Three Forms of Boson Expansions for the $su(2)$-Spin System and Their $c$-Number Counterparts

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In the framework of the Marumori·Yamamura·Tokunaga boson expansion, three forms for the $su(2)$-spin system are presented. The first is related with the well-known Holstein·Primakoff boson expansion and the second is the expansion reinvestigated in a previous paper by the present authors. The third is, in some sense, the original part of this paper. Corresponding with three forms, $c$-number counterparts are obtained by using three forms of wave packets defined in the $su(2)$-spin space. With the help of appropriate quantization rule, they go back to the original boson expansions. Furthermore, it is discussed if the $c$-number counterparts can be regarded as the classical counterparts. Finally, the relation to the $su(1,1)$-algebraic model is discussed.

§ 1. Introduction

Boson realization of Lie algebra has played a crucial role in many-body theory. Especially, the boson expansion of the $su(2)$-spin system, which is based on the $su(2)$-algebra, enables us not only to get schematic understandings of various many-body methods but also to perform realistic analyses of various phenomena. In response to the above situation, recently, the present authors reinvestigated the boson expansion of the $su(2)$-spin system, which will be referred to as (A). The basic idea can be found in a general method for the boson expansion which has been proposed by Marumori, Yamamura (one of the present authors) and Tokunaga (MYT). The boson expansion starts in the mapping of the original Hilbert space to a certain subspace in the whole boson space. In (A), the MYT method has given us two forms of the boson expansions in the $su(2)$-spin system. The first is, needless to say, closely related with the Holstein-Primakoff (HP) expansion, which is a simple application of the Belyaev-Zelevinsky-Marshalek (BZM) method. This form helped us to investigate the relation between the original quantal system and its classical counterpart. The second is of the form in which the boson operators are arranged in the normal order product. In (A), it was stressed that this form may be suitable for the use of the boson coherent state. With the aid of this form, we can describe time-development of the $su(2)$-spin system in the frame of classical mechanics without worrying with the forbidden range of the variables. Moreover, recently, the MYT method was applied to the rederivation of the $q$-boson expansion of the algebras, the $su_q(2)$ and the $su_q(1,1)$. On the other hand, the time-dependent variational approach has also played a crucial role in many-body theory. One of the typical examples can be found in the time-dependent Hartree-Fock (TDHF) theory in canonical form.
a wave packet which is parametrized by complex parameters, we obtain a c-number counterpart of the original quantal system. In the TDHF theory, a Slater determinant is adopted for the wave packet. If the parameters obey a certain condition, which is called the canonicity condition, the c-number counterpart is expressed in terms of canonical variables of classical mechanics. Further, by requantization, the c-number counterpart goes back to the original quantal system in disguise. The HP expansion of the \(su(2)\)-spin system, i.e., the first form, is a typical example. This form is obtained under the well-known wave packet

\[
|c\rangle = (|w|^2 + 1)^{-\frac{s}{2}} \exp(w \hat{S}_+) |s, -s\rangle , \\
\hat{S}_- |s, -s\rangle = 0, \quad \hat{S}_0 |s, -s\rangle = -s |s, -s\rangle .
\] (1.1)

Here, \(w\) denotes a complex parameter and the set \(\hat{S}_{\pm,0}\) obeys the \(su(2)\)-algebra. However, until the present, we do not know which form of the wave packet can reproduce the second form of the boson expansion for the \(su(2)\)-spin. In addition to the \(su(2)\)-spin, the above scheme was applied to the \(su(1,1)\)-spin system obeying the \(su(1,1)\)-algebra by the present authors (A. K., Y. T. and M. Y.) and thermal properties of the harmonic oscillator and various nuclear models based on the \(su(2)\)-algebra could be described with various interesting results.

The starting Hamiltonian of the \(su(1,1)\)-algebraic approach developed in Ref. 10) is as follows:

\[
\hat{H}_{su(1,1)} = \hat{K}_c - \hat{K}_d + \hat{V}_{su(1,1)} ,
\] (1.2)

\[
\hat{K}_c = F(\hat{c}^* \hat{c}) , \quad \hat{K}_d = F(\hat{d}^* \hat{d}) ,
\]

\[
\hat{V}_{su(1,1)} = -\gamma i(\hat{c}^* \hat{d}^* - \hat{d}^* \hat{c}) . \quad (\gamma: \text{constant})
\] (1.2a)

Here, \((\hat{c}^*, \hat{c})\) and \((\hat{d}^*, \hat{d})\) denote boson operators and \(F(\hat{c}^* \hat{c}) = F(\hat{d}^* \hat{d})\) denotes function of \(\hat{c}^* \hat{c}(\hat{d}^* \hat{d})\). For example, \(F(\hat{c}^* \hat{c}) = \omega \hat{c}^* \hat{c}\) corresponds to the harmonic oscillator. The Hamiltonian (1.2) can be expressed in terms of three components of the \(su(1,1)\)-spin. This implies that the system under investigation is a kind of the \(su(1,1)\)-spin systems and \(\hat{H}_{su(1,1)}\) is of the form which is a natural extension of the form shown in Ref. 11), where \(F(\hat{c}^* \hat{c}) = \omega \hat{c}^* \hat{c}\) is adopted. With the aid of \(\hat{H}_{su(1,1)}\), we are able to describe thermal properties of many-boson system expressed in terms of \((\hat{c}^*, \hat{c})\) (c-part) through the interaction with the degree of freedom expressed in terms of \((\hat{d}^*, \hat{d})\) (d-part). The Hamiltonian (1.2) gives us a constant of motion in the present \(su(1,1)\)-spin system. However, it should be noted that the constant of motion cannot result in the energy of the total system consisting of the c- and the d-part with the mutual interaction. In other words, the thermal properties induced by the Hamiltonian (1.2) do not originate in the energy flow between the c- and the d-part. Therefore, it may be quite interesting to investigate the thermal properties of the c-part in terms of the energy flow from (or to) the d-part. Since the present system is composed of two kinds of bosons, a possible candidate for this problem may be the use of the \(su(2)\)-algebra. The Hamiltonian in this case may be expressed in the form

\[
\hat{H}_{su(2)} = \hat{K}_c + \hat{K}_d + \hat{V}_{su(2)} .
\] (1.3)
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Here, $\vec{K}_c$ and $\vec{K}_d$ are of the same form as that used in the $su(1,1)$-spin system. Needless to say, it is essential to investigate which form is suitable for $\hat{V}_{su(2)}$. However, the wave packet (1·1) seems not to be suitable for the above-mentioned aim, and we must find other forms of the wave packet.

The aim of the present paper is to point out that, in addition to the first and the second, a third form for the boson expansion of the $su(2)$-spin can be derived from the MYT method. In this form, the generators $\hat{S}_{\pm,0}$ are expressed in terms of finite expansions for certain operators which are expressed in terms of boson operators. Furthermore, for the second and the third, the $c$-number counterparts can be derived by using certain wave packets. Concerning the second form, the results are very similar to those in the $su(1,1)$-algebra,\textsuperscript{10} we can guess that the wave packet which reproduces the third form is useful for the description of thermal properties of various many-body systems.

In the next section, the boson representation of the $su(2)$-algebra presented by Schwinger\textsuperscript{12} and its reformation will be reviewed. In § 3, three forms of the boson expansions of the $su(2)$-spin system will be presented with the help of the MYT method. Section 4 will be devoted to giving a general scheme for obtaining the $c$-number counterpart with application to the first form and the quantization rule. In this section, the three forms of the wave packets will be shown. In § 5, the $c$-number counterparts of the second and the third form will be presented. In § 6, some comments will be given for several examples. Finally, the relation to the $su(1,1)$-algebraic approach developed by the present authors will be discussed mainly in § 7.

§ 2. The $su(2)$-spin and its Schwinger boson representation

The $su(2)$-spin system is treated in terms of a set of three operators $\hat{S}_{\pm,0}$ which obey the $su(2)$-algebra:

\begin{equation}
\hat{S}^+ = \hat{S}^-, \quad \hat{S}^z = \hat{S}_0, \tag{2·1a}
\end{equation}

\begin{equation}
[\hat{S}^+, \hat{S}^-] = 2\hat{S}_0, \quad [\hat{S}_0, \hat{S}_z] = \pm \hat{S}_z. \tag{2·1b}
\end{equation}

The state with the magnitude $s$ and the $z$-component $s_0$, which we denote as $|s, s_0\rangle$, is a solution of the following eigenvalue equations:

\begin{equation}
\hat{S}^2|s, s_0\rangle = s(s+1)|s, s_0\rangle, \quad (s = 0, 1/2, 1, \cdots) \tag{2·2a}
\end{equation}

\begin{equation}
\hat{S}^z = s_0^2 + (\hat{S}^+_0 + \hat{S}^-_0)/2, \tag{2·2b}
\end{equation}

\begin{equation}
\hat{S}_0|s, s_0\rangle = s_0|s, s_0\rangle, \quad (s_0 = -s, -s+1, \cdots, s-1, s) \tag{2·2b}
\end{equation}

Here, the Casimir operator $\hat{S}^2$ denotes the square of the magnitude of the spin. We call the space spanned by the set $\{|s, s_0\rangle\}$ the $su(2)$-space with the magnitude $s$.

A concrete example of the above $su(2)$-spin system can be found in a boson representation given by Schwinger.\textsuperscript{12} Its outline was given recently by the present authors in (A). In this section, under slightly different notations and formalism, we will sketch its outline. In terms of two kinds of boson operators, $(\hat{c}^*, \hat{c})$ and $(\hat{d}^*, \hat{d})$, the three components $\hat{S}_{\pm,0}$ are expressed in the form

...
\[ \hat{S}_+ = \hat{c}^* \hat{d} , \quad \hat{S}_- = \hat{d}^* \hat{c} , \quad \hat{S}_0 = (\hat{c}^* \hat{c} - \hat{d}^* \hat{d})/2. \tag{2.3} \]

The square of the magnitude of the spin is given by
\[ \hat{S}^2 = \hat{S}(\hat{S} + 1) , \quad \hat{S} = (\hat{c}^* \hat{c} + \hat{d}^* \hat{d})/2. \tag{2.4} \]

The operator \( \hat{S} \) denotes the magnitude of the \( su(2) \)-spin. The state \(|s, s_0\rangle \) is of the form
\[ |s, s_0\rangle = (\sqrt{(s+s_0)!(s-s_0)!})^{-1} \hat{c}^{s+s_0} \hat{d}^{s-s_0} |0, 0\rangle , \tag{2.5} \]
\[ \hat{c} |0, 0\rangle = \hat{d} |0, 0\rangle = 0. \tag{2.6} \]

The vacuum \(|0, 0\rangle \) denotes the state with \( s = s_0 = 0 \). Especially, two states \(|s, +s\rangle \) and \(|s, -s\rangle \) are given, respectively, in the following forms:
\[ |s, +s\rangle = (\sqrt{(2s)!(s-s_0)!})^{-1} \hat{c}^{2s} |0, 0\rangle , \quad (\hat{d} |s, +s\rangle = 0) \tag{2.7} \]
\[ |s, -s\rangle = (\sqrt{(2s)!})^{-1} \hat{d}^{2s} |0, 0\rangle , \quad (\hat{c} |s, -s\rangle = 0) \tag{2.7} \]

The above is the Schwinger boson representation of the \( su(2) \)-spin.

Now, in order to make a reformation of the Schwinger boson representation, we introduce two sets of operators \( \hat{B}_\pm \) and \( \hat{T}_{\pm,0} \), which are defined in terms of \( \hat{S}_{\pm,0} \) and \( \hat{S} \):
\[ \hat{B}_+ = (\sqrt{\hat{S} - \hat{S}_0 + 1})^{-1} \hat{S}_+ , \quad \hat{B}_- = \hat{S}_- (\sqrt{\hat{S} - \hat{S}_0 + 1})^{-1} , \quad \hat{B}^* = \hat{B}_+ , \tag{2.8} \]
\[ \hat{T}_+ = (\sqrt{\hat{S} - \hat{S}_0 + 1})^{-1} \hat{S}_+ , \quad \hat{T}_- = (\sqrt{\hat{S} + \hat{S}_0 + 2t}) \hat{S}_- (\sqrt{\hat{S} - \hat{S}_0 + 1})^{-1} , \tag{2.9} \]
\[ \hat{T}_0 = \hat{S} + \hat{S}_0 + t , \quad \hat{T}^* = \hat{T}_- , \quad \hat{T}^* = \hat{T}_0 . \tag{2.9} \]

Here, the real parameter \( t \) runs in the region
\[ t = 1/2, 1, 3/2, 2, \ldots. \tag{2.9a} \]

The reason why the parameter \( t \) is restricted to the integer or the half-integer values shown in Eq. (2.9a) will be mentioned later. Since \( (\hat{S} - \hat{S}_0 + 1) \) is positive definite, the inverse of the square root can be defined. The operators \( \hat{B}_\pm \) have been introduced in \( \text{(A)} \) and they obey the commutation relation
\[ [\hat{B}_-, \hat{B}_+] = 1 - (\hat{N}_c + 1) \cdot \hat{D} . \tag{2.10} \]
Here, \( \hat{D} \), together with \( \hat{N}_c \) and \( \hat{N}_d \), is defined in the following form:
\[ \hat{D} = 1 - \hat{d}^* (\hat{N}_d + 1)^{-1} \hat{d} , \quad \hat{N}_c = \hat{c}^* \hat{c} , \quad \hat{N}_d = \hat{d}^* \hat{d} . \tag{2.11} \]

The operator \( \hat{D} \) is rewritten as and satisfies the relation
\[ \hat{D} = \sum_{s=0}^{\infty} |s, s\rangle \langle s, s| , \quad \hat{D}^2 = \hat{D} . \tag{2.11a} \]

The set \( \hat{T}_{\pm,0} \) obeys the commutation relations
\[ [\hat{T}_+, \hat{T}_-] = -2 \hat{T}_0 + (\hat{N}_c + 2t)(\hat{N}_c + 1) \cdot \hat{D} , \tag{2.11b} \]
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\[ [\hat{T}_0, \hat{T}_z] = \pm \hat{T}_z, \quad (2.12a) \]

Further, the set \( \hat{T}_{z,0} \) satisfies
\[ \hat{T}_z^2 = \hat{T}_{z,0}^2 - (\hat{T}_+ + \hat{T}_- + \hat{T}_+ \hat{T}_-)/2 \]
\[ = t(t-1) + (\hat{N}_c + 2t)(\hat{N}_c + 1)/2 \cdot \hat{D}. \quad (2.12b) \]

In the relations (2.10) and (2.12), we can see that if the terms related with \( \hat{D} \) can be neglected, the operators \( \hat{B}_\pm \) are reduced to boson creation and annihilation operators and the set \( \hat{T}_{z,0} \) to the the \( su(1,1) \)-spin with the magnitude \( t = 1/2, 1, 3/2, 2, 5/2, \ldots \), respectively. The details of the \( su(1,1) \)-spin can be seen in Refs. 10) and 11). The above is the reason why we set up the restriction (2.9a) for the range of the parameter \( t \). Including the well-known form, with the use of the three sets \( \hat{S}_\pm, \hat{B}_\pm \) and \( \hat{T}_\pm \), the state \( |S, S_0\rangle \) is rewritten in the forms:
\[ |S, S_0\rangle = \frac{1}{(2s)!} (\hat{S}_+)^{s+S_0} |S, -S_0\rangle, \quad \hat{S}_- |S, -S_0\rangle = 0, \quad (2.13) \]
\[ |S, S_0\rangle = \frac{1}{(s+S_0)!} (\hat{B}_+)^{s+S_0} |S, -S_0\rangle, \quad \hat{B}_- |S, -S_0\rangle = 0, \quad (2.14) \]
\[ |S, S_0\rangle = \frac{1}{(s+S_0)!} (\hat{D} + 2t - 1) (\hat{T}_+)^{s+S_0} |S, -S_0\rangle, \quad \hat{T}_- |S, -S_0\rangle = 0. \quad (2.15) \]

Hereafter, our concern will be restricted to the \( su(2) \)-space with a definite value of the magnitude \( s(s = 0, 1/2, 1, 3/2, \ldots) \).

§ 3. Marumori-Yamamura-Tokunaga boson expansion of the \( su(2) \)-spin in three forms

The most familiar boson expansion of the \( su(2) \)-spin may be Holstein-Primakoff (HP) representation. Three components \( \hat{S}_{z,0} \) for a fixed value of \( s \) are shown as
\[ \hat{S}_+ = \hat{b}^* \sqrt{2s - \hat{b}^* \hat{b}}, \quad \hat{S}_- = \hat{b} \sqrt{2s - \hat{b}^* \hat{b}}, \quad \hat{S}_0 = \hat{b}^* \hat{b} - s. \quad (3.1) \]

Here, \( (\hat{b}^*, \hat{b}) \) denote boson operators:
\[ [\hat{b}, \hat{b}^*] = 1. \quad (3.2) \]

The above form (3.1) is constructed so as to make it satisfy the relations (2.1).

In this section, let us begin with recapitulating the Marumori-Yamamura-Tokunaga (MYT) boson expansion for the \( su(2) \)-spin system under a way slightly different from that shown in (A). Starting point of the MYT expansion is one to one correspondence between the original \( su(2) \)-space with the magnitude \( s \) and a certain subspace in the whole boson space. This subspace is called the physical space and the space orthogonal to the physical one unphysical. The whole boson space is spanned by the set \( \{|n\rangle\} \), which is composed of the physical and the unphysical space in the following form:
\[ |n\rangle = (\sqrt{n})^{-1} \cdot \hat{b}^* |n\rangle, \quad (\hat{b} |0\rangle = 0. \quad (3.3) \]

The state \( |n\rangle \) is rewritten as
\[ |n\rangle = (\sqrt{n!(2t-1)!})^{-1} \sqrt{(n+2t-1)!} \cdot \hat{\beta}^n |0\rangle. \quad (3.4) \]

Definition of \((\hat{\beta}^*, \hat{\beta})\), together with the property, is given by

\[ \hat{\beta}^* = \hat{b}^* (\sqrt{\hat{b}^* \hat{b} + 2t})^{-1}, \quad \hat{\beta} = (\sqrt{\hat{b}^* \hat{b} + 2t})^{-1} \cdot \hat{b}, \]  \[ (3.5) \]

\[ [\hat{\beta}, \hat{\beta}^*] = (1 - \hat{\beta}^* \hat{\beta})^2 / 2t + \varepsilon \cdot \hat{\beta}^* \hat{\beta} (1 - \hat{\beta}^* \hat{\beta})^2 / 2t (2t - \hat{\beta}^* \hat{\beta}). \]  \[ (3.6) \]

Here, \(\varepsilon = 1\). Especially, in the case \(t = 1/2\), we have

\[ [\hat{\beta}, \hat{\beta}^*] = (1 - \hat{\beta}^* \hat{\beta})^2 + \varepsilon \cdot \hat{\beta}^* \hat{\beta} (1 - \hat{\beta}^* \hat{\beta}). \]  \[ (3.7) \]

Let the state \(|s, s\rangle\) correspond to the boson state \(|n = s + s_0\rangle\). Then, the physical space consists of the states \(|n\rangle\) with \(n = 0, 1, 2, \ldots, 2s\). Under this correspondence, we define the following transformation from the original to the physical space:

\[ U = \sum_{s_0 = -s}^{s} |s + s_0\rangle \langle s, s_0|. \]  \[ (3.8) \]

The properties of \(U\) are listed up as

\[ U^* U = \sum_{s_0 = -s}^{s} |s, s_0\rangle \langle s, s_0| = 1, \quad UU^* = \sum_{n=0}^{2s} |n\rangle \langle n| = \hat{P}. \]  \[ (3.9) \]

The operator \(\hat{P}\) plays a role of projecting on the physical space. Then, we have

\[ U|s, s_0\rangle = |s + s_0\rangle, \]  \[ (3.10a) \]

\[ U^* |s + s_0\rangle = |s, s_0\rangle, \quad U^* |2s + k\rangle = 0. \quad (k = 1, 2, 3, \ldots) \]  \[ (3.10b) \]

The state \(|2s + k\rangle\) belongs to the unphysical space. With the use of the transformation \(U\), the set \(S_{s,0}\) is transformed into

\[ \hat{S}_+ = U\hat{S}_+ U^* = \sum_{n=0}^{2s} \sqrt{(n+1)(2s-n)} |n+1\rangle \langle n|, \]

\[ \hat{S}_- = U\hat{S}_- U^* = \sum_{n=0}^{2s} \sqrt{(n+1)(2s-n)} |n\rangle \langle n+1|, \]

\[ S_0 = U\hat{S}_0 U^* = \sum_{n=0}^{2s} (n-s) |n\rangle \langle n|. \]  \[ (3.11a) \]

\[ \hat{S} = U\hat{S} U^* = \sum_{n=0}^{2s} n |n\rangle \langle n|. \]  \[ (3.11b) \]

It should be noted that the forms (3.11) satisfy the same relations as those given in Eqs. (2.1) and (2.4) in the physical space, and in the unphysical one, they lead us to trivial relations, which do not give us any meaningful content.

The expressions (3.11a) can be formally rewritten into three forms. The first form was originally presented by Marshalek and the main part of this form has been discussed in (A). By substituting the states (3.3) into the expressions (3.11a) and, further, by using the formula \(f(\hat{\beta}^* \hat{\beta})|n\rangle = f(n)|n\rangle\) or \(\langle n|f(\hat{\beta}^* \hat{\beta}) = \langle n|f(n), \hat{S}_{s,0}\) can be written as

\[ \hat{S}_{s,0} = \hat{P} \cdot \hat{S}_{s,0} \cdot \hat{P}. \]  \[ (3.12) \]
Here, the set $S_{\pm,0}$ is given in Eq. (3·1). The form (3·12) has been regarded as that showing the equivalence between the HP, or more generally, the BZM expansion and the MYT expansion. Hereafter, we will call the form (3·12) the form (i). Next, we note that the operator $|0\rangle \langle 0|$, which is denoted as $\hat{P}_0$, can be expressed as

$$\hat{P}_0 = |0\rangle \langle 0| = \exp(-\hat{b}^* \hat{b}) = \sum_{k=0}^{\infty} (-)^k/k! \cdot \hat{b}^k \hat{b}^*.$$  \hspace{1cm} (3·13)

Then, substituting the states (3·3) into the expressions (3·11a) and, further, using the relation (3·13), $\hat{S}_{\pm,0}$ can be rewritten as

(ii) \hspace{.5cm} \hat{S}_{+} = \sum_{n=0}^{2s} \sqrt{2s-n} \cdot (2s-n)! \cdot \hat{b}^* \hat{b}+1 \hat{P}_0 \hat{b}^n, \hspace{.5cm} \hat{S}_{-} = \sum_{n=0}^{2s} \sqrt{2s-n} \cdot (2s-n)! \cdot \hat{b}^n \hat{P}_0 \hat{b}^* \hat{b}+1, \hspace{.5cm} \hat{S}_0 = \sum_{n=0}^{2s} (n-s) \cdot (2s-n)! \cdot \hat{b}^n \hat{P}_0 \hat{b}^*(2s-n)! \cdot \hat{b}^* \hat{b}. \hspace{1cm} (3·14)$$

The above is the second form, hereafter, called the form (ii). It is expressed in terms of the power series for the normal order product of ($\hat{b}^* \hat{b}$). The forms (i) and (ii) were investigated in (A). Finally, we will show the third form, which belongs to a central part of the present paper. Substituting the states (3·4) into the relations (3·11a), the set $\hat{S}_{\pm,0}$ can be transformed into the following form:

(iii) \hspace{.5cm} \hat{S}_{+} = \sum_{n=0}^{2s} \sqrt{(2s+n)\cdot (2s-n)!} \cdot (n+2s-1)!/n!(2s+1)! \cdot \hat{b}^n \hat{P}_0 \hat{b}^* \hat{b}, \hspace{.5cm} \hat{S}_{-} = \sum_{n=0}^{2s} \sqrt{(2s+n)\cdot (2s-n)!} \cdot (n+2s-1)!/n!(2s+1)! \cdot \hat{b}^* \hat{b} \hat{P}_0 \hat{b}^* \hat{b}, \hspace{.5cm} \hat{S}_0 = \sum_{n=0}^{2s} (n-s) \cdot (2s-n)! \cdot \hat{b}^n \hat{P}_0 \hat{b}^* \hat{b}. \hspace{1cm} (3·15)$$

Here, $\hat{P}_0$ is given by

$$\hat{P}_0 = |0\rangle \langle 0| = (1-\hat{b}^* \hat{b})^{2s} = \sum_{k=0}^{2s} (-)^k (2s)!/k!(2s-k)! \cdot \hat{b}^k \hat{b}^*. \hspace{1cm} (3·16)$$

We can see that the form (3·15) is of the normal order product for ($\hat{b}^* \hat{b}$) and, further, it is noted that the expansion is finite. Hereafter, the third is called the form (iii). Thus, we know that there exist three forms of the boson expansions in the $su(2)$-spin. Needless to say, they are connected to each other through the general relation for any function $f(x)$:

$$f(\hat{b}^* \hat{b}) = \sum_{n=0}^{\infty} f(n) \cdot 1/n! \cdot \hat{b}^n \hat{P}_0 \hat{b}^*. \hspace{1cm} (3·17)$$

It should be noted that even if we use the relation (3·17), the form (3·1) cannot be reduced to the forms (3·14) and (3·15).
§ 4. General scheme for obtaining c-number counterpart and its application to the case of the form (i)

In the previous section, three forms of the representations were shown in the framework of the MYT boson expansion. Corresponding to these three forms, we will investigate three forms of wave packets in the \(su(2)\)-space with the magnitude \(s\):

(i) \( |c\rangle = (\sqrt{N_0})^{-1} \exp(w \hat{S}_+) |s, -s\rangle \), \( (4 \cdot 1) \)

(ii) \( |c\rangle = (\sqrt{N_0})^{-1} \exp(w \hat{B}_+) |s, -s\rangle \), \( (4 \cdot 2) \)

(iii) \( |c\rangle = (\sqrt{N_0})^{-1} \exp(w \hat{T}_+) |s, -s\rangle \). \( (4 \cdot 3) \)

Here, \( w \) denotes a complex parameter and \((\sqrt{N_0})^{-1}\) is a normalization constant which is determined through the condition \( \langle c|c\rangle = 1 \). Clearly, the wave packets \( |c\rangle \) are of the exponential type superpositions of the states \(|s, s_0\rangle\) shown in Eqs. \((2 \cdot 13), (2 \cdot 14)\) and \((2 \cdot 15)\), respectively. For each form, \( N_0 \) is given by

(i) \( N_0 = \sum_{n=0}^{2s} (2s)!/n!(2s-n)! \cdot (|w|^2)^n \), \( (4 \cdot 4) \)

(ii) \( N_0 = 1/n! \cdot (|w|^2)^n \), \( (4 \cdot 5) \)

(iii) \( N_0 = \sum_{n=0}^{2s} (n+2t-1)!/n!(2t-1)! \cdot (|w|^2)^n \). \( (4 \cdot 6) \)

It may be quite convenient for the later discussion to rewrite the states \(|c\rangle\) shown in Eqs. \((4 \cdot 1) \sim (4 \cdot 3)\) in the following forms:

\[ |c\rangle = (\sqrt{N_0})^{-1} \sum_{s_0=-s}^{s} C(s, s_0) w^{s+s_0} |s, s_0\rangle, \] \( (4 \cdot 7) \)

(i) \( C(s, s_0) = \sqrt{(2s)!}/(s+s_0)!(s-s_0)! \), \( (4 \cdot 8) \)

(ii) \( C(s, s_0) = \sqrt{1/(s+s_0)!} \), \( (4 \cdot 9) \)

(iii) \( C(s, s_0) = \sqrt{(s+s_0+2t-1)!/(s+s_0)!(2t-1)!} \). \( (4 \cdot 10) \)

The expectation values of \( \hat{S}_{\pm,0} \) for each \(|c\rangle\) are given as

(i) \( \langle \hat{S}_+ \rangle_c = N_0^{-1} \sum_{n=0}^{2s} (2s-n) \cdot (2s)!/n!(2n-n)! \cdot w^* (|w|^2)^n \), \( (4 \cdot 11) \)

\( \langle \hat{S}_- \rangle_c = N_0^{-1} \sum_{n=0}^{2s} (2s-n) \cdot (2s)!/n!(2s-n)! \cdot w (|w|^2)^n \),

\( \langle \hat{S}_0 \rangle_c = N_0^{-1} \sum_{n=0}^{2s} (n-s) \cdot (2s)!/n!(2s-n)! \cdot (|w|^2)^n \).

(ii) \( \langle \hat{S}_+ \rangle_c = N_0^{-1} \sum_{n=0}^{2s} \sqrt{2s-n} \cdot 1/n! \cdot w^* (|w|^2)^n \), \( (4 \cdot 11) \)

\( \langle \hat{S}_- \rangle_c = N_0^{-1} \sum_{n=0}^{2s} \sqrt{2s-n} \cdot 1/n! \cdot w (|w|^2)^n \),
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\[ (\bar{S}_0) = N_0^{-1} \sum_{n=0}^{2} (n-s) \cdot 1/n! \cdot (|w|^2)^n. \]  

(iii) \[ (\bar{S}_+) = N_0^{-1} \sum_{n=0}^{2} \sqrt{(2t+n)(2s-n)} \cdot (n+2t-1)!/n!(2t-1)!) \cdot w^* (|w|^2)^n, \]

\[ (\bar{S}_-) = N_0^{-1} \sum_{n=0}^{2} \sqrt{(2t+n)(2s-n)} \cdot (n+2t-1)!/n!(2t-1)!) \cdot w (|w|^2)^n, \]

\[ (\bar{S}_0) = N_0^{-1} \sum_{n=0}^{2} (n-s) \cdot (n+2t-1)!/n!(2t-1)! \cdot (|w|^2)^n. \]

Here, $(\bar{S}_{\pm,0})_c$ denote the expectation values of $\bar{S}_{\pm,0}$ with respect to the wave packet $|\psi\rangle$, $<\psi|\bar{S}_{\pm,0}|\psi\rangle$. Hereafter, the forms (4.11), (4.12) and (4.13) are called the forms (i), (ii) and (iii), respectively. By using the relation (4.4) and the binomial theorem, the form (i) becomes quite simple:

\[ N_0 = (|w|^2 + 1)^{2s}, \]

\[ (\bar{S}_+) = 2sw^* / (|w|^2 + 1), \quad (\bar{S}_-) = 2sw / (|w|^2 + 1), \quad (\bar{S}_0) = s(|w|^2 - 1) / (|w|^2 + 1). \]

We call the set $(\bar{S}_{\pm,0})_c$ c-number counterpart of the set $\bar{S}_{\pm,0}$.

As was stressed in the TDHF theory in canonical form, the c-number counterpart of quantal system under investigation can be determined through the parametrization of $|\psi\rangle$ in terms of canonical variables instead of $w$ itself. For example, we can adopt the variables, which is denoted by $(b^*, b)$ and obey the condition

\[ [b, b^*] = -i. \]

The symbol $[,]_p$ represents the Poisson bracket. For the parametrization of $|\psi\rangle$ in terms of $(b^*, b)$, we require a condition, which is called the canonicity condition:

\[ <\psi|\partial b^*|\psi> = b^*/2, \quad <\psi|\partial b|\psi> = -b/2. \]

The above condition has been used widely in the TDHF theory in canonical form. In common with the three forms, the canonicity condition can be written down explicitly as

\[ N_0 / N_0 \cdot (w^* \partial w / \partial b - w \partial w^* / \partial b)/2 = b^*/2, \]

\[ N_0 / N_0 \cdot (w^* \partial w / \partial b^* - w \partial w^* / \partial b^*)/2 = -b/2, \]

Here, $N_0$ denotes the derivative $dN_0(|w|^2)/d(|w|^2)$. Then, for the case $N_0 / N_0 > 0$, a possible solution of the condition (4.18) leads us to the following relations:

\[ \sqrt{N_0 / N_0} w = b^*, \quad \sqrt{N_0 / N_0} w = b. \]

The above gives us the solution

\[ (N_0 / N_0)|w|^2 = |b|^2, \quad \text{i.e.,} \quad |w|^2 = (N_0 / N_0)|b|^2. \]

Since the constant $N_0$ is a definite function of $|w|^2$ and by solving Eq. (4.20), we can determine $|w|^2$ as a function of $|b|^2$, i.e., $|w|^2 = F(|b|^2)$. Then, $N_0 / N_0$ can be expressed
as a function of $|b|^2$, i.e., $N_0/N_0 = G(|b|^2)$. Thus, we have the following form:

$$w^* = b^* \sqrt{G(|b|^2)}, \quad w = b \sqrt{G(|b|^2)}.$$  (4.21)

In the form (i), we have $N_0/N_0 = 2s/(|w|^2 + 1)$. Then, $F(|b|^2)$ and $G(|b|^2)$ are given as $|b|^2/(2s - |b|^2)$ and $1/(2s - |b|^2)$, respectively. By substituting these results into the relations (4.21), we can determine $w^*$ and $w$ as functions of the canonical variables:

$$w^* = b^*/\sqrt{2s - |b|^2}, \quad w = b/\sqrt{2s - |b|^2}.$$  (4.22)

Needless to say, the term $(2s - |b|^2)$ should be positive. With the use of the relation (4.22), $(\tilde{S}_{\pm,0})_c$ can be determined in the following form:

$$(\tilde{S}_+)_c = b^* \sqrt{2s - |b|^2}, \quad (\tilde{S}_-)_c = b \sqrt{2s - |b|^2}, \quad (\tilde{S}_0)_c = (b)^2 - s.$$  (4.23)

The above is nothing but well-known $c$-number representation of the HP boson expansion. Concerning the derivation of the form (4.23), there exist some methods and, in this paper, we will apply the above method to the forms (ii) and (iii).

As a final task, let us investigate quantization rule which enables us to regard the form (4.23) as the $c$-number counterpart of the $su(2)$-spin. Conventionally, the quantal version of the $c$-number representation (4.23), $(\tilde{S}_{\pm,0})_c$, is regarded as $\tilde{S}_{\pm,0}$ shown in the relation (3.1). However, as was already mentioned, the form (3.1) cannot be reduced to the forms (3.14) and (3.15). Therefore, if including the forms (3.14) and (3.15) in the present argument, the quantal version of $(\tilde{S}_{\pm,0})_c$ shown in Eqs. (4.23) should be regarded as $\tilde{S}_{\pm,0}$ shown in Eq. (3.12). This requirement permits us to write down

$$(\tilde{S}_{\pm,0})_c = P \cdot (\tilde{S}_{\pm,0})_c \cdot P,$$  (4.24)

$$P = \exp(-|b|^2) \cdot \sum_{n=0}^{2s} 1/n! \cdot (|b|^2)^n$$

$$= 1 - 1/(2s + 1) \cdot (|b|^2)^{2s+1} \cdot \sum_{n=0}^{\infty} (2s + 1)!/(2s + 1 + n)! \cdot (|b|^2)^n \exp(-|b|^2).$$  (4.25)

Then, by replacing $(b^*, b)$ with $(\tilde{b}^*, \tilde{b})$ and, further, by making an appropriate ordering of $(b^*, b)$ in $(\tilde{S}_{\pm,0})_c$, we arrive at the expression (3.12), i.e., the form (i). In the above sense, we can regard the set (4.23) as the $c$-number counterpart of the $su(2)$-spin.

§ 5. The $c$-number counterparts in the forms (ii) and (iii)

In the previous section, we showed a method for solving the canonicity condition (4.17) and, as a demonstration, the solution of the form (i) was given. Following this method, let us investigate a method to get the solutions of the forms (ii) and (iii). First, we can give $N_0/N_0$ in the form (ii) as follows:

$$(\text{ii}) \quad N_0/N_0 = 1 - K,$$  (5.1)

$$K = [1/(2s)! \cdot (|w|^2)^{2s}] \cdot [1 + \sum_{n=1}^{2s} 1/n! \cdot (|w|^2)^n]^{-1}.$$
\[\begin{align*}
&= \exp(-|w|^2) \cdot \frac{1}{(2s)!} \cdot (|w|^2)^{2s} \\
&\times [1 + 1/(2s+1)! \cdot (|w|^2)^{2s+1} \cdot (1 + \sum_{n=1}^{\infty} G_{s,n}(|w|^2)^n)].
\end{align*}\]

(5.1a)

For the later discussion, it is not necessary to give the form of \(G_{s,n}\). Concerning the result (4·20), the form (ii) gives us

\[|w|^2 = |b|^2/(1 - K).\]

(5·2)

By adopting an iterative method with the initial condition \(|w|^2 = |b|^2\), we obtain the following solution:

\[|w|^2 = |b|^2 \cdot G(|b|^2), \quad G(|b|^2) = (1 - L)^{-1},\]

(5·3)

\[L = P_0 \cdot \frac{1}{1/(2s)! \cdot (|b|^2)^{2s}} \left[1 + \sum_{n=1}^{\infty} F_{s,n}(|b|^2)^n\right],\]

(5·3a)

\[w^* = b^*/\sqrt{1 - L}, \quad w = b/\sqrt{1 - L},\]

(5·4)

\[P_0 = \exp(-|b|^2).\]

(5·5)

Here, the coefficient \(F_{s,n}\) can be determined iteratively. It should be noted that the leading term of \(L\) is of the order of \(|b|^{2s}\) except \(P_0\).

In order to treat the form (iii) in a way analogous to that in the form (ii), it may be necessary to make a modification for the form (iii). First, let us define new variables \((\beta^*, \beta)\) in the following form:

\[\beta^* = b^*/\sqrt{2t + |b|^2}, \quad \beta = b/\sqrt{2t + |b|^2}.\]

(5·6)

For the above variable, the Poisson bracket of \((\beta^*, \beta)\) can be calculated as

\[[\beta, \beta^*]_p = (-i) \cdot (1 - |\beta|^2)^2/2t.\]

(5·7)

Associating with the variables \((\beta^*, \beta)\), we introduce a quantity \(N_0/(2tN_0 + |w|^2N_0)\) in the form

(iii) \(N_0/(2tN_0 + |w|^2N_0) = 1 - K,\)

(5·8)

\[K = [(2s + 2t)!(2s)!(2t)! \cdot (|w|^2)^{2s}] \cdot \left[1 + \sum_{n=1}^{2s} (n + 2t)!/n!(2t)! \cdot (|w|^2)^n\right]^{-1}\]

\[= (1 - |w|^2)^{2s} \cdot [(2s + 2t)!(2s)!(2t)! \cdot (|w|^2)^{2s}]\]

\[\times (1 - |w|^2) \cdot [1 + (2s + 1 + 2t)!/(2s + 1)!(2t)! \cdot (|w|^2)^{2s+1}]\]

\[\times (1 + \sum_{n=1}^{\infty} G_{s,n}(|w|^2)^n)].\]

(5·8a)

We can show that the relation between \(|w|^2\) and \(|\beta|^2\) is given by

\[|w|^2 = |\beta|^2/(1 - K).\]

(5·9)

Then, in the same way as the case of the form (ii), we obtain

\[|w|^2 = |\beta|^2 \cdot G(|\beta|^2), \quad G(|\beta|^2) = (1 - L)^{-1},\]

(5·10)
\[ L = P_0 \cdot [2(2s+2t)!/(2s)!(2t)! \cdot (|\beta|^2)^{2s}] \cdot [1 + \sum_{n=1}^{\infty} F_{s,n} (|\beta|^2)^n] , \quad (5.10a) \]

\[ w^* = \beta^*/\sqrt{1-L} , \quad w = \beta/\sqrt{1-L} , \quad (5.11) \]

\[ P_0 = (1 - |\beta|^2)^{2t} . \quad (5.12) \]

It is also noted that the leading term of \( L \) is of the order of \((|\beta|^2)^{2s}\) except \( P_0 \). Further, we can see that the form (iii) is of the same structure as that of the form (ii).

In order to obtain the expectation values \((S_{\pm 0})_c\), we must determine \( N_0^{-1} \) as a function of \(|b|^2\) or \(|\beta|^2\). Substituting the expressions (5.3) and (5.10) into the relations (4.5) and (4.6), respectively, we have

\[ N_0^{-1} = P_0 \cdot (1 - I) , \quad (5.13) \]

(ii) \[ I = 2s/(2s+1) ! \cdot (|b|^2)^{2s+1} \cdot [1 + \sum_{n=1}^{\infty} I_{s,n} (|b|^2)^n] , \quad (5.14) \]

(iii) \[ I = 2s(2s+2t)!/(2s+1)!(2t-1)! \cdot (|\beta|^2)^{2s+1} \cdot [1 + \sum_{n=1}^{\infty} I_{s,n} (|\beta|^2)^n] . \quad (5.15) \]

Here, the quantity \( P_0 \) is defined for each form in Eqs. (5.5) and (5.12), respectively. The coefficients \( I_{s,n} \) can be determined in the framework of the present method. Then, by substituting the above results (5.4), (5.5), (5.11) and (5.12), together with Eqs. (5.13) and (5.14), we can express \((S_{\pm,0})_c\) for the forms (ii) and (iii) as functions of the variables \((b^*, b)\) and \((\beta^*, \beta)\), respectively. In the next section, we will show some special cases in more detail.

Finally, let us investigate the quantization of \((S_{\pm,0})_c\). The variables \((b^*, b)\) and \((\beta^*, \beta)\) obey the relations (4.16) and (5.7), respectively and, further, the commutation relations of \((\hat{b}^*, \hat{b})\) and \((\hat{\beta}^*, \hat{\beta})\) are of the form shown in Eqs. (3.2) and (3.6), respectively. Then, it may be permitted setting up the following correspondence:

\[ (b^*, b) \rightarrow (\hat{b}^*, \hat{b}) , \quad (\beta^*, \beta) \rightarrow (\hat{\beta}^*, \hat{\beta}) . \quad (5.16) \]

It should be noted that the form of the commutation relation (3.6) is formally different from the Poisson bracket (5.7). We can prove that the term related to \( e(-1) \) on the right-hand side of Eq. (3.6) expresses quantum effect. Therefore, we have

\[ P_0 \rightarrow \hat{P}_0 = |0\rangle\langle 0| , \quad P \rightarrow \hat{P} = \sum_{n=0}^{2s} |n\rangle\langle n| . \quad (5.17) \]

Next, we will search for the ordering of the variables. As can be seen from the relation (5.4), the variable \( w^* \) has the following form:

\[ w^* = b^*[1 + P_0 \sum_{r=0}^{\infty} X_{s,r} (|b|^2)^{2s+r}] = b^* + \sum_{r=0}^{\infty} X_{s,r} b^{2s+1+r} P_0 b^{2s+r} . \quad (5.18) \]

Then, \( w^* \) can be quantized as

\[ \hat{w}^* = \hat{b}^* + \sum_{r=0}^{\infty} X_{s,r} \hat{b}^{2s+1+r} \hat{P}_0 \hat{b}^{2s+r} = \hat{b}^* + \sum_{r=0}^{\infty} X_{s,r} \hat{b}^{2s+1+r} |0\rangle\langle 0| \hat{b}^{2s+r} . \quad (5.19) \]

Further, \( N_0^{-1} \) is treated in the following way:
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\[ N_{0}^{-1} = P_{0}[1 - \sum_{r=0}^{\infty} Y_{s,r} (|b|^{2})^{2s+1+r}] = P_{0} - \sum_{r=0}^{\infty} Y_{s,r} b^{* s+1+r} P_{0} b^{2s+1+r}. \quad (5.20) \]

Then, \( N_{0}^{-1} \) is quantized as

\[ \hat{N}_{0}^{-1} = \hat{P}_{0} - \sum_{r=0}^{\infty} Y_{s,r} \hat{b}^{* 2s+1+r} \hat{P}_{0} \hat{b}^{2s+1+r} = |0\rangle \langle 0| - \sum_{r=0}^{\infty} Y_{s,r} \hat{b}^{* 2s+1+r} |0\rangle \langle 0| \hat{b}^{2s+1+r}. \quad (5.21) \]

The form (iii) is also in the same situation as that in the case of the form (ii). Under the above preparation, let us consider the quantization of \((\hat{S}_{\pm,0})_{c}\). For this aim, we use the form

\[ \hat{P}_{0} \cdot N_{0}^{-1} w^{*} (|w|^{2})^{n} \cdot P = \hat{P}_{0} \cdot w^{* n+1} N_{0}^{-1} w^{n} \cdot P, \]
\[ \hat{P}_{0} \cdot N_{0}^{-1} w (|w|^{2})^{n} \cdot P = \hat{P}_{0} \cdot w^{* n} N_{0}^{-1} w^{n+1} \cdot P, \]
\[ \hat{P}_{0} \cdot N_{0}^{-1} (|w|^{2})^{n} \cdot P = \hat{P}_{0} \cdot w^{* n} N_{0}^{-1} w^{n} \cdot P. \quad (5.22) \]

Then, the quantization of the above forms for the case \(n \leq 2s\) is given as follows:

\[ \hat{P}_{0} \cdot w^{* n+1} \hat{N}_{0}^{-1} \hat{w}^{n} \cdot \hat{P} = \hat{b}^{* n+1} \hat{P}_{0} \hat{b}^{n}, \]
\[ \hat{P}_{0} \cdot w^{* n} \hat{N}_{0}^{-1} \hat{w}^{n+1} \cdot \hat{P} = \hat{b}^{* n} \hat{P}_{0} \hat{b}^{n+1}, \]
\[ \hat{P}_{0} \cdot \hat{w}^{* n} \hat{N}_{0}^{-1} \hat{w}^{n} \cdot \hat{P} = \hat{b}^{* n} \hat{P}_{0} \hat{b}^{n}. \quad (5.23) \]

For the case \(n \geq 2s+1\), the above quantities are all null operators. Thus, \((\hat{S}_{\pm,0})_{c}\) shown in Eqs. (4·12) become \(\hat{S}_{\pm,0}\) given in Eq. (3·14). The form (iii) is also treated in the same way as that in the form (ii). In this way, we can obtain two forms of the \(c\)-number counterparts of the su(2)-spin.

§ 6. Concrete expressions for some special cases

Let us show concrete expressions of some cases. Using them, we acquire physical implication of the \(c\)-number counterparts of \(\hat{S}_{\pm,0}\) presented in this paper. First, we will give a short comment on the form (i). For any value of \(s\), the form (i) satisfies the relations

\[ (\hat{S}_{+})_{c}^{*} = (\hat{S}_{-})_{c}, \quad (\hat{S}_{0})_{c}^{*} = (\hat{S}_{0})_{c}, \quad (6.1a) \]
\[ [(\hat{S}_{+})_{c}, (\hat{S}_{-})_{c}]_{P} = (-i) \cdot 2(\hat{S}_{0})_{c}, \quad (6.1b) \]
\[ [(\hat{S}_{0})_{c}, (\hat{S}_{\pm})_{c}]_{P} = (-i) \cdot \pm (\hat{S}_{\pm})_{c}, \quad (6.1c) \]
\[ (\hat{S}_{0})_{c}^{2} + (\hat{S}_{+})_{c}(\hat{S}_{-})_{c} = s^{2}. \quad (6.2) \]

We know the quantal relations (2·1) and (2·2) which correspond to the classical relations (6·1) and (6·2). In this sense, we could call the form (i) the classical counterpart of the su(2)-spin in (A). For our present discussion, this fact is quite important. Focussing on this aspect, we will investigate the forms (ii) and (iii) concretely for some special cases.

Let us start from the case \(s=1/2\). For the form (ii) and the form (iii) with arbitrary \(t\), the results are completely identical to those given in the form (i):
The results are quite trivial. Therefore, the next interest may be in the case \( s = 1 \). In the form (ii), the function \( G(|b|^2) \) introduced in (4.20) is given in the form
\[
G(|b|^2) = \frac{2}{1 - |b|^2 + \sqrt{1 + 2|b|^2 - (|b|^2)^2}}.
\]
In the region \( 0 \leq |b|^2 \leq 2 \), the function \( G(|b|^2) \) is positive. Three components \( (\tilde{S}_\pm, \tilde{S}_0) \) are expressed as functions of \( (w^*, w) \) in the following forms:
\[
(\tilde{S}_+)_c = w^* \left( \sqrt{2 + |w|^2} \right) \left[ 1 + |w|^2 + (|w|^2)^2 \right]^{-1},
(\tilde{S}_-)_c = w \left( \sqrt{2 + |w|^2} \right) \left[ 1 + |w|^2 + (|w|^2)^2 \right]^{-1},
(\tilde{S}_0)_c = - \left[ 1 - (|w|^2)^2 \right] \left[ 1 + |w|^2 + (|w|^2)^2 \right]^{-1},
\]
(6.5)
With the use of the function \( G(|b|^2) \), we can express \( (\tilde{S}_\pm, \tilde{S}_0)_c \) in terms of functions of the variables \( (b^*, b) \). Then, with the aid of the following relation, we calculate the Poisson bracket:
\[
[A, B]_p = (-i) \left( \frac{\partial A}{\partial w^*} \cdot \frac{\partial B}{\partial w} - \frac{\partial A}{\partial w} \cdot \frac{\partial B}{\partial w^*} \right) / M,
M = \frac{3b}{\partial w^* \cdot \partial b^* / \partial w - \partial b / \partial w^* \cdot \partial b^* / \partial w} = \frac{N_o N_0 + |w|^2 (N_0 N_o - N_o^2)}{N_o^2}.
\]
(6.6)
It is possible to prove that the expressions (6.5) satisfy the relations (6.1a) and (6.1c). However, they do not obey the relations (6.1b) and (6.2):
\[
[(\tilde{S}_+, \tilde{S}_-)_c]_p = (-i) \cdot 2 (\tilde{S}_0)_c (1 - \delta),
(\tilde{S}_0)_c^2 + (\tilde{S}_+)_c (\tilde{S}_-)_c = s^2 \cdot (1 - \Delta), \quad s = 1.
\]
(6.7)
(6.8)
Here, \( \delta \) and \( \Delta \) are given as follows:
\[
\delta = (3 - 2\sqrt{2}) |w|^2 \left[ 1 + 2 |w|^2 + (|w|^2)^2 \right]^{-1},
\Delta = (3 - 2\sqrt{2}) (|w|^2)^2 \left[ 1 + |w|^2 + (|w|^2)^2 \right]^{-2}.
\]
(6.9)
For the value \( |w|^2 = 0 \), we obtain \( (\tilde{S}_\pm)_c = 0 \) and \( (\tilde{S}_0)_c = -1 \) with \( \delta = \Delta = 0 \). If \( |w|^2 = \sqrt{2} \), \( |(\tilde{S}_\pm)_c| = \sqrt{1 - \delta} \) and \( (\tilde{S}_0)_c = 0 \). For this value of \( |w|^2 \), \( \delta \) and \( \Delta \) are the maximum for the change of \( |w|^2 \) in which \( \delta = (10 - 7\sqrt{2})/2 (\approx 0.05025) \) and \( \Delta = 17 - 12\sqrt{2} \approx 0.02944 \). The condition \( |w|^2 \to \infty \) leads us to \( (\tilde{S}_\pm)_c = 0 \) and \( (\tilde{S}_0)_c = 1 \) with \( \delta = \Delta = 0 \). In the form (iii) with \( s = 1 \), \( (\tilde{S}_\pm, \tilde{S}_0)_c \) are expressed as
\[
(\tilde{S}_+)_c t = w^* \sqrt{2t} \left[ \sqrt{2 + \sqrt{2} t(2t + 1)} |w|^2 \right] / W_t,
(\tilde{S}_-)_c t = w \sqrt{2t} \left[ \sqrt{2 + \sqrt{2} t(2t + 1)} |w|^2 \right] / W_t,
(\tilde{S}_0)_c t = - \left[ 1 + t(2t + 1) (|w|^2)^2 \right] / W_t,
W_0 = 1 + 2t |w|^2 + t(2t + 1) (|w|^2)^2.
\]
(6.10)
Here, we distinguish each case with the su(1,1)-spin \( t \) by the subscript \( t \). The relation between \( (w^*, w) \) and \( (b^*, b) \) is given by the function \( G(|b|^2) \):
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Thus, \((S_\pm)_{c,t}\) are expressed in terms of the canonical variables \((b^*, b)\). In this region \(0 \leq |b|^2 \leq 2\), the function \(G(|b|^2)\) is positive. In this case, for any \(t\), the relations (6·1a) and (6·1c) are also satisfied, while the relations (6·1b) and (6·2) are not. We can derive the relations similar to Eqs. (6·7) and (6·8) in which \(\delta\) and \(\Delta\) are given by

\[
\delta_t = |w|^2 \cdot \left\{ 2(3t + 1) - 4(2t + 1) \right\} / W_t ,
\]

\[
\Delta_t = |w|^2 \cdot 2t \left\{ 2(3t + 1) - 4(2t + 1) \right\} / W_t^2 .
\]

For any \(t(\geq 1/2)\), \(\delta_t\) and \(\Delta_t\) are positive. If \(|w|^2 = 0\), then, \((S_z)_{c,t} = 0\) and \((S_\theta)_{c,t} = -1\) with \(\delta_t = \Delta_t = 0\). It can be shown that \(\delta_t\) and \(\Delta_t\) take the maximum values for \(|w|^2 = 1 / (2t + 1)\). We express them as \(\delta_t(\text{max})\) and \(\Delta_t(\text{max})\). In this case, \(|(S_z)_{c,t}| = \sqrt{1 - \Delta_t}\) and \(|(S_\theta)_{c,t}| = 0\) are obtained. Here, \(\delta_t = \delta_t(\text{max}) = [(t + 1) / (\sqrt{2t + 1} + \sqrt{t})]^2 \times [2t + 1 + \sqrt{(2t + 1)}]^2 \) and \(\Delta_t = \Delta_t(\text{max}) = (t + 1)^2 / (\sqrt{2t + 1} + \sqrt{t})^2\). If we regard \(\delta_t(\text{max})\) and \(\Delta_t(\text{max})\) as the functions of \(t(\geq 1/2)\), these functions are monotonically decreasing with respect to \(t\). Thus, \(\delta_t(\text{max})\) and \(\Delta_t(\text{max})\) take the maximum values for \(t = 1/2\) in the change of \(t(\geq 1/2)\): \(\delta_{1/2}(\text{max}) = 1/6 = (0.1667)\) and \(\Delta_{1/2}(\text{max}) = 1/9 = (0.1111)\) with \(|w|^2 = 1\). Further, if \(t \to \infty\), then, \(\delta_{\infty}(\text{max}) = (10 - 7\sqrt{2}) = (0.05025)\) and \(\Delta_{\infty}(\text{max}) = 17 - 12\sqrt{2} = (0.02944)\), that is, the same values are obtained as those of the case (ii). Finally, if \(|w|^2 \to \infty\), \((S_z)_{c,t} = 0\) and \((S_\theta)_{c,t} = +1\) with \(\delta_t = \Delta_t = 0\) for any \(t\).

From the above analysis, it may be concluded that the forms (ii) and (iii) for the case \(s = 1\) cannot give the classical counterparts of the \(su(2)\)-spin system in the exact sense. However, in the wide range for the parameter \(|w|^2(0 \leq |w|^2 \leq \infty)\), \(\delta\) and \(\Delta\) are not so large and, then, the forms (ii) and (iii) can present approximate classical counterparts in high accuracy.

Next, we will investigate the case with sufficiently large \(s\). In this case, the asymptotic behavior with respect to the parameter \(s\) is interesting. In the form (ii), for sufficiently large \(s\), the normalization constant, \(N_0\) and the derivative for \(|w|^2\), \(N_0^s\) are given as \(N_0 = N_0^s = \exp|w|^2\). Then, the function \(G(|b|^2)\) is of the form

\[
G(|b|^2) = 1 .
\]

For arbitrary \(s\), the expectation value \((S_\pm)_{c}\) can be expressed as follows:

\[
(S_\pm)_{c} = w^* \sum_{n=0}^{\infty} (-)^n / n! \sum_{r=0}^{n} (-)^r / r! (n - r)! \cdot \sqrt{2s - r} / |w|^2 .
\]

In order to see that asymptotic behavior of \((S_\pm)_{c}\) for the sufficiently large value of \(s\), as examples, the terms with \(s = 0\), \(1\) and \(2\) can be rewritten as

\[
(-)^0 / 0! \cdot (-)^0 / 0! (2s - 0)! \cdot \sqrt{2s - 0} / |w|^2 = \sqrt{2s} , \quad (n = 0)
\]

\[
(-)^1 / 1! \cdot \sum_{r=0}^{1} (-)^r / r! (1 - r)! \cdot \sqrt{2s - r} / |w|^2
\]

\[
= - (\sqrt{2s - \sqrt{2s - 1}} / |w|^2 = -1 / (\sqrt{2s + \sqrt{2s - 1}}) \cdot |w|^2
\]

\[
\to -1 / 2 \sqrt{2s} \cdot |w|^2 , \quad (n = 1)
\]
Here, the difference is replaced with the differential and through this procedure, we obtain the same relations as those in the form (i):

\[
\begin{align*}
(\hat{S}_+)_c & \rightarrow b^* \sqrt{2s - |b|^2}, \quad (\hat{S}_-)_c \rightarrow b \sqrt{2s - |b|^2}, \\
(\hat{S}_0)_c & \rightarrow |b|^2 - s \quad \text{for the sufficiently large } s.
\end{align*}
\] (6·16)

Therefore, it is concluded that this case satisfies the conditions (6·1) and (6·2) asymptotically. We can see that the form (ii) is quite similar to the c-number counterpart of the second form given under the boson coherent state, which was discussed in (A).

In the form (iii) with \( t = 1/2 \), we investigate the behavior of \( N_0/N_\circ \) as a function of \( |w|^2 \) more carefully:

\[
N_0/N_\circ = \frac{1}{(1 - |w|^2) - (2s + 1)(|w|^2)^2/[1 - (|w|^2)^2s]}.
\] (6·17)

Then, the behavior of \( N_0/N_\circ \) for three regions is given as

\[
\begin{align*}
N_0/N_\circ & = \begin{cases} 
1/(1 - |w|^2), & (|w|^2 < 1) \\
|s|, & (|w|^2 = 1) \\
2s/|w|^2, & (|w|^2 > 1)
\end{cases}
\end{align*}
\] (6·18)

The following form satisfies the behavior (6·18) approximately:

\[
N_0/N_\circ = \begin{cases} 
1/[1 - |w|^2 + (|w|^2)^2/s], & (|w|^2 \leq 1) \\
1/[1 - (\sqrt{1 + 4/s} + 1)/2 \cdot |w|^2] + s(\sqrt{1 + 4/s} + 3)/2 \cdot 1/|w|^2, & (|w|^2 \geq 1)
\end{cases}
\] (6·19)

It is noted that \( N_0/N_\circ \) in Eq. (6·19) does not coincide with the exact derivative \( (N_0/N_\circ)' \) at \( |w|^2 = 1 \) which is equal to \( s(s - 2)/3 \). The derivatives \( (N_0/N_\circ)' \) in Eq. (6·19) at \( |w|^2 = 1 - \epsilon \) and \( |w|^2 = 1 + \epsilon \) are equal to \( s(s - 2) \) and \( (3s^2/8 + 1)/(s^2/8 + 1) \sqrt{1 + 4/s} \), respectively. Here, \( \epsilon \) is infinitely small positive parameter. Then, \( N_0/N_\circ \) is not smooth at \( |w|^2 = 1 \), but, the orders of \( (N_0/N_\circ)' \) are the same as that of the exact one. In this sense, the form (6·18) may be workable. With the use of Eq. (6·18), \((w^*, w)\) and \((b^*, b)\) are connected to each other through \( G(|b|^2) \) shown as

\[
G(|b|^2) = \begin{cases} 
2/[|b|^2 + 1 + \sqrt{(|b|^2 + 1)^2 - 4(|b|^2)^2/s}], & (|b|^2 \leq s) \\
(\sqrt{1 + 4/s} + 3)/2(\sqrt{1 + 4/s} + 1) - 1/|b|^2 + \sqrt{1 + 4/s} - 1)/2(\sqrt{1 + 4/s} + 1) - 1/(2s - |b|^2), & (s \leq |b|^2 \leq 2s)
\end{cases}
\] (6·20)

The above is an analytically expressed approximate form of \( G(|b|^2) \). It should be
noted that the behavior must be investigated in the two regions \(|w|^2 \leq s|b|^2 \leq 1\) and \(1 \leq |w|^2 \leq |b|^2 \leq 2s\), separately. Thus, any quantity is expressed in terms of \((b^*, b)\).

Next, we investigate the relations (6·1) and (6·2). It is easy to prove that the form (iii) satisfies the relations (6·1a) and (6·1c). For the relations (6·1b) and (6·2), we can prove that the behavior of \(|(\tilde{S}_z)_c|\) is the same as that of the case \(s = 1\). The condition \(|w|^2 = 0\) gives us \((\tilde{S}_z)_c = 0\) and \((\tilde{S}_0)_c = -s\) with \(\delta = \Delta = 0\). If \(|w|^2 = 1\), \(|(\tilde{S}_z)_c| \to s\sqrt{1 - \Delta}\) and \((\tilde{S}_0)_c = 0\). This case leads us to the situation that \(\delta\) and \(\Delta\) are the maximum with \(\delta = 1 - 3\pi^2/32(=0.07472)\) and \(\Delta = 1 - \pi^2/16(=0.3831)\). The condition \(|w|^2 \to \infty\) gives us \((\tilde{S}_z)_c = 0\) and \((\tilde{S}_0)_c = +s\) with \(\delta = \Delta = 0\). The above conclusion is obtained by using the exact relation between \((w^*, w)\) and \((b^*, b)\). For the above investigation, for example, the following formula with respect to sufficiently large \(s\) is useful:

\[
\sum_{n=0}^{2s} \sqrt{(n+1)(2s-n)} \to \int_{-1}^{2s} \sqrt{(x+1)(2s-x)} \, dx \to \pi s^2/2. \tag{6·21}
\]

Then, it may be concluded that the form (iii) with \(t = 1/2\) is also an approximate classical counterpart for the case where the quantity \(s\) is sufficiently large. The case \(t > 1/2\) may also be in the same situation as that in the case \(t = 1/2\).

§ 7. Discussion

Until the present, we have discussed the boson expansion of the \(su(2)\)-spin system in three forms and their \(c\)-number counterparts. In these three forms, mainly, we were concerned with the set \(\tilde{S}_{z,0}\). In this section, we will sketch the boson representation of \(\tilde{B}_{\pm}\) and \(\tilde{T}_{z,0}\) and their \(c\)-number counterparts.

Let us start in the case of \(\tilde{B}_+\). In the boson space defined in § 3, \(\tilde{B}_+\) are transformed to the following forms:

\[
\tilde{\mathcal{B}}_+ = U\tilde{B}_+ U^* = \sum_{n=0}^{2s-1} \sqrt{n+1} |n+1\rangle \langle n|, \quad \tilde{\mathcal{B}}_- = U\tilde{B}_- U^* = \sum_{n=0}^{2s-1} \sqrt{n+1} |n\rangle \langle n+1|. \tag{7·1}
\]

In the same technique as that adopted in the case of \(\tilde{S}_{z,0}\), \(\tilde{B}_\pm\) can be expressed as

\[
\tilde{B}_+ = \tilde{P} \cdot \tilde{B}_+ \cdot \tilde{P}, \quad \tilde{B}_- = \tilde{b}^* \quad \text{and} \quad \tilde{B}_+ = \tilde{b}. \tag{7·2}
\]

The operator \(\tilde{B}_+ \), \(\tilde{B}_-\) are nothing but the boson operators. We can see that if \(s\) is sufficiently large, \(\tilde{P} \to 1\) and we have

\[
(\tilde{B}_+, \tilde{B}_-) \to (\tilde{b}^*, \tilde{b}). \tag{7·4}
\]

This implies that in the case of sufficiently large \(s\), \(\tilde{B}_\pm\) may be regarded as the boson operators. However, if \(s\) is not so large, for example, if \(s = 1/2\), we have

\[
\tilde{B}_+ = \tilde{b}^* |0\rangle \langle 0|, \quad \tilde{B}_- = |0\rangle \langle 0| \tilde{b}. \tag{7·5}
\]

The above relations tell us that \(\tilde{B}_\pm\) cannot be regarded as the boson operators, but the fermion-like operators from the following anti-commutation relation:
\[
\{\hat{B}_-, \hat{B}_+\} = |0\rangle\langle 0| + \hat{b}^* |0\rangle\langle 0| \hat{b} = \hat{P} \quad \text{for} \quad s = 1/2.
\]

The above suggests to us that the existence of the projection operator \(\hat{P}\) in Eq. (7·2) is quite important. In order to demonstrate this fact, we will calculate the expectation values of \(\hat{B}_\pm\), i.e., the c-number counterparts of \(\hat{B}_\pm\) for the wave packet \(|c\rangle\) in the form (ii):

\[
(\hat{B}_+)_c = w^* N_0 / N_0 = b^* \sqrt{1 - L} \quad , \quad (\hat{B}_-)_c = w N_0 / N_0 = b \sqrt{1 - L}.
\]

Here, \(L\) is given in Eq. (5·3a). If \(s = 1/2\), \((\hat{B}_\pm)_c\) are reduced to

\[
(\hat{B}_+)_c = b^* \sqrt{1 - |b|^2} \quad , \quad (\hat{B}_-)_c = b \sqrt{1 - |b|^2}.
\]

On the other hand, if \(s\) is sufficiently large, we have the following relations:

\[
(\hat{B}_+)_c \rightarrow b^*, \quad (\hat{B}_-)_c \rightarrow b.
\]

The relations (7·8) tell us that the behavior of \((\hat{B}_\pm)_c\) in the case where \(s\) is small is quite different from that of \((\hat{B}_\pm)_c\) with sufficiently large \(s\). It is in the same situation as that in the HP expansion of \(\hat{S}_\pm\) for \(s = 1/2\).

Next, we discuss the case of \(\hat{T}_\pm, 0\). In the boson space, \(\hat{T}_{\pm, 0}\) are transformed to

\[
\hat{T}_+ = U \hat{T}_+ U^+ = \sum_{n=0}^{2n+1} \sqrt{(n+1)(n+2t)} |n+1\rangle\langle n|,
\]
\[
\hat{T}_- = U \hat{T}_- U^+ = \sum_{n=0}^{2n+1} \sqrt{(n+1)(n+2t)} |n\rangle\langle n+1|,
\]
\[
\hat{T}_0 = U \hat{T}_0 U^+ = \sum_{n=0}^{2n} (n+t) |n\rangle\langle n|.
\]

The above form is rewritten as

\[
\hat{T}_{\pm, 0} = \hat{P} \cdot \hat{T}_{\pm, 0} \cdot \hat{P},
\]

\[
\hat{T}_+ = b^* \sqrt{2t + b^* b} \quad , \quad \hat{T}_- = b^* b \quad , \quad \hat{T}_0 = b^* b^* + t.
\]

We can prove that the set \(\hat{T}_{\pm, 0}\) composes the \(su(1, 1)\)-algebra with the magnitude \(t = 1/2, 1, 3/2, \cdots\), i.e., the relations (7·12) give us the HP representation of the \(su(1, 1)\)-spin with the magnitude \(t\), which is parallel to the HP representation of the \(su(2)\)-spin. If \(s\) is sufficiently large, \(P \rightarrow 1\) and we have

\[
\hat{T}_{\pm, 0} \rightarrow \hat{T}_{\pm, 0}.
\]

However, if \(s = 1/2\), \(\hat{T}_{\pm, 0}\) are reduced to

\[
\hat{T}_+ = \sqrt{2t} \cdot \hat{b}^* |0\rangle\langle 0| \quad , \quad \hat{T}_- = \sqrt{2t} \cdot |0\rangle\langle 0| \hat{b},
\]
\[
\hat{T}_0 = t \cdot |0\rangle\langle 0| + (t+1) \cdot \hat{b}^* |0\rangle\langle 0| \hat{b}.
\]

The set \(\hat{T}_{\pm, 0}\) in the case \(s = 1/2\) cannot be regarded as composing the \(su(1, 1)\)-algebra. Rather, \(\hat{T}_s\) plays a role of the fermion-like operators in the same sense as that in the case of \(\hat{B}_s\). The c-number counterparts of \(\hat{T}_{\pm, 0}\) for the state \(|c\rangle\) in the form (iii) can be expressed as...
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\[ (\hat{T}_+)_c = w^*N_0/N_0 = b^*\sqrt{2t+|b|^2}\cdot\sqrt{1-L} , \]
\[ (\hat{T}_-)_c = wN_0/N_0 = b\sqrt{2t+|b|^2}\cdot\sqrt{1-L} , \]
\[ (\hat{T}_0)_c = |w|^2\cdot N_0/N_0 + t = |b|^2 + t . \] (7.15)

The quantity $L$ is given in Eq. (5.10a). Two extreme cases are expressed explicitly as follows:
\[ (\hat{T}_+)_c = \frac{b^*}{2t}2t + |b|^2 , \quad (\hat{T}_-)_c = \frac{b}{2t}2t + |b|^2 , \]
\[ (\hat{T}_0)_c = |b|^2 + t . \quad (s=1/2) \] (7.16)
\[ (\hat{T}_+)_c = b^*\sqrt{2t+|b|^2} , \quad (\hat{T}_-)_c = b\sqrt{2t+|b|^2} , \]
\[ (\hat{T}_0)_c = |b|^2 + t . \quad (s: \text{sufficiently large}) \] (7.17)

The behaviors of the above forms are completely different from each other and similar to those of $\hat{T}_{\pm,0}$. The $c$-number counterpart (7.17) is identical with that given in the frame of the $su(1,1)$-algebra.

As a final problem, let us investigate coherent nature of the wave packets $|c\rangle$ in the three forms with respect to $\hat{B}_-$, which behaves as boson annihilation operator for the case of sufficiently large $s$. Through this investigation, we can understand the role of the parameter $t$. The operation of $\hat{B}_-$ on the state $|c\rangle$ leads us to the following form:

\[ \hat{B}_-|c\rangle = w(\sqrt{N_0})^{-1}\sum_{s_0=-\infty}^{\infty} \sqrt{s+s_0+1} C(s, s_0+1) w^{s+s_0}|s, s_0\rangle . \] (7.18)

For each case, we have

(i) \[ \hat{B}_-|c\rangle = \sqrt{2s} \cdot w||c\rangle - (\sqrt{N_0})^{-1} C(s, s) w^{2s}|s, s\rangle \]
\[ - \sum_{k=1}^{2s-1} [\sqrt{2s}(\sqrt{2s-k+1} + \sqrt{2s-k})]^{-1} \]
\[ \times (\sqrt{N_0})^{-1}\sum_{s_0=-\infty}^{\infty} C(s, s_0) w^{s+s_0}|s, s_0\rangle , \] (7.19)

(ii) \[ \hat{B}_-|c\rangle = w||c\rangle - (\sqrt{N_0})^{-1} C(s, s) w^{2s}|s, s\rangle \}
\[ - \sum_{k=1}^{2s-1} [\sqrt{2t}(\sqrt{2t+k-1} + \sqrt{2t+k})]^{-1} \]
\[ \times (\sqrt{N_0})^{-1}\sum_{s_0=-\infty}^{\infty} C(s, s_0) w^{s+s_0}|s, s_0\rangle . \] (7.20)

(iii) \[ \hat{B}_-|c\rangle = \sqrt{2t} \cdot w||c\rangle - (\sqrt{N_0})^{-1} C(s, s) w^{-2s}|s, s\rangle \]
\[ + \sum_{k=1}^{2s-1} [\sqrt{2t}(\sqrt{2t+k-1} + \sqrt{2t+k})]^{-1} \]
\[ \times (\sqrt{N_0})^{-1}\sum_{s_0=-\infty}^{\infty} C(s, s_0) w^{s+s_0}|s, s_0\rangle . \] (7.21)

If $s$ is sufficiently large, Eqs. (7.19) and (7.20) are reduced to

\[ \hat{B}_-|c\rangle \sim b|c\rangle . \] (7.22)

The above relation tells us that the wave packets in the forms (i) and (ii) are coherent states for $\hat{B}_-$ in the case where $s$ is sufficiently large. However, in the form (iii), we
must add one more condition: The parameter $t$ is also sufficiently large. Then, we get the relation (7·22) in the form (iii). Further, under the same condition as that in the above cases, we have

$$\langle c|\tilde{D}|c\rangle \sim 0.$$  (7·23)

Here, $\tilde{D}$ is defined in Eq. (2·11). The relation (7·23) tells us

$$\langle c|\tilde{B}_-\cdot \tilde{B}_+|c\rangle \sim 1.$$  (7·24)

With the aid of the relations (7·22) and (7·24), the wave packets (i) and (ii) obey the minimum uncertainty for $\hat{q} = (\hat{B}_+ + \hat{B}_-)/\sqrt{2}$ and $\hat{p} = i(\hat{B}_+ - \hat{B}_-)/\sqrt{2}$, if $s$ is sufficiently large. However, in the case of the form (iii), the minimum uncertainty appears only in the case where $t$ is sufficiently large. Depending on the value of $t$, the uncertainty relation changes. This situation realizes in the case of the $su(1,1)$-algebraic model.

In Ref. 10), we learned that the $su(1,1)$-algebra enables us to describe the damped and amplified harmonic oscillation in relation to thermal effects. Therefore, in the frame of our present form, we can expect to describe thermal effects of various systems, for example, under the interaction between the $c$- and the $d$-part

$$\tilde{V}_{su(2)} = -\gamma \cdot i(\tilde{T}_+ - \tilde{T}_-) .$$  (7·25)

The above is analogous to the form in the $su(1,1)$-algebraic model. In a forthcoming paper, we will investigate this problem.

References

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