is included, whether it is ordinary differential
equation or difference equation, can be used as a model of intermittency and it describes the physical situation well, the results obtained by the numerical simulation for such an equation are preferable to those obtained through unrealistic assumption for the distribution of reinjection points. Although the scaling law $\langle N \rangle \propto \varepsilon^{-1}$ which was derived through the numerical simulation is quite simple, the dynamics which the complex logarithmic map exhibits is rather complicated.

The one-dimensional distribution of reinjection points is associated with the mechanism in which burst states occur, and the burst states are brought about by the restriction $-\pi/2 < \arg(Z) < \pi/2$ for the map. We may note only the contour of iterated points around $A$ for burst states; a laminar state ceases to exist at the instant when the contour expanding out of $A$ in the iteration process touches on the wall at $X=0$. The shape which the spiraling orbits are cut off by the $y$-axis looks like a crescent. Hence, it may be a general characteristic that reinjection points are one-dimensionally distributed.

Let us consider the reason why the reinjection region in Fig. 6 does not approach the unstable fixed point $A$ or $B$ even in the limit $\varepsilon \to 0$, although this point turns into the stable fixed point for $\varepsilon < 0$. The eigenvalue of this map is given by $\lambda = |F'(Z^*)| = 1 + \varepsilon > 1$ for $\varepsilon > 0$. Hence, the iterative points go out but they are inevitably restored near the unstable fixed point due to the discontinuity of the map at $X=0$. Thus the iterative points continue an intermittent motion. On the other hand, for $\varepsilon < 0$, the iterative points always tend to the fixed point because $\lambda < 1$, and discontinuity of the map at $X=0$ insignificantly affects the centripetal motion. For the type-II intermittency in the difference system including the reinjection mechanism there is no necessity for the reinjection point to approach the unstable fixed point in the limit $\varepsilon \to 0$.

The existence of 31 quasi-periodic sequences around $A$ and $B$ which correspond to the spiraling trajectories in the continuous dynamical system exhibits very characteristic features of type-II intermittency. There exists no physical meaning in a complex logarithmic map, as there is in a familiar logistic map and a tent map. But its iterates exhibit physically interesting structures as mentioned above and reproduce the scaling law $\langle N \rangle \propto \varepsilon^{-1}$ which has been experimentally observed, but has not been numerically derived by any other maps as yet. This is the reason why we have used a complex logarithmic map. However, there remains the problem whether it is true or not that low order non-linear terms are removed through the quasi-periodic behavior in type-II intermittent chaos, and the present results should be ascertained by maps other than the complex logarithmic map.

References

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