An experiment to invert *Seasat* altimetry for the Mediterranean and Black Sea mean surfaces

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**SUMMARY**

The high accuracy mapping of the mean sea surface (MSS) from satellite altimetry requires efficiency, flexibility and mathematical transparency from the data analysis. This paper argues that so does the least squares inverse method. Prior to the data inversion, models of covariance functions are constructed for the expected residual mean sea surface relative to a starting map and for the various error sources in the data including the radial orbit errors, instrument noise and sea height variability. The unique optimum solution is then restored from the data in a single step analysis, with formal error estimates and, possibly, local *a posteriori* covariances ('resolving kernels'). Numerical experiments are performed to obtain the Mediterranean and Black Sea mean surfaces by an inversion of the *Seasat* altimeter data.

In our present computational environment, the constraint raised by the cost of a run leads us to degrade the rigorous data analysis strategy. Owing to these limitations, and to the assumed covariance models and data coverage, the MSS is mapped with an accuracy of ~12 cm for wavelengths greater than 335 km. The supplementary error related to shorter scales is ~70 cm. The inverse MSS solution is compared with another model computed at the Bureau Gravimétrique International in a classical way (crossing arc analysis plus data filtering) from *Geos3* and *Seasat*. Then, neglecting the dynamic heights of the sea circulation, we invert the *Seasat* data set to map the equivalent free air gravity anomalies. The cross-covariance used is consistently derived from the *a priori* power spectrum of the MSS. The gravity anomalies are recovered with an accuracy of 1 mGal. Moreover, a large error of ~25 mGal is expected from the smoothing of the shorter scales. An external comparison made with the gravimetric map of Torge, Weber & Wenzel (1983) broadly agrees with our formal error estimates. Several prospective specifications concerning the covariance models and the numerical procedure are drawn from these preliminary experiments and comparisons.

**Key words:** Black Sea, inverse method, mean sea surface, Mediterranean Sea, satellite altimetry

**1 INTRODUCTION**

As a subject for space geodesy, the mean sea surface (MSS) does not fit a simple geometrical shape. Even with wavelengths shorter than 1000 km troughs and bumps are over 2.3 m rms. Only satellite altimetry provides sufficient data sampling and accuracy to map the absolute heights of the MSS on a basin scale. Considered as improved determinations of the marine geoid, these heights, or their equivalent gravity anomalies, are currently included in an issue of earth gravity field models (Rapp 1986a). The implicit approximation, and the subsequent earth models, are inadequate for oceanography as the departure between the MSS and the geoid stems from the vertically integrated general circulation. Because the dynamic heights of the circulation do not generate satellite orbit perturbations, its separation from the geoid and the simultaneous improvements of both are formally possible from altimetry alone (Wagner 1986). This also can be made efficiently with complementary data (Wunsch & Gaposchkin 1980). Here we restrict this general problem to a few basic questions: how precisely can we map the absolute MSS from satellite altimetry and by what procedure? The answers are not obvious. First, the radial orbit errors are larger than the desired MSS model accuracy (and even more precise sea surface maps will be required in oceanography from improved satellite missions like *Topex/Poseidon*); second, the geophysical corrections to the measurements cast non-negligible residuals with complex space and time variabilities.

Many regional or global models of the MSS have already been obtained from (combined) sets of *Geos3* and *Seasat* data (e.g. Bernard *et al.* 1983; Marsh *et al.* 1984, 1986). In a first, and independent step, a crossing arc analysis is performed to minimize radial orbit errors by polynomial fitting. Then the MSS heights are computed on a grid with a
classical interpolation scheme. This method is economical but provides no error bounds on the solution. The overall level of the mapped MSS can also be biased and tilted when a few altimeter profiles ('master arcs') are arbitrarily selected and fixed in order to stabilize the crossing arc analysis. Linear optimum estimators—Kalman predictors, collocation—are sometimes applied to the gridding step with a posteriori error analyses (e.g. Balmino et al. 1979; Rapp 1986b). But as long as the crossing arc analysis, using polynomial or even Fourier type basis functions, is performed in a previous and independent adjustment, optimality of the height predictions breaks down. In particular, the radial orbit errors are correlated in the along and cross-track directions (Anderle & Hoskin 1977) and geographically so that the reduction of the altimeter crossover differences releases a significant fraction of orbit errors in the MSS solution (Tapley & Rosborough 1985; Mazzega 1986).

These orbit error properties and the inherent difficulties are naturally taken into account in the least squares inversion described in this paper. This method was first applied by Wunsch & Zlotnicki (1984) to simulated satellite tracks for a sensitivity study. Here we analyse a real data set and slightly generalize their approach: introducing a ‘first order’ solution, the inverse problem is translated to the restitution of a residual signal of the MSS with a shorter correlation length so that the separation from the radial orbit error is made easier. The total inverse optimally combines all the available information (Tarantola & Valette 1982): the altimeter data themselves, the starting map and the a priori covariance functions for the residual MSS heights, the radial orbit errors and the other error sources. The complete three months of *Seasat* data over the Mediterranean and Black Seas are inverted to directly restore point values of the MSS heights and the equivalent free air gravity anomalies on a 6' × 6' grid.

In the following section, the data editing and the starting maps (e.g. the GEML2 model complete up to degree and order 20) are described. In Section 3, we discuss the choice of the various covariance functions and try to give an insight of their geometrical properties for the space–time sampling of the sea by the orbiting altimeter. Section 4 is devoted to the reconstruction of the inverse solutions for the MSS and the gravity anomalies from the previous information. The main numerical difficulties are also tackled. Our nominal solutions with their formal errors are presented in Section 5 and compared with maps obtained by other authors. Finally the overall method is discussed and further improvements sketched (Section 6). One should immediately note that if this MSS map is optimum with regard to the assumed covariances, it is not the best possible; because the a priori statistics of the residual MSS, built globally, can be refined, the present solution must not be considered as definitive. Moreover the limited power of our computer compelled us to degrade drastically the inversions in this numerical experiment.

## 2 ALTIMETER DATA PROCESSING

The altimeter data base is from the JPL-distributed Geophysical Data Record (GDR; Lorell, Parke & Scott 1980). We use the complete *Seasat* coverage of the Mediterranean and Black Seas as depicted in Fig. 1. The profiles were reproduced visually and the outlines deleted by hand. Then averages over five consecutive 1 s data were performed (for a ~35 km spatial resolution) to lower point-to-point altimeter noise. This is justified as the along-track resolution capability of *Seasat* data is of the same order (Marks & Sailor 1986). We applied the corrections for the atmospheric propagation effects and earth tides given on the GDR. Considering the actual uncertainty on the right way to correct the data for electromagnetic bias (e.g. Barrick & Lipa 1985) and the relative quietness of the sea state for summer 1978 (see the $H_{1/2}$ histogram in Fig. 2A), we disregard this time dependent effect. We adopt the same attitude for the atmospheric pressure at sea-level which only exceptionally shows 10 mbar anomalies (or 10 cm sea height variability in the inverse barometer approximation; see Fig. 2B) in our space–time window.

The *Seasat* ephemerides were generated at the Goddard Space Flight Center with the PGS-S4 earth gravity field model (Lerch et al. 1982), and substituted for the GDR.

![Figure 1. *Seasat* altimeter data coverage of the Mediterranean and Black Seas after the editing process. Along the profiles, each datum represents a ~35 km instantaneous sea height average.](https://academic.oup.com/gji/article-abstract/96/2/259/610582/figure1)
An inverse MSS from satellite altimetry

The nominal radial accuracy of \(~70\) cm will be slightly underestimated in order to prevent contradictory observational constraints: during the data inversion, we specify a standard deviation of 1 m for the radial orbit error. The data analysis rigour is not strictly preserved because Seasat altimetry itself is mixed in the PGS-S4 geopotential model used to compute the Lerch et al.’s reference orbit.

This partial rigour is then maintained by choosing an altimetry-free geoid for an \textit{a priori} MSS map. The GEML2 earth potential benefits from several years of laser tracking on the high altitude Lageos satellite (Lerch et al. 1985a). At the time of our computations this model provided the best description of the large scale potential and its error spectrum is well documented (see the next section). So we selected for the \textit{a priori} map the GEML2 geoid complete up to degree and order 20 with a correction for the Honkasalo permanent deformation (Moritz 1979) (such a tide, in equilibrium, does not generate a perturbing potential so that it is not contained in a gravity model derived from satellite dynamics). Like the altimeter data, the GEML2 \textit{a priori} map is expressed in geodetic coordinates with ellipsoidal corrections (Moritz 1980) to the GRS 80 ellipsoid (see Fig. 3). The corresponding \textit{a priori} map for the gravity anomalies is also constructed from the GEML2 model.

3 CHOICE OF COVARIANCE FUNCTIONS

The altimeter data must improve our starting map, the GEML2 (20,20) geoid, (Fig. 3) or gravity anomalies. Further information is included in this process, namely the data errors and expected signal covariance functions. They carry in space or time the information from one point to another. Furthermore, they ensure the stability of the computation and uniqueness of the solution.

The \textit{a priori} covariance function of the MSS heights is deduced from the geoid plus permanent dynamic height power spectra. In a spherical approximation, the degree

Figure 2. (A) Histogram of the \(H_{15}\) values for the whole data set of Fig. 1 (0.5 m classes in abscissa). (B) Histogram of the atmospheric pressure at sea level for the whole data set (1 mbar classes in abscissa).

Figure 3. The \textit{a priori} mean sea level map is obtained from the GEML2 gravity field model complete up to degree and order 20. The reference ellipsoid is the GRS 80.
The free air gravity anomalies $\Delta g$ are related to the anomalous earth potential (the fundamental equation of physical geodesy) or to the geoid by a linear differential operator (see the Appendix). The residual MSS covariance $C_{hh}$ consistently propagates into the residual gravity anomalies cross-covariance $C_{h\Delta g}(\psi)$ [respectively autocovariance $C_{\Delta g\Delta g}(\psi)$] by applying once (respectively twice) this operator (see Moritz 1980, chapter 15). The expected variance of the residual $\Delta g$ is (17 mGal)$^2$ including the errors of the GEML2 model and the anomalies with wavelengths greater than 335 km ($l = 120$). Note that the assumption that the dynamic heights can be neglected is made only to deduce the gravity anomalies (and not the MSS) from altimetry.

The choice of a covariance for the radial orbit errors is less obvious because an exhaustive study of their statistical behaviour is lacking. For arcs of 6 days, the orbit error is dominated by frequencies around one cycle per revolution due mainly to errors in the gravity field and erroneous initial elements for the orbit integration. Non-linearities and non-gravitational forces acting on the spacecraft complicate this heuristic representation. Nevertheless, following Wunsch & Zlotnicki (1984), we write the orbit error autocovariance $D_1$ as a cosine with an exponential damping, e.g.:

$$D_1(\Delta t) = \sigma_{\text{orb}}^2 \cos \left( \frac{2\pi \Delta t}{T} \right) \exp \left( -\frac{\Delta t^2}{\Delta T^2} \right),$$

where $\Delta t$ is the time lag between two data points, $\sigma_{\text{orb}}^2$ the orbit error variance here set to (1 m)$^2$, $T$ is the Seasat orbital period and $\Delta T$ a characteristic decorrelation time. We have no clear evidence to fix this last parameter. But as we shall see, owing to our $a \ posteriori$ analysis this indetermination has actually (with Seasat) no dramatic effect on the solution. We set to zero the $D_1$ covariance for pair of points belonging to independent arcs of ephemeres as summarized in Lorell et al. (1980; Table 5-3).

Figure 4. (A) Degree root mean square of the geoid (metres) as a function of the degree $l$. Curve ‘T’ plots the GEML2 spectrum complete up to (20, 20); Curve ‘R’ plots the error spectrum of GEML2: both curves are completed up to $l = 120$ by an empirical law. The total variances are (30.4 m)$^2$ for $T$ and (2.3 m)$^2$ for $R$. (B) Auto-correlation functions for the total $T$ and residual $R$ geoid heights corresponding to the above spectra. The use of the GEML2 geoid $a \ priori$ map for the inversion reduces the e-folding correlation length $\psi$, from $\sim 30^\circ$ (curve T) down to $\sim 3^\circ$ (curve R).
The correlation level of the radial orbit error between these two consecutive tracks is 96 per cent. On the other hand, the crossover points at these mid-latitudes are overflown with time lags larger than one revolution period. Thus the orbit errors of the ascending and descending arcs at crossovers are almost decorrelated. For example, during the ‘Bermuda’ orbit, the crossovers situated at 36.8°N and 43.2°N are sampled with 835 min and 580 min time lags between the ascending and descending passes. The corresponding orbit error correlations are down to 7 and 27 percent, respectively. We see that the in-time description of the orbit error is non-stationary: we set to zero the auto-correlation between two points belonging to independent arcs (in this particular sketch, for a time lag of ~465 min).

No other covariances were added to complete the error budget of altimetry. But any further information on the statistical properties of the residual of geophysical corrections applied to the data (e.g. tropospheric or ionospheric corrections) could be easily implemented in the same way in the inverse problem.

4 THE MSS INVERSE MODEL

We here follow the total inverse method of Tarantola & Valette (1982) but our problem is simply linear. The complete equations, for the MSS and gravity anomalies \(\Delta g\), with explicit operators are given in Appendix. The optimal estimate of the MSS height \(h(r)\) at any geographical point \(r(\varphi, \lambda)\) (\(\varphi\) and \(\lambda\) are the geographical coordinates) is obtained from the starting GEML2 geoid \(h_0(r)\) plus a linear combination of the reduced data by:

\[
\hat{h}(r) = h_0(r) + \sum_{i=1}^{N} \alpha_i(r) [d(r', t') - h_0(r')],
\]

with the coefficients

\[
\alpha_i(r) = \sum_{i=1}^{N} C_{hh}(\psi') [S^{-1}]^\psi
\]

the summations being performed over the number \(N\) of data. The argument \(\psi'\) is the spherical distance between the \(i\)th data \(d(r', t')\) and the prediction point. The \((i, j)\) elements of the symmetric positive definite matrix \(S\) are computed from:

\[
S^{ij} = C_{hh}(\psi'^o) + D(\psi'^o, \Delta t^o),
\]

where \(\psi'^o\) and \(\Delta t^o\) are the distance and time lag between the \(i\)th and \(j\)th data points. \(D\) is the function of the data error covariance (see equation 6) and \(C_{hh}\) the residual MSS covariance function as constructed in Section 3. The \(a\ priori\) covariance \(C_{hh}\) using Legendre polynomials is tabulated once and interpolated by splines during the run.

The \(a\ posteriori\) signal covariance between \(r\) and \(r'\)

\[
\hat{C}_{hh}(r, r') = C_{hh}(\psi) - \sum_{i=1}^{N} \alpha_i(r) C_{hh}(\psi'), \quad \overline{\psi'} = \psi
\]

is determined by the data sampling, not by their ‘physical’ values (altimeter data \(d\)). The square roots of the \(C_{hh}\) diagonal elements (when \(r' = r\)) provide the error on the corresponding individual points of the solution \(\hat{h}(r)\).

The inverse map of the free air gravity anomalies \(\Delta g(r)\) is constructed with the auto-covariance \(C_{\Delta g \Delta g}\) and cross-covariance \(C_{h \Delta g}\) in the same way (see the Appendix), e.g.

\[
\Delta g = \Delta g_0 + C_{h \Delta g}[S^{-1}](d - h_0).
\]
where \( \Delta g_0 \) is the GEML2 (20,20) reference map. The corresponding \textit{a posteriori} covariance is then:

\[
\hat{\Delta} = C_{\Delta} - C_{\Delta}[S^{-1}]C_{\Delta}
\]  

(11)

The same square matrix \( S \) enters in the reconstruction of the MSS (equations 7–9) and of the gravity anomalies (equations 10–11).

It must be realized that contrary to the classical optimum estimators which would use only local data points to calculate \( h(r) \), the present method uses all the data points. Due to the orbit error covariance, data geographically far from position \( r \) but on a consecutive orbit do contribute to the estimation of \( h(r) \). As a consequence \( S \) cannot be reduced to a small matrix, and a high computational cost results from the estimation of \( S^{-1} \). The limited capacity of the computer at present necessitates storage of the full matrix on tape. \( S^{-1} \) is effectively computed with an updated Cholesky method (Balmino & Moynot 1977). To lower drastically the processing time, the entire domain (Fig. 1) is partitioned into the western basin, two parts in the eastern Mediterranean basin and the Black Sea, with overlaps of \( \pm 4^\circ \). Actually, such a cutting up strongly lowers the potential of our method: e.g. \textit{Seasat} overflies the Black Sea only 101 min after passing off Corsica, but these two tracks with highly correlated orbit errors (see Fig. 5) are treated separately.

Before analysing the inverse MSS accuracy, we discuss its sensitivity to a few parameters. The influence of the characteristic time \( \Delta T \) of the orbit error decorrelation has been tested over the solution for the western basin. Fig. 6(A) shows the information extracted from the data set, a map of \( (h - h_0) \), with \( \Delta T \) fixed to five orbit periods; then we

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**Figure 6.** (A) Western basin map of the residual mean sea surface heights (m) extracted from the Seasat data set by the inversion procedure. The reference surface is the \textit{a priori} GEML2 map of Fig. 3. The decorrelation time of the radial orbit error was set to five revolution periods (\( \sim 505 \) min). (B) Map of the differences (cm) between the above map and an inverse solution obtained with an orbit error time decorrelation of 14 revolution periods (\( \sim 1 \) day). For that particular data set, the differences are everywhere lower than the 1 s.d. \textit{a posteriori} uncertainties (\( \sim 10-15 \) cm) except where very few data or even no data at all are available (respectively in the Alboran Sea and south of Sicilia).
have dropped $\Delta T$ to 14 orbit periods and computed a new solution. The departures from the previous map are almost everywhere lower than 10 cm (see Fig. 6B), except where the data coverage is poor (no data at all south-east of Sicilia) or when the altimeter profiles are very short as in the Alboran sea. The differences are still generally contained in the error bars of the solutions (see the next section) so that we keep a $\Delta T$ of five orbit periods. In other words, the choice of the orbit error decorrelation time $\Delta T$ is not critical, with regard to the Seasat coverage and data error budget.

We deliberately avoid conclusions which might be applied to the 'Topex/Poseidon' or 'ERSI' missions.

The sea tides can be partly aliased in the MSS (Mazzega 1986) and generate errors in the prediction. A sensitivity test has been performed in the north east Atlantic, off France and Spain, where the M2 constituent reaches 1–2 m. With the three months of Seasat data, similar inversions in a 1000 km square area with and without tidal corrections show that the errors in the MSS never exceed 10 per cent of the total tide (error $\leq 0.1$ m in that zone). This test suggests that the tidal residuals are lower than $\sim 2$ cm in the Mediterranean MSS except in a few harbours with exceptional tidal harmonics (Rickards 1985).

The inverse solutions for the MSS and gravity anomalies, computed on $6' \times 6'$ grids, are drawn in Figs 7 and 8. These maps are optimal with regard to the Seasat data set and the assumed covariance functions of the signals and errors. The smoothness of the inverse MSS map is a priori controlled by the correlation length $\psi_s$ of 3' prescribed to the expected signal. This correlation length deduced from global data is certainly too large for the Mediterranean and Black Seas; first, because we deliberately assumed for this numerical experiment no power in the MSS for wavelengths shorter than 335 km ($l > 120$) as it appears in the rms spectrum of Fig. 4(A), and second because in this area the geoid heights are statistically rougher than the global geoid. Another way

Figure 7. Inverse mean sea surface heights (m) relative to the GRS80 ellipsoid ($6' \times 6'$ spatial gridding). The correlation length of the signal extracted from the altimeter data was $\psi_s \sim 3'$. This map is a concatenation of 'independent' solutions computed over the four regions with $\sim 4'$ overlaps.

Figure 8. Inverse solution for the free air gravity anomalies. This map is consistently derived from the inverse mean sea surface map (Fig. 7).
to reduce the signal correlation length $\psi_s$ would be to choose, for the starting map $h_0$, a gravity field with a higher spatial resolution (e.g. complete up to $l = 100$) if a realistic error power spectrum was also available for such a model. But this translation would only be a formality as the new map $h_0$ would also be removed from the data before the inversion and would not physically change the properties of the MSS model. However, the danger of perturbing the residual MSS signal by the sea height mesoscale variability would increase, as the wavelength of both signals would be similar. In the present case, this could be critical as the duration of the altimeter mission was commensurable with the decorrelation time of the mesoscale variability.

5 PREDICTION ERRORS AND EXTERNAL COMPARISONS

The Seasat inverse MSS was calculated on a $6' \times 6'$ grid. An assessment of its absolute accuracy is attempted by the computation of the formal errors and by a comparison with another MSS model supplied by the Bureau Gravimetric International (BGI). A similar work made with the inverse gravity anomalies is briefly exposed at the end of the section.

Following equation (9), the entire a posteriori covariance function (for a range of distance lags) could have been determined for each prediction point. The width of the principal peak of the ‘resolving kernels’ would provide an estimate of the inverse model resolution (the $6' \times 6'$ spacing just corresponds to the gridding of the solution). In any case the model resolution is here limited by the $35$ km along-track data averaging done during the editing process (see Section 2). To save computational time, we restricted our error analysis to the estimation of the standard deviation (s.d.) of the solution along a few profiles drawn in Fig. 9(A). Starting from the GEML2 map with an homogeneous uncertainty of $2.5$ m, the a posteriori s.d. would quantify the absolute accuracy of the inverse model if the prescribed MSS spectrum of Fig. 4(A) were the real one. The estimated s.d. are homogeneous in the whole domain, even

Figure 9. (A) The formal errors of the inverse MSS model are computed along the four profiles. Even near the land, the errors are $10-12$ cm in the Mediterranean Sea (profiles AA', BB' and DD') and up to $15$ cm in the Black Sea (profile CC'). The west, central and east basin solutions (respectively WMSS, CMSS and EMSS) are connected at the lines $R_1$ and $R_2$. (B) Heights of the WMSS and CMSS along the $R_1$ line (in metres on the left y-axis). The curve of the difference (in metres on the right y-axis) are within the $+1$ s.d. error. (C) Heights of the CMSS and EMSS along the $R_2$ line (in m on the left y-axis). The curve of the difference (in m on the right y-axis) exceeds the formal errors in this area where the MSS gradient is strong.
for the near coast sides of the profiles of Fig. 9(A); the formal errors are between 10 and 12 cm in the Mediterranean Sea (profiles AA', BB' and DD') and slightly higher in the Black Sea where they reach 15 cm. In view of the heavy Seasat error budget and oceanic height variability variance, these errors are surprisingly low. The measurement of almost 3 months, including two orbits with respectively 17 and 3 day repeat periods and crossing tracks, provides an efficient observation set to separate the MSS from the various errors. The method would be certainly even more powerful if consecutive tracks were analysed together. Note that we do not observe complex structures of the errors associated with the data gaps as those depicted by Wunsch & Zlotnicki (1984). A supplementary 0.70 m uncertainty, hereafter called the truncation error, corresponds to the MSS root sum square for wavelengths shorter than 335 km (estimated from the Wagner & Colombo, 1979, power law from the degree $l = 121$ up to 1000).

The inverse MSS heights of the regional solution situated on each side of the connection lines of the three subdomains of the Mediterranean Sea are drawn in Fig. 9(B and C). The different solutions seem to be slightly shifted downward from west to east. The differences are contained in the $\pm 1$ s.d. errors (see Fig. 9B) except where the MSS gradient is strong as in the south of Greece (Fig. 9C). In this case, the truncation error may be legitimately invoked to explain these height differences or, following the same idea, the data coverage becomes crucial in this kind of area.

All these error properties are again illustrated by the comparison of the inverse MSS with an independent estimate. The latter is based upon a MSS model computed at the BGI from both Geos3 and Seasat altimeter data. This MSS model is restored in two separate steps, following the usual procedure (Balmino & Rouquet, private communication): first a crossing arc analysis to minimize the radial orbit errors; second a filtering performed with a function of the type $[1 + 1/(X/0.2°)^2]$ where $X$ is the distance between the datum and the grid point, and the 0.2° a typical decorrelation length. The resulting MSS map (see Fig. 10A) is naturally less smooth than the present inverse solution (Fig. 7) obtained with a $\psi \sim 3°$ decorrelation length.

Figure 10. (A) Mean sea surface heights (m) relative to the GRS 80 ellipsoid computed at the BGI (6' x 6' spatial gridding). Both Geos3 and Seasat data are used. After the editing process, the radial orbit errors are minimized through a crossover analysis. In a second step, the MSS is restored by a spatial filtering of the data screened in a $1°$ square around each prediction point. (B) Map of the departures (m) between the above MSS supplied by the BGI and the inverse MSS of Fig. 7. The mean discrepancy increases from the western Mediterranean basin to the central basin, eastern basin and Black sea (see Table 1).
Table 1. Overview of the errors of the MSS inverse solution and statistics of the external comparison.

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<tr>
<th></th>
<th>Complete MSS inverse solution</th>
<th>Western basin solution</th>
<th>Central basin solution</th>
<th>Eastern basin solution</th>
<th>Black Sea solution</th>
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<tr>
<td>A posteriori formal</td>
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<td>10–12</td>
<td>12–15</td>
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<td>Departures mean from the</td>
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<td>96</td>
<td>57</td>
<td>80</td>
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</table>

The differences between the BGI MSS and the inverse solution (Fig. 7) are drawn in Fig. 10(B). The discrepancies between the two solutions exceed our estimation of the formal errors augmented by the truncation error. Two distinct components are observed in this discrepancy: a larger scale component and a small-scale one.

First, the difference between the two MSS is clearly increasing from west to east (Table 1), except in the Adriatic Sea where the difference remains small. We actually have no explanation for this tendency. It could result from the distinct analysis methods: a selection of master arcs is implemented in the BGI analysis for the stability of the crossovers analysis which might introduce an arbitrary large distance scale tilt in the MSS. It could also result from the use of the Geos 3 data with their specific errors (orbit errors, residuals of data corrections, etc). The compatibility between the two MSS would be better if the means of the differences were removed over each basin.

![Figure 11. MSS heights of the inverse solution (full line) and BGI solution (dashed line) along the four profiles of Fig. 8(a). The vertical lines locate the Seasat crossing tracks. The main discrepancies between the two solutions generally rise when Geos 3 samples a MSS feature that Seasat does not. Note that the inverse solution formal errors are ~10–12 cm for profiles AA', BB', DD' and ~12–15 cm along CC'; the supplementary truncation (smoothing) error is ~70 cm.](image-url)
suggesting that basin scale errors vitiate one or both solutions. Any further interpretation would be arbitrary as long as an inversion of the complete data set is not performed in a single step. Also, the comparison should be made with totally independent data such as absolute positioning of the mean sea levels from tide gauges.

The difference of the two MSS also shows a small-scale pattern. First some large differences are apparently associated with the steep geoid slope of the eastern Mediterranean subduction zone. Second, the small scale MSS features appear only in the BGI surface. This is clearly apparent in Fig. 11 where four profiles are extracted from the two solutions. The smoothness of our inverse solution could be a consequence of the prescribed MSS covariance. It also results from the Seasat undersampling, as the small scale differences mainly appear in the western basin, mainly sampled by Geos 3.

The map of free air gravity anomalies (Fig. 8) has been produced with the consistent crosscorrelation between Δg and the geoid heights. The resolution of the climatological hydrographic atlas (Levitus 1982) being too coarse to represent correctly the Mediterranean Sea dynamic topography, we presently assume that the mean sea surface coincides with the geoid. The relation between the equipotential level and the free air anomalies through a derivative operator, here directly applied to the MSS autocovariance function, implies that the gravity anomalies are more energetic at short wavelengths. As a matter of fact, the inverse Δg solution is more obviously penalized by the smoothing and Seasat undersampling. The a priori variance of the residual Δg is (17 mGal)² and the truncation error (25 mGal)², compared with the (2.4 m)² residual MSS variance and (0.7 m)² truncation error. As previously observed with the inverse MSS, the a posteriori formal errors (equation 11) of the inverse Δg are homogeneous in the whole basin and never exceed 1 mGal. The inverse solution is then compared with the gravimetric map published by Torge et al. (1983). The comparison made separately for each basin and for the whole domain are summarized in Table 2. We obtain a mean departure of 3.6 mGal with a s.d. of 26 mGal. This discrepancy can be reasonably attributed to the smoothing of the inverse solution. Even our crude treatment of the radial orbit errors seems rather satisfactory: no satellite ground tracks appear in the Δg map as frequently found with the usual analysis (e.g. the comments in Rapp 1986b).

### Table 2. Overview of the errors of the gravity anomaly inverse solution and statistics of the external comparison with the model published by Torge et al. (1983).

<table>
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<th>Complete gravity anomaly inverse solution</th>
<th>Western basin solution</th>
<th>Central basin solution</th>
<th>Eastern basin solution</th>
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</tr>
<tr>
<td>Departures mean from the</td>
<td>8.3</td>
<td>1.4</td>
<td>0.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Torge et al. rms model (mGal)</td>
<td>26.3</td>
<td>19.0</td>
<td>26.0</td>
<td>32.1</td>
<td></td>
</tr>
</tbody>
</table>

6 DISCUSSION

This first numerical experiment is a good basis to specify the necessary further improvements. Two major difficulties actually limit the capabilities of the inverse method: the choice of simplified covariance functions and the necessity to process the data in several independent sets due to the restricted computational power used in this study. Under the stationariness assumption, the MSS spectrum (including the dynamic heights and the geoid) could easily be extended to the short wavelengths with a power law, so decreasing the signal correlation length. Providing definiteness and positivity of the S matrix are verified, the equations of the inverse problem (equations 7-11) also allow relaxation of the assumptions of stationariness (e.g. by a regional scaling of the variance) and isotropy of the covariances. However, the practical estimation of more realistic covariances may be a heavy task.

A refinement of the radial orbit error covariance is a central question which should be supported by the ephemeris numerical computation itself. No real progress seems imminent. The usually published statistic of the altimeter crossover differences (even in separate domains of the world ocean) is an insufficient criterion, because (1) all the crossover discrepancy cannot be attributed to orbit errors; (2) all the radial orbit errors do not generate crossover discrepancies; (3) no information is given about the error correlations along a track and between consecutive and crossing tracks; and (4) nothing is said at all about the arc discontinuities. Even with a (nominal) radial orbit error below the 15 cm level, as projected for the Topex/Poseidon mission, this refinement of the orbit error covariance will still be required as higher accuracies will be requested for the altimeter derived products.

A means to handle smaller matrices in the inverse equations (equations 7-11) might start from a better resolving map h₀, leaving a MSS signal with shorter correlation lengths. Then only the data immediately surrounding the prediction point would have to be processed together. Remember, however, that due to the satellite sampling, the altimeter 'errors' are correlated over large distances (e.g. the orbit errors) and long time gaps (e.g. the oceanic mesoscale variability) so that the matrix to invert cannot be partitioned. On the contrary, it is desirable to perform one-step analyses over domains including at least three consecutive tracks (say ~6000 km square areas): the
larger the network, the stronger the constraints to reject the orbit errors and to reach the absolute MSS. Finally, only the data from several satellites or from a long term 'drifting' orbit supply a sufficiently dense coverage to map the MSS with a 10 cm absolute accuracy: for such an accuracy to be reached, the MSS must be observed at a half wavelength at least down to 30 km. In any case, accurate inverse computation of the MSS and its error covariance matrices require a high speed and large direct memory computer.

The inverse method constitutes a flexible tool to mix data of different natures (by the use of cross-correlations) or from other measurement techniques. In particular the combination with altimetry of tide gauges data tied to a global geodetic frame by differential doppler positioning techniques (Carter et al. 1986) is very promising as suggested by the simulation study made by Wunsch (1986). Two strategies are possible. The tide gauge mean sea level, relative to the same ellipsoid, is directly introduced in the inverse equation (7) but with its specific error covariances. This high accuracy information (say ~10 cm errors) would locally separate the MSS from the altimetric orbit errors. This separation would be propagated into the whole domain with the satellite motion. An alternative method would be to analyse the hourly measurements of the tide gauges in order to estimate the instantaneous orbit errors of the nearby altimeter tracks and to restore at the same time the MSS from the whole data set.

CONCLUSION

Estimating an unbiased MSS from altimeter data can only be made by a direct inverse calculation taking into account orbital and measurement error covariance and a priori knowledge of the expected signal. Decorrelation of the MSS from orbit errors is best achieved by estimating only the residuals of the MSS relative to a long wavelength reference geoid. We have implemented and tested this technique for recovering the MSS for the Mediterranean and Black Seas, from Seasat data. At wavelengths greater than the selected cutoff of 335 km, the a posteriori error is of the order of 12 cm. The equivalent inverse solution for the free air gravity anomalies is recovered with an error of 1 mGal.

The inverse method presented here can be greatly improved by (1) a refinement of the basic physics contained in the observation equations, in the description of the error budget, and in the various assumed statistics; and (2) an implementation of in-core algorithms on a computer specially designed to cope with 'large' problems. Several consecutive satellite tracks over a basin scale domain would constitute a framework for an efficient rejection of the radial orbit errors (regardless of their geographical correlation) and of the sea height variabilities from the inverse MSS solution. On the other side, refinement of the covariance function adapted to the particular domain and a dense data coverage would allow one to map the short scale MSS undulations (mainly associated with bathymetric features). On the basis of the present numerical experiments, we believe that the mixing in the inverse analysis of altimeter data from past and future satellite missions, in which the different error statistics are rigorously specified, will deliver MSS maps with a decimetric absolute accuracy.

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REFERENCES

APPENDIX

Let the MSS height \( h(r) \) relative to the reference ellipsoid be a continuous function of the geographic allocation \( r(\varphi, \lambda) \). An altimeter datum \( d(r, t) \) is a discrete space-time sample of the MSS:

\[
d(r, t) = G_r(r, t) h(r),
\]

with the linear operator \( G_r(r, t) \) defined by

\[
G_r(r, t) = \int_A \int_T \delta(r - r_i) \delta(t - t_i) \, dr \, dt,
\]

where \( \delta(x) \) is the Dirac function and the integrals are taken over the oceanic area \( A \) and time axis \( T \) respectively. First of all we reduce the data to a starting map \( h_0(r) \) of the MSS and intend to recover the residual MSS heights characterized by the \textit{a priori} covariance function \( C_{hh}(r, r') \). The radial orbit error, the mesoscale variability of the sea surface and the radar noise are then considered as errors in the theoretical equation (A1). Their respective \textit{a priori} covariance functions are noted \( D_1(t, t'), D_2(r, r'; t, t') \) and \( D_3(t, t') \). Based upon the least squares criterion (see Tarantola & Valette 1982) the optimal inverse solution \( \hat{h}(r) \) is obtained from the N altimeter data and from the starting map \( h_0(r) \) by:

\[
\hat{h}(r) = h_0(r) + \sum_{i=1}^{N} \alpha_i(r)[d(r_i, t_j) - h_0(r_j)]
\]

with the coefficients

\[
\alpha_i(r) = \sum_{j=1}^{N} G_i(r'', t') C_{hh}(r, r'') [S^{-1}]^{(42)}.
\]

The matrix \( S \) is given by

\[
S^{(42)} = G_i(r, t) G_j(r', t') [D_i(t, t') + D_2(r, r'; t, t') + D_3(t, t')]^{-1}.
\]

The \textit{a posteriori} covariance \( \hat{C}_{hh}(r, r') \) of the inverse solution \( \hat{h}(r) \) is

\[
\hat{C}_{hh}(r, r') = C_{hh}(r, r') - \sum_{i=1}^{N} \alpha_i(r) G_i(r'', t'') C_{hh}(r', r'').
\]

Applying the explicit expression (A2) of the operators to the covariance functions we obtain

\[
\alpha_i(r) = \sum_{j=1}^{N} C_{hh}(r, r_j) [S^{-1}]^{(43)}.
\]

\[
S^{(43)} = D_1(t_i, t_j) + D_2(r_i, r_j; t_i, t_j) + D_3(t_i, t_j) + C_{hh}(r_i, r_j).
\]

and

\[
\hat{C}_{hh}(r, r') = C_{hh}(r, r') - \sum_{j=1}^{N} \alpha_i(r) C_{hh}(r_j, r').
\]

Remembering that the \textit{a priori} covariance functions only depend on the spherical distance \( \psi = \overrightarrow{r}' \) or time lag \( \Delta t = |t - t'| \), the equations (A3) and (A7–A9) are transformed in the inverse equations of Section 4. The \textit{a posteriori} covariance \( \hat{C}_{hh}(r, r') \) is not necessarily stationary nor isotropic so that we keep the two arguments \( r \) and \( r' \) in (A9).

To compute the inverse map of the free air gravity anomalies we assume that the MSS in the Mediterranean and Black Seas is a fair approximation of the geoid. Using Bruns' formula, we express this assumption by:

\[
h(r, z)|_{z=R} \neq [T(r, z)/\gamma]|_{z=R}.
\]

where \( z \) is the radial spherical coordinate, \( R \) the earth radius, \( \gamma \) the normal gravity and \( T \) the anomalous potential. Now from the fundamental equation of physical geodesy (Moritz 1980, chapter 2), we obtain the gravity anomaly at the surface of the earth as a linear functional of \( h(r, z) \):

\[
\Delta g(r, z = R) \neq L(r, z) h(r, z)
\]

with the differential operator

\[
L(r, z) = - \frac{\partial}{\partial z} \frac{2}{\overrightarrow{r}} \left[ \frac{\partial}{\partial \overrightarrow{r}} \right]_z = R.
\]

As we presently consider the function \( h(r, z) \) to be an equipotential of \( T(r, z) \), the \textit{a priori} covariance \( C_{hh} \) depends on the geographical locations \( r \) and \( r' \) but also on the radial coordinates \( z \) and \( z' \):

\[
C_{hh}(r, r'; z, z') = \sum_{i=1}^{N} \alpha_i^2 \left( \frac{R^2}{zz'} \right)^{(47)} P(\psi, \psi = \overrightarrow{r}', \overrightarrow{r}'),
\]

where \( \alpha_i^2 \) are the \textit{a priori} degree variances of \( h(r, z = R) \).
The \textit{a priori} covariance of the MSS given in Section 3 (equation 2) is derived from (A13) in the particular case $z = z' = R$.

Applying the operator $L(r, z)$ to the inverse equation (A3), we readily obtain:

$$\Delta g(r) = \Delta g_0(r) + \sum_{j=1}^{N} \beta_j(r)[d(r_j, t_j) - h_0(r_j)]$$  \hspace{1cm} (A14)

with the coefficients

$$\beta_j = \sum_{i=1}^{N} G_i(r^*, r^*) C_{h \Delta g}(r, r^*) [S^{-1}]_{ji}$$  \hspace{1cm} (A15)

where the cross-covariance $C_{h \Delta g}(r, r^*)$ is consistently derived from (A12) and (A13) by

$$C_{h \Delta g}(r, r^*) = L(r, z) C_{hh}(r, r^*; z, z').$$  \hspace{1cm} (A16)

The \textit{a posteriori} covariance of the $\Delta g$ map is obtained in the same way:

$$\hat{C}_{\Delta g \Delta g}(r, r^*) = C_{\Delta g \Delta g}(r, r^*) - \sum_{j} \beta_j(r) G_j(r^*, r^*)$$  \hspace{1cm} (A17)

with the \textit{a priori} auto-covariance

$$C_{\Delta g \Delta g}(r, r^*) = L(r, z) C_{hh}(r, r^*; z, z') L(r^*, z').$$  \hspace{1cm} (A18)

Then, introducing in (A15) and (A17) the expression (A2) of the operator $G_i(r, t)$, we obtain easily the inverse equations for the gravity anomalies given in Section 4.