We demonstrate an equivalence between Dirac quantization and reduced phase space quantization. The equivalence of the two quantization methods determines the operator ordering of the Hamiltonian. Some examples of the operator ordering are shown in simple models.

§ 1. Introduction

In recent years several authors\(^1\) have discussed Dirac quantization and reduced phase space quantization. Their arguments are that the reduced phase space quantization and Dirac quantization may be different in the constraint system with a non-trivial metric. In order to clarify the problem, let us consider the simplest model as an example.

The Lagrangian is given by

\[ L = \frac{1}{2} \dot{x}^2 + \frac{f(x)}{2} (\dot{y} - \lambda)^2, \tag{1} \]

where \( \lambda \) is a Lagrange multiplier. There is a non-trivial metric here, \( f(x) \). This is not a field theory but quantum mechanics. The Hamiltonian of this system is

\[ H = \frac{1}{2} p_x^2 + \frac{1}{2 f(x)} p_y^2 + \lambda p_y, \tag{2} \]

and there are two constraints,

\[ p_y = 0, \tag{3} \]

\[ p_x \approx 0. \tag{4} \]

These are first-class constraints. We set \( p_y = 0 \) in the Hamiltonian before the quantization. Then the Hamiltonian reduces to

\[ H = \frac{1}{2} p_x^2, \tag{5} \]

and the Hamiltonian operator is

\[ \hat{H} = -\frac{1}{2} \hat{\partial}_x^2. \tag{6} \]

This is the reduced phase space quantization. The procedure of the reduced phase space quantization is to reduce first and then quantize.

In the case of Dirac quantization, the procedure is to quantize first and then reduce. The Hamiltonian in this model is defined on the two-dimensional space of \( x \)
and $y$ without a constraint term. To ensure the invariance under the coordinate transformation, the Hamiltonian operator is written as

$$\hat{H} = -\frac{1}{2\sqrt{f}}\partial_x \sqrt{f} \partial_x - \frac{1}{2\sqrt{f}}\partial_y \sqrt{f} \frac{1}{f} \partial_y,$$

(7)

where $\sqrt{f}$ is $\sqrt{\text{det}g_{\mu\nu}}$. The metric $g_{\mu\nu}$ is the two-dimensional metric of $x$-$y$ space. Since $p_y = (1/i)\partial_y \approx 0$, $y$ derivatives in the Hamiltonian operator are eliminated. Then the Hamiltonian operator in Dirac quantization is

$$\hat{H} = -\frac{1}{2\sqrt{f}}\partial_x \sqrt{f} \partial_x.$$

(8)

This is not the same as the result of the reduced phase space quantization. This is a problem of the inconsistency of the reduced phase space quantization and Dirac quantization.

In § 2 we show the equivalence of the two quantization methods. It is shown that the Hamiltonian operator of Dirac quantization should include a constraint term and be invariant under the three-dimensional coordinate transformation of $x$, $y$, and a configuration variable conjugate to the Lagrange multiplier.

In § 3 we discuss the problem of the operator ordering. If the Hamiltonian has a non-trivial metric,

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu,$$

(9)

the Hamiltonian operator may have a function of scalars like $R$, $R^{\mu\nu} R_{\mu\nu}$, $R^{\mu\nu\sigma\tau} R_{\mu\nu\sigma\tau}$, ... from the invariance of the coordinate transformation, in addition to the Laplacian

$$\hat{H} = -\frac{1}{2} \Delta + F(R, R^{\mu\nu} R_{\mu\nu}, R^{\mu\nu\sigma\tau} R_{\mu\nu\sigma\tau}, \ldots).$$

(10)

The Laplacian is indispensable from the invariance and derived from the discretized path integral. The additional function is called the quantum mechanical potential. The problem of operator ordering is to determine the quantum mechanical potential. Using the equivalence between the reduced phase quantization and Dirac quantization, we determine this potential in simple models.

Section 4 is devoted to conclusions and discussion.

§ 2. The reduced phase space quantization and Dirac quantization

Let us reconsider the Hamiltonian (2). We take the gauge condition $\lambda = 0$ to quantize this system in the path integral formalism. The Hamiltonian form path integral is given by

$$Z = \int d\mu \exp[iS],$$

$$d\mu = [dx dp, dy dp, d\pi d\lambda].$$
\[
S = \int dt p_x \dot{x} + p_y \dot{y} - \pi \dot{\lambda} - \frac{1}{2} p_x^2 - \frac{1}{2 f(x)} p_y^2 - \lambda p_y. \tag{11}
\]

\(\lambda\) is a momentum variable so that the sign of the gauge fixing term is negative. After partial integration, it becomes usual one. Since this gauge is an Abelian, we need not introduce any ghost. After the integration of \(\pi, \lambda, p_y\) and \(y\), the partition function becomes

\[
Z = \int d\mu' \exp[iS'],
\]

\[
d\mu' = [dx dp_x],
\]

\[
S' = \int dt p_x \dot{x} - \frac{1}{2} p_x^2. \tag{12}
\]

This is nothing but the partition function of a free particle. Then the Hamiltonian operator is Eq. (6). This implies that the Schrödinger picture corresponding to the stage of the path integral (12) is represented by reduced phase quantization. On the other hand, Dirac quantization is within the Schrödinger picture corresponding to the stage of the path integral (11). In Eq. (11) no variable is integrated and constraint variables are still alive. The symmetry of this path integral is the coordinate transformation of the whole configuration space including \(\pi\), which is a configuration variable conjugate to \(\lambda\). Therefore, the Hamiltonian operator should be made invariant under the three-dimensional coordinate transformation, not the two-dimensional one. Then the Hamiltonian operator is

\[
\hat{H} = \frac{-1}{2 \sqrt{g}} \partial_{\mu} \sqrt{g} g^{\mu \nu} \partial_{\nu}
\]

\[
= -\frac{1}{2} \partial_x^2 - \frac{1}{2} \partial_y \partial_x - \frac{1}{2} \partial_y \partial_\pi - \frac{1}{2} \partial_\pi \partial_y, \tag{13}
\]

where \(g_{\mu \nu}\) is the inverse of \(g^{\mu \nu}\) in the Hamiltonian. The original Lagrangian (1) has a singular metric. However, the gauge fixed Lagrangian which is made by the integration of momentum variables in Eq. (11) has a regular metric and it coincides with the inverse of \(g^{\mu \nu}\).

Using the constraint \(\bar{p}_y = (1/i) \partial_y \approx 0\), we obtain the Hamiltonian

\[
\hat{H} = -\frac{1}{2} \partial_x^2. \tag{14}
\]

This is the Dirac quantization and we obtain the same Hamiltonian operator with the reduced phase space quantization. This is natural because we start from the same path integral (11). This simplest example indicates that Dirac quantization and the reduced phase space quantization should coincide.

The naive Dirac quantization given in the introduction is made by the requirement that the Hamiltonian operator is invariant under the coordinate transformation of \(x\) and \(y\). In that case the constraint term is treated separately. However, under some coordinate transformation, the net Hamiltonian and the constraint term are
mixed. The naive Dirac quantization does not represent the symmetry correctly. This is the reason why the naive Dirac quantization is different from the reduced phase space quantization.

In the general case with many variables, we can propose that Dirac quantization and the reduced phase space quantization are equivalent because both quantizations are within the Schrödinger picture of different forms of the same path integral as before. We can determine the quantum mechanical potentials with this property.

§ 3. The operator ordering

Let us now consider the Lagrangian

\[ L = \frac{1}{2} \hbar(x) \dot{x}^2 + \frac{g(x)^2}{2f(x)} \left( \dot{y} - \frac{\lambda}{g(x)} \right)^2. \]  

This Lagrangian leads to the Hamiltonian

\[ H = \frac{1}{2\hbar(x)} p_x^2 + \frac{f(x)}{2g(x)^2} p_y^2 + \frac{1}{g(x)} \lambda p_y, \]  

and the constraints

\[ p_x \approx 0, \]
\[ p_y \approx \pi \approx 0, \]  

as before. Reduced phase space quantization causes the Hamiltonian to become

\[ H = \frac{1}{2\hbar(x)} p_x^2 \]  

due to the constraints. Then the Hamiltonian operator is

\[ \hat{H} = -\frac{1}{2\sqrt{\hbar}} \frac{\partial_x}{\sqrt{\hbar}} - \frac{1}{\sqrt{\hbar}} \frac{\partial_x}{\partial_x}. \]  

While in Dirac quantization, we first consider the Hamiltonian in three dimensions. The invariance of the three-dimensional coordinate transformation allows the Hamiltonian operator to assume the form

\[ \hat{H} = -\frac{1}{2} \Delta + F(R, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu\sigma}R_{\mu\nu\sigma}, \ldots) \]

\[ = -\frac{1}{2g\sqrt{\hbar}} \frac{\partial_x}{g\sqrt{\hbar}} \frac{\partial_x}{\partial_x} - \frac{1}{2g\sqrt{\hbar}} \frac{\partial_y}{g} \frac{\partial_y}{\partial_y} - \frac{1}{2g\sqrt{\hbar}} \frac{\partial_y}{g\sqrt{\hbar}} \frac{\partial_y}{\partial_x} \]

\[ - \frac{1}{2g\sqrt{\hbar}} \frac{\partial_x}{g\sqrt{\hbar}} \frac{\partial_y}{g\sqrt{\hbar}} + F \]  

because in this model \( R, \ldots \) are not zeros. \( g_{\mu\nu} \) is an inverse matrix of \( g^{\mu\nu} \) of the Hamiltonian and is the same as that of the gauge fixed Lagrangian as before. The constraint \( \bar{p}_y = (1/i)\partial_y \approx 0 \) causes the Hamiltonian to take a simple form,
The inner product for the reduced phase quantization is defined as

$$\int \sqrt{h} \, d\xi \, \Psi^\ast_r \Psi_r ,$$

where $\Psi_r$ is a wave function of the reduced phase quantization. On the other hand, for Dirac quantization, it is written as

$$\int \sqrt{h} \, g \, dx \, dy \, d\pi \Psi^\ast \Psi_D ,$$

where $\Psi_D$ is a Dirac quantized wave function. Since the constraint $\pi \approx 0$ implies $\int d\pi \Psi_B^\ast \pi \Psi_D = 0$, $\Psi_B^\ast \Psi_D$ is proportional to $\delta(\pi)$ and could be written as $\Psi_B^\ast \Psi_D \delta(\pi)$. We rewrite $\Psi_D$ as $\Psi_r$ again and the inner product reads

$$\int dy \int \sqrt{h} \, g \, dx \, \Psi^\ast_B \Psi_D .$$

The integral $\int dy$ represents a gauge volume and should be ignored. For the inner products to agree, the equality

$$\Psi_D = \frac{1}{\sqrt{g}} \Psi_r$$

must be satisfied.

The expectation value of the energy for the reduced phase space quantization is

$$\langle E \rangle_r = \int \sqrt{h} \, d\xi \, \Psi^\ast_r \hat{H} \Psi_r ,$$

$$= \int \sqrt{h} \, d\xi \, \Psi^\ast_r \left( -\frac{1}{2g\sqrt{h}} \partial_{\xi} \frac{g}{\sqrt{h}} \partial_{\xi} + F \right) \frac{1}{\sqrt{g}} \Psi_r .$$

While in Dirac quantization it is given by

$$\langle E \rangle_D = \int \sqrt{h} \, g \, dx \, \Psi^\ast \hat{H} \Psi_D ,$$

$$= \int \sqrt{h} \, g \, dx \, \frac{1}{\sqrt{g}} \Psi^\ast \left( -\frac{1}{2g\sqrt{h}} \partial_x \frac{g}{\sqrt{h}} \partial_x + F \right) \frac{1}{\sqrt{g}} \Psi_r ,$$

$$= \langle E \rangle_r ,$$

$$+ \int \sqrt{h} \, dx \, \Psi^\ast_r \left( \frac{g''}{4hg} - \frac{g'^2}{8h^2g^2} - \frac{g'''}{8gh^2} + F \right) \Psi_r ,$$

where $'$ represents differentiation with respect to $x$. To be consistent with each other, the second term should be zero in the last equation. In other words, the function $F$ is determined so that the two quantization methods coincide.

In this space, $R$ and $R_{\mu\nu} R_{\mu\nu}$ are written as
\[ R = \frac{1}{\hbar} \left( -\frac{g'}{g} + \frac{g_{\mu}^2}{2g^2} + \frac{g' h'}{2gh} \right) - \frac{g''}{gh}, \tag{27} \]

\[ R_{\mu\nu} = \left( -\frac{g''}{g} + \frac{g_{\mu}^2}{2g^2} + \frac{g' h'}{2gh} \right)^2 \frac{1}{\hbar^2} + \frac{1}{2} \left( \frac{g''}{gh} \right)^2. \tag{28} \]

If we define

\[ A = \frac{1}{\hbar} \left( -\frac{g''}{g} + \frac{g_{\mu}^2}{2g^2} + \frac{g' h'}{2gh} \right), \tag{29} \]

\[ B = \frac{g''}{gh}, \tag{30} \]

\( R \) and \( R_{\mu\nu} \) are rewritten as

\[ R = A - B, \tag{31} \]

\[ R_{\mu\nu} = A^2 + \frac{1}{2} B^2. \tag{32} \]

From these equations, we obtain

\[ A = \frac{R + \sqrt{6 R_{\mu\nu} R_{\rho\sigma} - 2 R^2}}{3}. \tag{33} \]

Then if we take

\[ F = \frac{R - \sqrt{6 R_{\mu\nu} R_{\rho\sigma} - 2 R^2}}{12}, \tag{34} \]

\( \langle E \rangle_r \) coincides with \( \langle E \rangle_p \). Here we take the negative sign of the root. We discuss the reason for this later. The operator ordering for the Hamiltonian (16) is, then,

\[ \hat{H} = -\frac{1}{2} A + \frac{R - \sqrt{6 R_{\mu\nu} R_{\rho\sigma} - 2 R^2}}{12}. \tag{35} \]

Let us consider another example. The Hamiltonian in this case is

\[ H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \]

\[ = \frac{1}{2} \gamma^{ij} p_i p_j - \frac{N^i}{M} p_i p_j + \frac{N^2}{2 M^2} p_j^2 + \frac{1}{M} p_j \lambda, \tag{36} \]

where \( p_i \) denotes \( p_{xi} \) and \( i \) runs from 1 to \( n \). \( \lambda \) is a Lagrange multiplier. The metric \( g^{\mu\nu} \) of the Hamiltonian and its inverse which accords with the metric \( g_{\mu\nu} \) of the gauge fixed Lagrangian are

\[
g^{\mu\nu} = \begin{pmatrix}
\gamma^{ij} & -\frac{N^j}{M} & 0 \\
-\frac{N^i}{M} & \frac{N^2}{M^2} & \frac{1}{M} \\
0 & \frac{1}{M} & 0
\end{pmatrix}, \tag{37}
\]
The metric $g_{\mu\nu}$ depends only on $x$. The constraints are $p_\mu = \pi \approx 0$ and $p_\nu \approx 0$ as before.

Since the Hamiltonian of the reduced phase space quantization,

$$H = \frac{1}{2} \gamma^\mu p_\mu \gamma_\nu,$$

has a non-trivial metric in this case, the Hamiltonian operator is

$$\hat{H} = -\frac{1}{2\sqrt{\gamma}} \partial_\nu \gamma^{\nu\mu} \partial_\mu + F(R, \cdots).$$

Here $F$ is a function of $R, \cdots, \gamma_\nu$. In this model the reduced phase space quantization may also have an additional function $F$.

While the Hamiltonian operator of Dirac quantization is

$$\hat{H} = \frac{1}{2\sqrt{g}} \partial_\nu \sqrt{g} g^{\mu\nu} \partial_\mu + G(R, \cdots)$$

$$= -\frac{1}{2M\sqrt{\gamma}} \partial_\nu \gamma^{\mu\nu} \partial_\nu + \frac{1}{2M\sqrt{\gamma}} \partial_\nu \gamma N^\nu \partial_\nu + \frac{1}{2M\sqrt{\gamma}} \partial_\nu \gamma N^\nu \partial_\nu$$

$$-\frac{1}{2M\sqrt{\gamma}} \partial_\nu \gamma \frac{N^2}{M} \partial_\nu - \frac{1}{2M\sqrt{\gamma}} \partial_\nu \gamma \partial_\nu - \frac{1}{2M\sqrt{\gamma}} \partial_\nu \gamma \partial_\nu$$

$$+ G(R, \cdots).$$

Here $G$ is a function of $R, \cdots, g_{\mu\nu}$. The constraint $\bar{p}_\nu = (1/i) \partial_\nu \approx 0$ makes the Hamiltonian

$$\hat{H} = -\frac{1}{2M\sqrt{\gamma}} \partial_\nu \gamma^{\mu\nu} \partial_\mu + G.$$

The inner product for the reduced phase quantization is defined as

$$\int \sqrt{\gamma} dx \Psi_r^* \Psi_r.$$

On the other hand, for Dirac quantization, the inner product is written as

$$\int \sqrt{\gamma} M dx dy \pi^* \Psi_d \Psi_d = \int dy \int \sqrt{\gamma} M dx \Psi_r^* \Psi_r,$$

as before. For the two inner products to agree, the relation

$$\Psi_d = \frac{1}{\sqrt{M}} \Psi_r$$

must be satisfied in this case.

The expectation value of the energy for the reduced phase space quantization is
While in Dirac quantization it is given by

\[ \langle E \rangle_r = \int \sqrt{\gamma} \, dx \, \Psi^* \left( -\frac{1}{2\gamma} \partial_i \sqrt{\gamma} \gamma^{ij} \partial_j + F \right) \Psi_r. \]  

(46)

where $\nabla_i$ is a covariant derivative with respect to $\gamma_{ij}$. For the two quantizations to be equivalent, the second term should be zero in the second equation.

To simplify the problem, suppose that $\gamma_{ij}$ is a two-dimensional metric. The dimension of the space on which Dirac quantization is performed is four. The four-dimensional $R_{ijkl}$, $R^{ijkl}$, and $R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma}$ of the metric (37) and (38) are related with the two-dimensional counterparts for the metric $\gamma_{ij}$ as

\[ R = R^{(2)} + 4 \gamma^{zz} \frac{\nabla x \nabla x \sqrt{M}}{\sqrt{M}} - 2 \gamma^{zz} \frac{\nabla x \nabla z \sqrt{M}}{\sqrt{M}}, \]  

(48)

\[ R^\mu_{\nu} R^\nu_{\mu} = R^{(2)} R^{(2)} + 4 R^{zz} \frac{2 \nabla x \nabla x \sqrt{M}}{\sqrt{M}} + 2 \left( 2 \gamma^{zz} \frac{\nabla x \nabla z \sqrt{M}}{\sqrt{M}} \right)^2 + 8 \left( \gamma^{zz} \right)^2 \frac{\nabla x \nabla x \sqrt{M}}{\sqrt{M}} \frac{\nabla z \nabla z \sqrt{M}}{\sqrt{M}} \right) + \frac{1}{2} \left( \gamma^{zz} \frac{2 \nabla x \nabla z \sqrt{M}}{\sqrt{M}} \right)^2, \]  

(49)

\[ R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma} = R^{(2)} R^{(2)} + 4 \left( 2 \gamma^{zz} \frac{\nabla x \nabla x \sqrt{M}}{\sqrt{M}} \right)^2 + 16 \left( \gamma^{zz} \right)^2 \frac{\nabla x \nabla x \sqrt{M}}{\sqrt{M}} \frac{\nabla x \nabla x \sqrt{M}}{\sqrt{M}} \right) + \left( 4 \gamma^{zz} \frac{\nabla x \nabla z \sqrt{M}}{\sqrt{M}} \right)^2 - \gamma^{zz} \frac{2 \nabla x \nabla z \sqrt{M}}{\sqrt{M}}, \]  

(50)

where $R^{(2)}$, $R^{(2)}$, and $R^{(2)}_{ijkl}$ are two-dimensional. We use a complex coordinate in two dimensions where $\gamma_{zz} \neq 0$ and $\gamma_{zz} = \gamma_{zz} = 0$. Using the relations $R^\mu_{\nu} R_{\nu} = (1/2) R^2$, $R^{ijkl} R_{ijkl} = R^2$, and $R^{zz} = (1/2) R^{zz}$ in two dimensions and defining

\[ a = 2 \gamma^{zz} \frac{\nabla x \nabla x \sqrt{M}}{\sqrt{M}}, \]  

(51)

\[ b = (\gamma^{zz})^2 \frac{\nabla x \nabla x \sqrt{M}}{\sqrt{M}} \frac{\nabla z \nabla z \sqrt{M}}{\sqrt{M}}, \]  

(52)

\[ B = 2 \gamma^{zz} \frac{\nabla x \nabla z \sqrt{M}}{M}, \]  

(53)

we can rewrite Eqs. (48)~(50) as
From these equations, we obtain
\[ a = \frac{R \pm \sqrt{R^2 - 6R^\mu R_\mu 3R^{\mu \nu \sigma} R_{\mu \nu \sigma}}}{12} \] (57)

Therefore, if we take this quantity as \( G \) in Eq. (47), the reduced phase space quantization result coincides with that for Dirac quantization. Since no two-dimensional quantity appears on the right-hand side of Eq. (57), \( F \) in Eq. (47) is zero in two dimensions.

Now we obtain two operator orderings. The Hamiltonian operator for Eq. (36) in four dimensions is
\[ \hat{H} = -\frac{1}{2} \Delta + \frac{R - \sqrt{R^2 - 6R^\mu R_\mu + 3R^{\mu \nu \sigma} R_{\mu \nu \sigma}}}{12} \] (58)

where we take the negative sign of the root, as before. And for the two-dimensional Hamiltonian of Eq. (39), the Hamiltonian operator is
\[ \hat{H} = -\frac{1}{2} \Delta . \] (59)

In two dimensions there does not appear any function of \( R \).

So far we have obtained three operator orderings. These operator orderings have the relations each other. In the three-dimensional constraint system of the metric of Eqs. (37) and (38), \( R^{\mu \nu \sigma} R_{\mu \nu \sigma} \) is written as
\[ R^{\mu \nu \sigma} R_{\mu \nu \sigma} = 4R^\mu R_\mu - R^2 . \] (60)

Substituting this equation into Eq. (58), we obtain Eq. (35). In two dimensions \( R^\mu R_\mu = (1/2)R^2 \) is satisfied. If we substitute this relation into Eq. (35), we obtain the two-dimensional trivial Hamiltonian operator (59). This is the reason why we take the negative sign of the root.

§ 4. Conclusion and discussion

We demonstrated the equivalence of reduced phase space quantization and Dirac quantization. These methods consist of different operator formalism for the same path integral. Using this equivalence and the reparametrization invariance, we determined operator orderings in three examples. However, these expressions are not unique, because scalars are expressible by other scalars. We can derive many equivalent forms.

In general, the \( n \)-dimensional Hamiltonian operator is determined by the equivalence with the artificially extended \((n + 2)\)-dimensional constraint system and the \( n \)
+2)-dimensional Hamiltonian operator of the constraint system is determined at the same time. However, it is difficult to determine a concrete form for the quantum mechanical potential.

In the case of the quantum gravity, the Hamiltonian operator is not positive definite. However, this method is applicable to the quantum gravity. For example, in the minisuperspace model with scale factor and scalar matter, the Wheeler-DeWitt equation reduces to $\square \Psi = 0$, because in the case of two dimensions there appears no quantum potential.

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