Coupled Integrable Systems Associated with a Polynomial Spectral Problem and Their Virasoro Symmetry Algebras

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(Received March 26, 1996)

An isospectral hierarchy of commutative integrable systems associated with a polynomial spectral problem is proposed. The resulting hierarchy possesses a recursion structure controlled by a hereditary operator. The nonisospectral flows generate the time first order dependent symmetries of the isospectral hierarchy, which constitute Virasoro symmetry algebras together with commutative symmetries.

§ 1. Introduction

The study of integrable systems has been approached from various points of view. Mathematically, a very successful analytic method is the inverse scattering transform (IST), which has become the standard method for solving nonlinear systems. This method is based mainly upon two linear spectral problems, i.e., a Lax pair, connected with nonlinear systems. At the beginning of development of IST, it is required that the eigenvalues of the associated spectral operator be left invariant when the potential evolves according to the nonlinear systems. Subsequent investigation (see, for example, Ref. 2) has shown that the nonisospectral nonlinear systems associated with Lax pairs are also important and may be yet solved by IST.

With the help of Lax pairs, some integrable systems of nonlinear differential and difference equations important in physics fields have been generated. A good skeleton to do so is Sato theory, in which commuting flows and \( \tau \)-functions are presented simultaneously. However, Sato theory is a general theory, though it involves many integrable systems of soliton equations as reductions, and thus it is still important to find concrete specific integrable systems, especially coupled integrable systems. In addition, the problem of constructing integrable systems itself is also interesting both from a mathematical as well as physical point of view.

In the present paper, we would like to present a class of isospectral \((\lambda_{tm}=0)\) coupled integrable systems by introducing a special spectral problem with the polynomial form of the spectral parameter \(\lambda\). All of these systems can be solved by following the standard IST. A simple calculation allows us to characterize them iteratively so that the whole integrable hierarchy may be written in a nice and neat form. Nonisospectral \((\lambda_{tm}=\lambda^{m+1})\) flows generate a centerless Virasoro master symmetry algebra for each system in the isospectral hierarchy. Therefore the resulting isospectral hierarchy provides us with some typical integrable systems of soliton equations. However, we have not been able to find Hamiltonian structures and to give rise to usual Darboux transformations for the proposed isospectral hierarchy.
§ 2. Isospectral integrable systems

Let us start from the spectral problem
\[ \phi_x = U\phi = U(u, \lambda)\phi, \quad \phi = (\phi_1, \phi_2)^T, \quad u = (v_0, v_1, \ldots, v_{q-1})^T \]  
with the spectral operator \( U \), being of polynomial form in \( \lambda \):
\[ U = Q\sigma_1 + i\sigma_2 + \sigma_3 = \begin{pmatrix} 1 & Q+1 \\ Q-1 & -1 \end{pmatrix}, \quad Q = Q(u, \lambda) = \sum_{i=0}^{q} v_i \lambda^i, \quad v_q = -1. \]  

Here \( q \) is an arbitrary integer, and the \( \sigma_i \) are 2x2 Pauli matrices. This spectral problem is a generalization of one appearing in Ref. 6) and has a multi-component potential \( u \). Our discussion here will be focused on constructing integrable systems.

In order to derive the related hierarchy of isospectral \( (\lambda_{tm}=0) \) systems, we introduce the auxiliary problem associated with (2.1):
\[ \phi_x = V\phi = V(u, \lambda)\phi, \]  
\[ V = a\sigma_1 + b\sigma_2 + c\sigma_3 = \begin{pmatrix} c & a+b \\ a-b & -c \end{pmatrix}. \]

It is readily worked out that
\[ [U, V] = (2b-2c)\sigma_1 + (2a-2Qc)i\sigma_2 + (2a-2Qb)\sigma_3. \]

Therefore we see that the compatibility condition \( U_t - V_x + [U, V] = 0 \) of the Lax pair (2.1) and (2.2) becomes
\[ \begin{cases} Q_x - a_x + 2b - 2c = 0, \\ -b_x + 2a - 2Qc = 0, \\ -c_x + 2a - 2Qb = 0. \end{cases} \]

The equalities (2.3b) and (2.3c) demand equivalently
\[ \begin{cases} a = \frac{1}{4} (b + c)_x + \frac{1}{2} Q(b + c), \\ (b - c)_x - 2Q(b - c) = 0. \end{cases} \]

From the second equality above, we obtain \( b-c = [(b(x_0) - c(x_0)] \exp(\int_{x_0}^{x} 2Qdx) \). We are interested in the most simple case, \( b = c \). For this case, we have \( a = (1/2)c_x + Qc \), and thus (2.3a) reads
\[ Q_t = a_x = \partial \left( \frac{1}{2} \partial + Q \right) c. \]

Assume that
\[ V = V^{(m)} = \left( \frac{1}{2} c_x^{(m)} + Qc^{(m)} \right) \sigma_1 + c^{(m)} i\sigma_2 + c^{(m)} \sigma_3, \quad c^{(m)} = \sum_{j=0}^{m} c_j \lambda^{m-j}, \quad m \geq 0. \]
Further for the sake of convenience, set

\[ R_0 = \frac{1}{2} \partial + v_0, \quad R_i = v_i, \quad 1 \leq i \leq q. \tag{2.6} \]

In this way, we have

\[ \partial \left( \frac{1}{2} \partial + Q \right) c^{(m)} = \partial \sum_{j=0}^{m} R_j \lambda^j \sum_{m-j=0}^{m-j} c_{j+m-j} \lambda^{m-j} = \partial \sum_{j=0}^{q-1} (\sum_{j=0}^{q-1} R_j \lambda^{j+m-i}) \lambda^{i} + \partial \sum_{i=0}^{q+m} (\sum_{j=0}^{q} R_j \lambda^{j+m-i}) \lambda^{i}, \tag{2.7} \]

where \( c_j = 0, j < 0 \). In order to generate isospectral integrable systems from (2.4), it is required that

\[ \partial \sum_{j=0}^{q} R_j \lambda^{j+m-i} = 0, \quad q \leq i \leq q + m. \]

These equations have a special solution,

\[ \sum_{j=0}^{q} R_j \lambda^{j+m-i} = \begin{cases} -1, & i = q + m, \\
0, & q \leq i \leq q + m - 1. \end{cases} \]

Noting that \( R_q = -1 \), this relation can be satisfied by choosing

\[ c_0 = 1, \quad c_j = \sum_{i=0}^{q-1} R_i \lambda^{i+j-q}, \quad 1 \leq j \leq \infty. \tag{2.8} \]

It follows that

\[ c_1 = v_{q-1}, \quad c_2 = v_{q-2} + v_{q-1}^2. \]

A general solution \( \{ c_j \} \) is a linear combination of the solution (2.8) with arbitrary functions of \( t \) as coefficients. This could not lead to new integrable systems in nature.

At this stage, we see from (2.7) that (2.4) engenders equivalently the isospectral \((\lambda_{tm} = 0)\) integrable hierarchy

\[ u_{tm} = K_m = \partial (R_0 c_m, R_0 c_{m-1} + R_1 c_m, \ldots, \sum_{j=0}^{q-1} R_j \lambda^{j+m-(q-1)})^T \]

\[ = \partial (R_0 c_m, R_0 c_{m-1} + R_1 c_m, \ldots, \sum_{j=0}^{q-2} R_j \lambda^{j+m-(q-2)}, c_{m+1})^T, \quad m \geq 0, \tag{2.9} \]

and the isospectral hierarchy possesses the Lax pairs

\[ \phi_x = U \phi = (Q \sigma_1 + i \sigma_2 + \sigma_3) \phi = \begin{pmatrix} 1 & Q+1 \\ Q-1 & -1 \end{pmatrix} \phi, \tag{2.10a} \]

\[ \phi_{tm} = V^{(m)} \phi = \left( \frac{1}{2} c^{(m)} + Q c^{(m)} \right) \sigma_1 + c^{(m)} i \sigma_2 + c^{(m)} \sigma_3 \phi \]
\[ \begin{pmatrix}
    \frac{1}{2}c_x^{(m)} + c^{(m)}(Q+1) & 0 \\
    \frac{1}{2}c_x^{(m)} + c^{(m)}(Q-1) & -c^{(m)}
\end{pmatrix} \phi, \quad (2.10b) \]

where \( c^{(m)} = \sum_{j=0}^{m} c_j \lambda^{m-j} \), with the \( c_j \) defined by (2.8). To show that the isospectral hierarchy (2.9) exhibits a kind of hereditary structure, we define an integro-differential operator

\[ \Phi = \begin{pmatrix}
    0 & 0 & \cdots & 0 & P_0 \\
    1 & 0 & \cdots & 0 & P_1 \\
    0 & 1 & \cdots & 0 & P_2 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & 1 & P_{q-1}
\end{pmatrix}, \quad P_0 = \frac{1}{2} \partial + \partial v_0 \partial^{-1}, \quad P_i = \partial v_i \partial^{-1}, \quad 1 \leq i \leq q. \quad (2.11) \]

Since, as is easily seen, \( P_q = -1, \ P_i \partial = \partial R_i, \ 0 \leq i \leq q \), we obtain

\[ u_{tm} = K_m = \Phi K_{m-1} = \cdots = \Phi^n K_0 = \Phi^n u_x, \quad m \geq 0. \quad (2.12) \]

That is to say,

\[ \begin{pmatrix}
    u_{0tm} \\
    u_{1tm} \\
    u_{2tm} \\
    \vdots \\
    u_{q-1,tm}
\end{pmatrix} = \begin{pmatrix}
    0 & 0 & \cdots & 0 & \frac{1}{2} \partial + \partial v_0 \partial^{-1} \\
    1 & 0 & \cdots & 0 & \partial v_1 \partial^{-1} \\
    0 & 1 & \cdots & 0 & \partial v_2 \partial^{-1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & 1 & \partial v_{q-1} \partial^{-1}
\end{pmatrix} \begin{pmatrix}
    u_{0x} \\
    u_{1x} \\
    u_{2x} \\
    \vdots \\
    u_{q-1,x}
\end{pmatrix}, \quad m \geq 0. \quad (2.13) \]

Noting the special form \( \partial v_i \partial^{-1} \) in \( P_i \), we see from the nice form (2.13) itself that every system in the hierarchy (2.13) is local in spite of the integro-differential character of the operator \( \Phi \). The multi-component integrable systems in (2.13) are similar, in appearance, to coupled KdV systems in Ref. 7). We shall show in § 4 that the flows generated by (2.13) commute with each other. But we shall also see later that these flows have different characteristics from those of coupled KdV systems.

§ 3. Nonisospectral integrable systems

Let us now turn to the deduction of nonisospectral (\( \lambda_{tn} = \lambda^{n+1} \)) integrable systems associated with the spectral problem (2.1). In order to apply the generating scheme in Ref. 8), we should verify that the characteristic operator equation

\[ [Q, U] + \partial_x = U'[\Phi X] - \lambda U'[X] \quad (3.1) \]

has solutions \( \Omega = \Omega(X) \) for any given vector field \( X = (X_1, \cdots, X_q)^T \) and need to search for a pair of solutions \( B_0 \) and \( \Phi_0 \) to a certain key operator equation

\[ [B_0, U] + B_{0x} = U'[\Phi_0] + \lambda^k U_x = U'[\Phi_0] + \lambda U_x, \quad U_x = \frac{\partial U}{\partial \lambda}. \quad (k=1) \quad (3.2) \]
Here the key operator equation with \( k=1 \) is imposed. In view of the former deduction of isospectral integrable systems, we may suppose that \( \Omega \) and \( B_0 \) possess the forms

\[
\Omega = \left( \begin{array}{cc}
\frac{1}{2} c(\Omega) + Q c(\Omega) & c(\Omega) i \sigma_2 + c(\Omega) \sigma_3 \\
c(\Omega) & -c(\Omega)
\end{array} \right),
\]

\[
B_0 = \left( \begin{array}{cc}
\frac{1}{2} c(B_0) + Q c(B_0) & c(B_0) i \sigma_2 + c(B_0) \sigma_3 \\
c(B_0) & -c(B_0)
\end{array} \right),
\]

where \( c(\Omega) \) and \( c(B_0) \) are two functions to be determined. Set \( \Phi X = ((\Phi X)_0, \ldots, (\Phi X)_q)^T, \quad \Phi \sigma = (\Phi \sigma_0, \ldots, \Phi \sigma_q)^T \). Then we have

\[
U'[\Phi X] - \lambda U'[X] = \sum_{i=0}^{q} [(\Phi X)_i - \lambda X_i] \lambda^{i-1} \sigma_i = \left[ \sum_{i=0}^{q} (P_i X_q) \lambda^{i} - X_q \lambda^q \right] \sigma_i,
\]

\[
U'[\Phi \sigma] + \lambda U[\Phi \sigma] = \left[ \sum_{i=0}^{q} (\Phi \sigma_0 + i iv_i) \lambda^{i} \right] \sigma_i.
\]

Now we easily find that \( c(\Omega) = 1/2 \partial^{-1} X_q \) and \( c(B_0) = qx \). Therefore we obtain the following solution to (3·1)

\[
\Omega = \left( \begin{array}{cc}
\frac{1}{4} X_q + \frac{1}{2} Q \partial^{-1} X_q & \frac{1}{2} \partial^{-1} X_q i \sigma_2 + \frac{1}{2} \partial^{-1} X_q \sigma_3 \\
\frac{1}{2} \partial^{-1} X_q & \frac{1}{4} X_q + \frac{1}{2} (Q + 1) \partial^{-1} X_q
\end{array} \right),
\]

\[
(3·3)
\]

and the following pair of solution to (3·2)

\[
B_0 = \left( \begin{array}{cc}
\frac{1}{2} q + Q qx \sigma_1 + qx i \sigma_2 + qx \sigma_3 \\
qx & \frac{1}{2} q + qx (Q + 1)
\end{array} \right),
\]

\[
\left( \begin{array}{c}
\frac{1}{2} q + qx (Q - 1) \sigma_1 - qx \\
qx
\end{array} \right).
\]

\[
ge_0 = (q v_0, (q - 1) v_1, \ldots, v_{q-1})^T + q x (v_0, v_1, \ldots, v_{q-1})^T.
\]

Now taking full advantage of the result in Ref. 8), we know that the Lax pairs...
\[
\begin{align*}
\phi_x &= U\phi, \quad \lambda_t = \lambda^{t+1}, \\
\phi_t &= W^{(n)}\phi, \quad W^{(n)} &= \lambda^n B_0 + \sum_{j=1}^{n} \lambda^{-j} \Omega(\Omega^{-1} g_0) 
\end{align*}
\]

lead to the following nonisospectral \((\lambda_t = \lambda^{t+1})\) hierarchy of multi-component integrable systems

\[
u_{tn} = \rho_n = \Phi^ng_0, \quad n \geq 0,
\]

or,

\[
\begin{pmatrix}
v_{0tn} \\
v_{1tn} \\
v_{2tn} \\
\vdots \\
v_{q-1,tn}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & \cdots & 0 & \frac{1}{2} \partial + \partial v_0 \partial^{-1} \\
1 & 0 & \cdots & 0 & \partial v_1 \partial^{-1} \\
0 & 1 & \cdots & 0 & \partial v_2 \partial^{-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & \partial v_{q-1} \partial^{-1}
\end{pmatrix}^n
\begin{pmatrix}
q v_0 \\
(q-1)v_1 \\
(q-2)v_2 \\
\vdots \\
v_{q-1}
\end{pmatrix} + q x
\begin{pmatrix}
v_{0x} \\
v_{1x} \\
v_{2x} \\
\vdots \\
v_{q-1,x}
\end{pmatrix},
\]

\(n \geq 0\).

These nonisospectral systems are nonlocal, except for the case \(n=0\). In the next section, we shall show that (3·7) yields a hierarchy of common first degree master-symmetries, and thus Virasoro symmetry algebras are generated for the isospectral integrable systems (2·13).

\section*{§ 4. Virasoro symmetry algebras}

In what follows, we would like to give Virasoro \(r\)-symmetry algebras for the isospectral hierarchy (2·13). To this end, let us first prove that \(\Phi\) in (2·11) is a hereditary symmetry\(^9\) (or the Nijenhuis operator\(^10\)). Namely, we must verify that \(\Phi'(\Phi K) S - \Phi\Phi' [K] S\) is symmetric with respect to two vector \(K\) and \(S\). In fact, for \(K = (K_1, \cdots, K_q)^T, S = (S_1, \cdots, S_q)^T\), we can compute that

\[
\Phi'(\Phi K) S - \Phi\Phi' [K] S = (\cdots, \partial(P K_q) \partial^{-1} S_q - P_q \partial K_q \partial^{-1} S_q, \cdots)^T, \quad 0 \leq i \leq q - 1,
\]

\[
\partial(P K_q) \partial^{-1} S_q = P_q \partial K_q \partial^{-1} S_q
\]

\[
= -\frac{1}{2} \delta_{i0} (K_{ax} S_q + K_q S_{ax}) + v_{ix} (\partial^{-1} K_q)(\partial^{-1} S_q) + v_{ix} [K_q (\partial^{-1} S_q) + (\partial^{-1} K_q) S_q],
\]

\(\delta_{i0}\) Kronecker symbol,

whereupon we see that \(\Phi\) is a hereditary symmetry, indeed. Second, a direct computation can show that

\[
[K_q, g_0] = q K_0, \quad L_{K_q} \Phi = 0, \quad L_{g_0} \Phi = \Phi,
\]

where the commutator of vector fields is defined by \([K, S] = K[S] - S[K]\), and the Lie derivative by \(L_K \Phi = \Phi'[K] - [K', \Phi]\) (see Ref. 11 for more information). Based on the basic result in Ref. 11), these two properties permit us to conclude that the \(s\)th isospectral coupled integrable system \(u_s = \Phi^s u_s\) has two hierarchies of \(K\)-symmetries
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(time independent symmetries) and $\tau$-symmetries (time first-order dependent symmetries),

\[ K_m = \Phi^m u_x, \quad m \geq 0; \quad \tau_n^{(s)} = (s + q) t K_{n+s} + \rho_n, \quad n \geq 0, \quad (4.2) \]

and that these symmetries constitute a $\tau$-symmetry algebra:

\[
\begin{align*}
[K_m, K_n] &= 0, \quad m, n \geq 0, \quad (4.3a) \\
[K_m, \tau_n^{(s)}] &= (m + q) K_{m+n}, \quad m, n \geq 0, \quad (4.3b) \\
[\tau_m^{(s)}, \tau_n^{(s)}] &= (m - n) \tau_{m+n}^{(s)}, \quad m, n \geq 0. \quad (4.3c)
\end{align*}
\]

This symmetry algebra also shows that $\rho_n, n \geq 0$, defined by (3.6), are the common first degree master-symmetries of the integrable hierarchy (2.13).

§ 5. Conclusion and remarks

We have constructed a hierarchy of coupled integrable systems (2.13), and each system in the hierarchy (2.13) has a $\tau$-symmetry algebra (4.3), in which master-symmetries are generated by nonisospectral flows of the spectral problem (2.1). The first nonlinear system in the integrable hierarchy (2.13) reads

\[
\begin{align*}
\frac{\partial}{\partial t} v_0 &= \frac{1}{2} v_{q-1,xx} + (v_0 v_{q-1})_x, \\
\frac{\partial}{\partial t} v_1 &= v_0, x + (v_1 v_{q-1})_x, \\
&\vdots \\
\frac{\partial}{\partial t} v_{q-2,1} &= v_{q-3,xx} + (v_{q-2} v_{q-1})_x, \\
\frac{\partial}{\partial t} v_{q-1,1} &= v_{q-2,xx} + (v_{q-1}^2)_x. \quad (5.1)
\end{align*}
\]

This resembles coupled Burgers equations, but it seems to us that it could not be simplified to Burgers equations. Also, it might be very difficult to find a reduction of the integrable hierarchy (2.13) to scalar Burgers equations $w_{tm} = (1/2 \partial + \partial w \partial^{-1})^m w_x, m \geq 0$.

When $q = 2$, the resulting isospectral hierarchy (2.13) becomes

\[
\begin{pmatrix}
\frac{\partial}{\partial t} v_0 \\
\frac{\partial}{\partial t} v_1
\end{pmatrix}
= \Phi^{m}
\begin{pmatrix}
\frac{\partial}{\partial x} v_0 \\
\frac{\partial}{\partial x} v_1
\end{pmatrix}, \quad \Phi = \begin{pmatrix}
0 & \frac{1}{2} \partial + \partial v_0 \partial^{-1} \\
1 & \partial v_1 \partial^{-1}
\end{pmatrix}, \quad m \geq 0. \quad (5.2)
\]

The first two nonlinear systems of the hierarchy (5.2) are as follows:

\[
\begin{align*}
\frac{\partial}{\partial t} v_0 &= \frac{1}{2} v_{1,xx} + (v_0 v_1)_x, \\
\frac{\partial}{\partial t} v_1 &= v_0 + 2 v_1 v_1 x, \quad (5.3) \\
\frac{\partial}{\partial t} v_{0,2} &= \frac{1}{2} v_{0,xx} + (v_1 v_1 x + v_0^2 + v_0 v_1^2)_x, \\
\frac{\partial}{\partial t} v_{1,2} &= \frac{1}{2} v_{1,xx} + (2 v_0 v_1 + v_1^3)_x. \quad (5.4)
\end{align*}
\]
The first system has already appeared in Ref. 6) and can also be solved by means of the WTC Painlevé analysis method. Moreover, the system (5·3) can be expressed as

$$\begin{pmatrix} v_{0t} \\ v_{1t} \end{pmatrix} = J_1 G_1 = \begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix} \begin{pmatrix} \frac{1}{2} v_{1x} + v_0 v_1 \\ v_0 + v_1^2 \end{pmatrix} = J_2 G_2 = \begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix} \begin{pmatrix} v_0 + v_1^2 \\ \frac{1}{2} v_{1x} + v_0 v_1 \end{pmatrix}. \quad (5·5)$$

Although $J_1$ and $J_2$ are both Hamiltonian operators, $G_1$ and $G_2$ are not gradients, i.e.,

$$G_i = \delta H_i / \delta u, \quad H_i = \int_0^1 u^T G_i(\lambda u) d\lambda, \quad u = (v_0, v_1)^T, \quad i=1,2.$$ 

Therefore the system (5·3) does not possess Hamiltonian structures with the form (5·5). With the same argument, it may be verified that the system (5·4) does not possess similar Hamiltonian structures either.

In general, the isospectral hierarchy (2·13) does not possess Hamiltonian structures with a simple Hamiltonian operator as above. This property is completely different from those of usual systems of soliton equations, for example, coupled KdV systems7) and WKI systems4), both of which possess Hamiltonian structures. Moreover, we conjecture that there exists a finite number of polynomial conserved densities for the hierarchy (2·13), and that each system in the hierarchy (2·13) is C-integrable (for definition, see Ref. 15). It is also interesting for us to check whether or not the hierarchy (2·13) may be derived from Sato theory3) as a reduction or may be transformed into a bilinear form16).

Acknowledgements

This work was supported by the Alexander von Humboldt Foundation of Germany, the National Natural Science Foundation of China, the Shanghai Science and Technology Commission of China, and Fok Ying-Tung Education Foundation of China. One (W.-X. Ma) of the authors would like to thank Professor B. Fuchssteiner and Dr. W. Oevel for valuable discussions at the Soliton Seminar at the University of Paderborn.

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