Electromagnetic Energy Shift of Nucleus between Conducting Plates

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Electromagnetic energy shifts of a single particle quantum state sandwiched between parallel conducting plates are studied. We find an energy shift of order $1/b^2$, with $b$ the interval of the boundary plates. Applying the calculation to a polarized nucleus in a uniform magnetic field, we find significant energy shifts which depend on the magnetic quantum number. The relation of this effect and the enhancement factor in $E1$ photo nuclear reactions is discussed.

§1. Introduction

In 1948, Casimir$^1$ predicted the possible observation of quantum fluctuations of a radiation field confined in a finite region. In field theories, quantum fluctuations in the vacuum are divergent, and therefore they are to be absorbed by the vacuum energy renormalization. Casimir, however, pointed out that vacuum radiative fluctuations restricted in a finite region by boundaries are different from those in free space and that this difference can be observed. Indeed, the long wavelength (infrared) part of the photon field should be quantized according to the boundary condition, and thus the zero point energy also depends on the boundary. Such Casimir energies of the confined vacuum have been observed experimentally$^2$ since Casimir's work, thus confirming the concept of field quantization.

Another type of observable quantum fluctuation is a set of radiative corrections in quantum electrodynamics (and other field theories). Again, the divergent radiative corrections to a free particle are to be absorbed in the renormalization of the particle mass and the charge, and therefore they cannot be observed. It is, however, known that an electron bound in an atom feels radiative corrections which are different from those for the free electron. This causes energy shifts of atomic levels known as the Lamb shift. The Lamb shift is caused by a radiative field of wavelengths comparable to and longer than the atomic radius. In both the Casimir energy and the Lamb shift, short wavelength radiation gives ultraviolet divergences which can be absorbed in the unrestricted vacuum or the free electron energy and do not cause extra energy shifts.

What happens if the radiative corrections to the atomic electrons are restricted by a boundary condition? We naturally expect the long wavelength part of the radiative energy shift to be altered by the boundary and therefore that a new type of energy shift (which we call the Lamb-Casimir shift) will appear. Several studies...
Fig. 1. A quantum system between parallel conducting plates.

have been done\textsuperscript{3), 4)} concerning this energy shift. The purpose of this paper is to solve some of the discrepancies appearing in the literature and to propose a new aspect of this problem.

We consider a quantum system consisting of a single particle moving in a static potential. We have atomic and (certain) nuclear systems in mind. When we place such a quantum system between a set of parallel conducting plates (Fig. 1), the radiation fields affected by the boundary will induce a Lamb-Casimir effect of the single particle energy level. In fact, the boundary of perfect conductors makes our argument unnecessarily complicated. We therefore carry out our study for a periodic boundary condition instead. Throughout this paper, we choose the Coulomb gauge,

\[
\phi = 0, \quad \text{(1.1)}
\]
\[
div \mathbf{A} = 0. \quad \text{(1.2)}
\]

Our boundary conditions read

\[
\mathbf{A}(x, y, 0) = \mathbf{A}(x, y, b), \quad \text{(1.3)}
\]
\[
\partial_i \mathbf{A}(x, y, 0) = \partial_i \mathbf{A}(x, y, b) \quad \text{for } i = \{x, y, z\}, \quad \text{(1.4)}
\]

while perfectly conducting walls instead require

\[
E_x = E_y = H_z = 0 \quad \text{(1.5)}
\]

or equivalently,

\[
A_x = A_y = 0, \quad \text{(1.6)}
\]
\[
\frac{\partial A_z}{\partial z} = 0 \quad \text{(1.7)}
\]

at \(z = 0\) and \(b\).

If the boundary interval \(b\) is a macroscopic laboratory scale, only the leading \(1/b\) contribution is relevant. The expansion parameter is in general given by \(\hbar/mbc\), where \(m\) is the mass of the particle. We find that to first order in \(1/b\) this term vanishes in general, contrary to the results appearing in some works.\textsuperscript{4) - 6)} Barton\textsuperscript{3), 7)} argued that the \(1/b^2\) terms also vanish due to the Thomas-Reiche-Kuhn sum rule if the particle moves in a spin and momentum independent potential. We
here confirm his result and, further, consider the case in which the residual spin and/or charge dependent interactions leave finite $1/b^2$ contributions.

The possibility for measuring the Lamb-Casimir shift in nuclear systems is studied. Recent microwave techniques will enable us to measure a precise energy difference between polarized nuclear states. It will be quite exciting to determine if the Lamb-Casimir shifts can be measured precisely in nuclear systems.

§2. The Lamb-Casimir effect

We start with a single particle hamiltonian of the form

$$H = H_0 + H_{\text{rad}}, \quad (2.1)$$

$$H_0 = \frac{1}{2m} p^2 + V, \quad (2.2)$$

$$H_{\text{rad}} = -\frac{e}{m} p \cdot A_q + \frac{e^2}{2m} A_q^2, \quad (2.3)$$

where $V$ represents a static potential in which the particle is in motion.

When the gauge field satisfies the periodic boundary conditions (1·3) and (1·4), the first term in Eq. (2·3) induces a radiative correction given by (Fig. 2)

$$\delta E = \frac{e^2}{m^2 8\pi^2 b} \sum_{\beta \lambda} \frac{|\langle \beta | p \cdot \vec{e}(\lambda) | \alpha \rangle|^2}{E_\alpha - E_\beta - \omega_k} \frac{1}{\omega_k}, \quad (2.4)$$

where $|\alpha\rangle = |q, J, M\rangle$, and $|\beta\rangle$ are eigenstates of $H_0$. The label $J$ denotes the total angular momentum, $M$ is its third component, $q$ represents other quantum numbers, and $\omega_k = k$. The summation over $k$ represents

$$\sum_k = \int_{-\infty}^{\infty} dk_x dk_y \sum_{n=-\infty}^{\infty}$$

for $k = \left( k_x, k_y, \frac{2n\pi}{b} \right)$.

In Eq. (2·4), we have used the electric dipole approximation which is valid in the long wavelength limit. This approximation is allowed in the present case because the boundary condition affects only the long wavelength part of the radiation. Indeed, the contribution from the next leading magnetic dipole term is of order $1/b^2$ and negligible.

![Fig. 2. Radiative correction given by $-\frac{e}{m} p \cdot A_q$.](https://academic.oup.com/ptp/article-abstract/97/5/749/1932934)
The summation over the polarization \( \lambda \) can be evaluated easily:

\[
\sum_\lambda |\langle J'M'|p \cdot \bar{e}(\lambda)|JM\rangle|^2
= \sum_\lambda \epsilon_\mu^e(\lambda) \epsilon_{-\mu}(\lambda) \langle JM|p_{-\mu}|J'M'\rangle \langle J'M'|p_\mu|JM\rangle
= \sum_\mu \left\{ 1 - \hat{k}_\mu \hat{k}_{-\mu}(-)^\mu \right\} |\langle J'M'|p_\mu|JM\rangle|^2,
\]

where \( \hat{k} \) is the unit vector, i.e., \( \hat{k} = k/|k| \). Then we obtain

\[
\delta E = \frac{\alpha}{2\pi b} \frac{1}{m^2} \sum_{q',J'M'\mu} |\langle q'J'M'|p_\mu|qJM\rangle|^2 \sum_k \frac{1}{(\Delta_0 - k)k} \left( 1 - \hat{k}_\mu \hat{k}_{-\mu}(-)^\mu \right)
= \delta E_1 + \delta E_2,
\]

where \( \Delta_0 \equiv E_\alpha - E_\beta \) is the excitation energy. The summations in the above equations are divergent. These divergences are not removed by the usual renormalization technique for the free particle, but we can extract a finite difference between the same summation with and without the boundary conditions (1·3) and (1·4) using the regularization method given in the Appendix.

We apply the type 1(2) regularization in the Appendix to \( \delta E_1(\delta E_2) \). It is, however, noted that in the present case, even if we do not introduce the smooth cutoff at higher momentum and instead make a sharp cutoff, terms depending on the cutoff momentum cancel each other. After the regularization, we get the following result for \( \delta E \):

\[
\text{Reg}[\delta E] = \frac{\alpha \pi}{3b^2} \frac{1}{m^2} \sum_{q',J'M'\mu} |\langle q'J'M'|p_\mu|qJM\rangle|^2 \frac{1 + (-)^\mu}{\Delta_0}
= \frac{2\alpha \pi}{3b^2} \frac{1}{m^2} \sum_{q',J'} |\langle q'JM|p_0|qJM\rangle|^2 \frac{1}{\Delta_0}.
\]

The second term in Eq. (2·3) also yields an energy shift called the seagull term (see Fig. 3) given by

\[
\delta E_S = \langle 0|H_S|0 \rangle = \frac{e^2}{2m} \langle 0|A^2(x)|0 \rangle = \frac{\alpha}{2\pi b m} \sum_k \frac{1}{\omega_k}.
\]
After the regularization (type 1 in the Appendix), we get the result

$$\text{Reg} [\delta E_s] = \frac{\alpha \pi}{3b^2 m}. \quad (2.10)$$

According to Barton if the potential is independent of the spin and the velocity, the contributions given by Eqs. (2.8) and (2.10) cancel each other, and the sum vanishes to order $1/b^2$. In fact if we use $p = im[H, r]$, the summation in Eq. (2.8) leads to

$$
\sum_{q'J'} \frac{|\langle q'J'M|p_0|qJM \rangle|^2}{E_{qJ} - E_{q'J'}} = -m^2 \sum_{q'J'} \frac{\langle qJ | [H, z] | q'J' \rangle \langle q'J' | [H, z] | qJ \rangle}{E_{qJ} - E_{q'J'}} = m^2 \sum_{q'J'} (E_{qJ} - E_{q'J'}) \langle qJ | z | q'J' \rangle \langle q'J' | z | qJ \rangle
$$

$$= \frac{m^2}{2} \langle qJ | [H, z], z | qJ \rangle$$

$$= \frac{m^2}{2} \left\langle qJ \left[ \frac{p_z}{m}, z \right] | qJ \right\rangle$$

$$= -\frac{m^2}{2} \frac{1}{m} = -\frac{1}{2} m.$$ 

Then the sum of Eqs. (2.8) and (2.9) vanishes to the order $1/b^2$ and we expect that the leading term is $O(1/b^4)$.

§3. Application to nuclear energy shift

In this section we apply the above formulation to a nuclear system. We calculate the radiative corrections for a nucleon in a nucleus that is placed between two conducting plates. We suppose that the nucleus is polarized and is subject to a constant magnetic field. We assume that the single particle potential $V$ is composed of a harmonic oscillator central potential and some residual interaction,

$$V = \frac{m \omega^2}{2} r^2 + U.$$ 

Then for a nucleon in the single particle state $|nLJM\rangle$, after some calculations, we obtain

$$\text{Reg} [\delta E] = \frac{2e^2 k_3 \alpha \pi}{3b^2} \frac{1}{m^2} \sum_{n'J'M} |\langle n'L'J'M|p_0|nLJM \rangle|^2 \frac{1}{\Delta_0}, \quad (3.1)$$
\[ \Delta_0 = E_{nLJM} - E_{n'J'M'}. \]

where the coefficient \( c_{E1} \) is the effective charge of the nucleon for the \( E1 \) transition.

A pioneering precise experiment is planned \(^8\) that measures the energy differences between the magnetic substates of a polarized nucleus placed between conducting plates. The plan proposes to use \(^{131}\)Xe\(_{77}\) as the nucleus. Therefore we apply the present formulation to the case of \(^{131}\)Xe\(_{77}\) as a concrete example. We assume that the system is described by the motion of a single neutron, which also carries the total angular momentum of the nucleus.

Thus we assume \( 1d_{3/2} \) (\( n = 1, L = 2, M, J = 3/2 \)) for the ground state, and we set

\[ X = \left| \left< n'L'J'M|p_0|1, 2, \frac{3}{2}, M \right> \right|^2. \]

The number of harmonic quanta for the \( 1d_{3/2} \) state is \( N = 2n + L = 4 \). For the \( E1 \) transition, both \( N \) and \( L \) change by one, while \( \Delta J = 0 \) or \( \pm 1 \). Thus the possible intermediate states are

\[ 2p_{1/2}, 2p_{3/2}, 1f_{5/2} \quad \text{for} \quad N' = 5, \]

\[ 1p_{1/2}, 1p_{3/2}, 0f_{5/2} \quad \text{for} \quad N' = 3. \]

Though the \( N' = 3 \) intermediate states are not allowed due to the Pauli exclusion principle, it is known that the Pauli blocking effect is compensated by the contribution of the occupied states, giving a result identical to that without the Pauli blocking. \(^*)\)

We calculate contributions from all the transitions

\[ X(1d_{3/2} \to 2p_{1/2}) = \frac{9 - 4M^2}{18B^2}, \]
\[ X(1d_{3/2} \to 2p_{3/2}) = \frac{8M^2}{225B^2}, \]
\[ X(1d_{3/2} \to 1f_{5/2}) = \frac{9(25 - 4M^2)}{200B^2}, \]
\[ X(1d_{3/2} \to 1p_{1/2}) = \frac{7(9 - 4M^2)}{72B^2}, \]
\[ X(1d_{3/2} \to 1p_{3/2}) = \frac{14M^2}{225B^2}, \]
\[ X(1d_{3/2} \to 0f_{5/2}) = \frac{25 - 4M^2}{100B^2}. \]

with \( B = 1/\sqrt{m\omega} \).

If the residual interaction is neglected, we find \( \Delta_0 = -\omega \) for \( N' = 5 \) states, and \( \Delta_0 = \omega \) for \( N' = 3 \) states. If we include all the intermediate states, neglecting the

\(^*)\) We thank Professor K. Yazaki for pointing this out to us.
Pauli principle, we obtain
\[
\sum_{n'L',J'} X(1d_{3/2} \rightarrow |n'L'J') \frac{1}{\Delta_0} = \frac{1}{2 B^2(-\omega)} = -\frac{1}{2} \frac{1}{m}.
\] (3.3)
Hence contributions of Eqs. (3.1) and (3.2) cancel each other, leaving
\[
\text{Reg} \left[ \delta E_{\text{total}} \right] = O \left( \frac{1}{b^4} \right).
\]
This is Barton's result derived by the use of the Thomas-Reiche-Kuhn sum rule.

If we take into account a strong \(LS\) potential as well as an \(L^2\) term in the residual interaction, \(^9\)
\[
V = \frac{1}{2} m \omega^2 r^2 - K \hbar \omega \left[ 2 \vec{L} \cdot \vec{S} + \mu \left( \vec{L}^2 - \frac{N(N+3)}{2} \right) \right],
\] (3.4)
we obtain
\[
\text{Reg}[\delta E_{\text{total}}] = \frac{2c_E^2 \alpha \pi}{3b^2 m} \left\{ \frac{1}{4} (K_3 - 3K_4 + 2K_5)
+ \frac{1}{5} (11K_4 - 5K_3 - 6K_5)M^2 + \left( \frac{43}{8} \mu'_4 - 22\mu'_4 + 18\mu'_5 \right)
- \left( \frac{13}{6} \mu'_3 - \frac{88}{15} \mu'_4 + \frac{24}{5} \mu'_5 \right) \right\} M^2.
\] (3.5)
Here, \(K_N\) and \(\mu_N\) denote \(K\) and \(\mu\) for the \(N\)-shell, respectively, and \(\mu'_N = \mu_N K_N\). Note that, if the values of \(K\) are the same for different \(N\), there are no contributions from the \(LS\) term.

Assuming that the neutron effective charge is given by \(c_E = -\frac{Z}{A} = -\frac{54}{131}\), taking values for \(K\) and \(\mu\) from Ref. 9), Eq. (3.5) gives
\[
\text{Reg}[\delta E_{\text{total}}] = \frac{2c_E^2 \alpha \pi}{3b^2 m} (0.01122 - 0.02696M^2)
= \frac{1}{b^2} (0.01122 - 0.02696M^2) \times 1.08 \times 10^{-1} [\text{MeV}] \text{ for } b[\text{fm}],
\] (3.6)
\[
\begin{align*}
\text{Reg}[\delta E_{\text{total}}(M = 1/2)] - \text{Reg}[\delta E_{\text{total}}(M = 3/2)] \\
= 5.8 \times 10^{-23} [\text{eV}] \text{ when } b = 1[\text{cm}].
\end{align*}
\] (3.7)

We note that if the boundary condition for the perfectly conducting wall is employed, the results should be modified. If we neglect the direct interactions between the quantum system and the walls, \(b\) in Eq. (3.6) should be replaced by \(2b\), because the number of excitation modes in the perfectly conducting wall boundary condition is half of that for the periodic boundary condition. Then the result (3.7) will be reduced to \(1/4\).

Although the result (3.7) above seems too small to be observed, we find a much larger effect from two-body charge exchange interactions. In order to see this, we use a relation between
\[
\text{Reg}[\delta E_{\text{total}}] = \frac{2\alpha \pi}{3b^2} \frac{1}{2} (qJ \vert[H, c_E z], c_E z] \vert qJ) + \text{Reg}[\delta E_S]
\] (3.8)
and the enhancement $\kappa$ in the $E1$ sum rule\textsuperscript{10} defined by

$$
\kappa = -4\pi\alpha \frac{A}{NZ} m \langle qJ|[[V, c_{E1} z], c_{E1} z]|qJ\rangle.
$$

The relation is given by

$$
\kappa = -12 \frac{Am}{NZ} b^2 \text{Reg}[\delta E_{\text{total}}].
$$

Here, $\kappa = -1.11 \times 10^{-5} + 2.65 \times 10^{-5} M^2$ for Eq. (3·6).

The enhancement factor $\kappa$ is actually larger than that obtained in the above. There is a relation between $\kappa$ and $\delta g_l$, the change of the orbital g-factor due to exchange currents. The relation $\kappa = 2\delta g_l$ of Fujita and Hirata\textsuperscript{10} suggests that $\kappa$ is $0.2 \sim 0.4$. The tensor correlation leads to an even larger value of $\kappa$, of order $1.11$. Nuclear photo-absorption experiments show that $\kappa = 0.8 \sim 1.0$ up to the 100 MeV excitation energy for a range of nuclei.

Here, we give the $M$-dependent part of $\kappa$ calculated from a two-body charge-exchange interaction taken here to be the one-pion-exchange potential, OPEP:

$$
\kappa = - \frac{A}{NZ} m \left[ \frac{3^+}{2} M \left[ \sum_{i>j} V_{ij}^{\text{OPEP}}, \frac{1}{2} \sum_k \tau_k^{(3)} z_k, \frac{1}{2} \sum_k \tau_k^{(3)} z_k \right] \right] \frac{3^+}{2} M,
$$

where $V_c(r) = (\mu/3)(f^2/4\pi)(e^{-\mu r}/\mu r)f^2(r)$ with $\mu$ the pion mass and $f$ the $\pi NN$ coupling constant, $f^2/4\pi = 0.081$. $f(r)$ is a short-range correlation function, which we take to be a cutoff function: $f(r) = 0$ for $r < r_c = 0.7$ fm and $f(r) = 1$ for $r \geq r_c$.

The contribution comes from the interaction between the valence neutron in the $1d_{3/2}$ orbit and the proton-core of $^{131}$Ce. The calculated value of $\kappa$ is

$$
\kappa = -0.0898 \left( M^2 - \frac{5}{4} \right),
$$

which gives

$$
\kappa(M = 1/2) - \kappa(M = 3/2) = 0.180.
$$

This is of the same order as $\kappa \sim 2\delta g_l$ for the $M$-independent part. This value corresponds to

$$
\text{Reg}[\delta E_{\text{total}}(M = 1/2)] - \text{Reg}[\delta E_{\text{total}}(M = 3/2)] = -1.10 \times 10^{2} \frac{1}{b^2} \{\kappa(M = 1/2) - \kappa(M = 3/2)\}[\text{MeV}] \text{ for } b[\text{fm}],
$$

$$
= -2.0 \times 10^{-19}[\text{eV}] \text{ when } b[1\text{cm}].
$$
If $\kappa$ is taken to be around 1, of the same order as the observation for the $M$-independent case, the energy difference above becomes $\sim 10^{-18}$ eV. These values $10^{-19} \sim 10^{-18}$ eV are large enough to be measurable in the present experimental facilities, which seem to be able to reach the precision of $10^{-23}$ eV. 8)

§4. Conclusion

In this paper, we considered the radiative corrections of a single particle quantum system restricted by the periodic boundary condition. The long wavelength part of the radiative correction is modified due to the boundary condition so that the differences from the radiative correction to the free system are observable quantities (the Lamb-Casimir effect). We estimated the Lamb-Casimir shift in the electric dipole approximation. We found that the sum of all the intermediate states results in vanishing $1/b^2$ contribution, while residual interactions with spin polarization may give finite $1/b^2$ contributions.

We applied our formulation to a nuclear system. We found that the Lamb-Casimir effect is intimately related to the $E1$ enhancement factor $\kappa$ observed in nuclear photo-absorption reactions. In other words, the Lamb-Casimir effect is a good tool to obtain $\kappa$, which is rather hard to determine precisely by photo-absorption reactions. Although the contribution from a single particle spin-orbit force is negligibly small, the one-pion exchange force is found to give a large $\kappa(\sim 0.2)$ and therefore predicts a large Lamb-Casimir effect. It would be extremely interesting if such effects could be measured in nuclear systems.

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Appendix A

--- Regularization ---

In the present calculation, we encounter the difference of an infinite sum and an integral of the form

$$\sum_{n=0}^{\infty} f(an) - \frac{1}{\alpha} \int_{0}^{\infty} f(x)dx,$$

$$\sum_{n=0}^{\infty} f(an) \equiv \frac{f(0)}{2} + \sum_{n=1}^{\infty} f(an).$$

Because both the sum and the integral are divergent we must regularize them to find a finite answer. In order to obtain a finite value which corresponds to a physical quantity, we introduce a smooth cutoff function $\vartheta_M(x)$ such that

$$\vartheta_M(x) = \begin{cases} 
1 & \text{for } x \in [0, M], \\
0 & \text{for } x > M(x \to \infty). 
\end{cases}$$
This corresponds to introducing a momentum cutoff, and at the end we remove the cutoff. We define the regularization with the function $\theta_M(x)$ by

$$\text{reg} \left[ \sum_{n=0}^{\infty} f(an) \right] \equiv \lim_{M \to \infty} \left\{ \sum_{n=0}^{\infty} f(an) \theta_M(an) - \frac{1}{\alpha} \int_0^\infty f(x) \theta_M(x) dx \right\}. \quad (A.1)$$

If $f(x)$ is a regular function, we can use the Euler-Maclaurin formula

$$\frac{f(a) + f(b)}{2} + \sum_{\nu=1}^{n-1} f \left( a + \frac{b-a}{n} \right) = \frac{n}{b-a} \int_a^b f(x) dx + \sum_{\mu=1}^{\nu} \frac{B_{2\mu}}{(2\mu)!} \left( \frac{b-a}{n} \right)^{2\mu-1} \left\{ f^{(2\mu-1)}(b) - f^{(2\mu-1)}(a) \right\},$$

where $B_{2\mu}$ is Bernoulli’s number, i.e., $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}$. Then from Eq. (A.2) we obtain

$$\sum_{\nu=0}^{\infty} f(\alpha \nu) \theta_M(\alpha \nu) - \frac{1}{\alpha} \int_0^\infty f(x) \theta_M(x) dx = -\sum_{\mu=1}^{\infty} \frac{B_{2\mu}}{(2\mu)!} \alpha^{2\mu-1} f^{(2\mu-1)}(0). \quad (A.3)$$

Because the right-hand side is independent of $M$, we can take the limit $M \to \infty$, and obtain

$$\text{reg} \left[ \sum_{\nu=0}^{\infty} f(\alpha \nu) \right] = -\sum_{\mu=1}^{\infty} \frac{B_{2\mu}}{(2\mu)!} \alpha^{2\mu-1} f^{(2\mu-1)}(0). \quad (A.4)$$

This is the desired result.

Next, we move on to the 3-dimensional case. We consider the form

$$\int_{-\infty}^{\infty} dx dy \left( \sum_{n} f(x, y, an) - \frac{1}{\alpha} \int_0^\infty dz f(x, y, z) \right). \quad (A.5)$$

Now we introduce the cutoff by requiring $x^2 + y^2 + (\alpha n)^2 \leq M^2$. We define the regularization

$$R \equiv \text{Reg} \left[ \int_{-\infty}^{\infty} dx dy \sum_{n} f(x, y, an) \right]$$

$$\equiv \lim_{M^2 \to \infty} \int dx dy \left[ \sum_{n=0}^{\infty} f(x, y, an) \theta_M(\sqrt{x^2 + y^2 + (\alpha n)^2}) \right. \left. - \frac{1}{\alpha} \int_0^\infty f(x, y, z) \theta_M(\sqrt{x^2 + y^2 + z^2}) dz \right]. \quad (A.6)$$

**Type 1**: $f = f(\sqrt{x^2 + y^2 + z^2})$

Using cylindrical coordinates, we introduce the function

$$F_M(z) \equiv \left\{ \begin{array}{ll}
0, & (z \geq M) \\
\int_0^{\sqrt{M^2 - z^2}} rdr \int_0^{2\pi} d\theta f(\sqrt{r^2 + z^2}), & (z < M) \end{array} \right.$$
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\[ R = \sum_{n} F'_{M}(\alpha n) \]

Then Eq. (A·6) becomes

\[ R = \text{reg} \left[ \sum_{n} F'_{M}(\alpha n) \right] \]

\[ = - \sum_{\mu=1}^{\infty} \frac{B_{2\mu}}{(2\mu)!} \alpha^{2\mu-1} F^{(2\mu-1)}_{M}(0). \quad (A·7) \]

Note that \( F^{(2\mu-1)}_{M}(0) \) is independent of \( M \) because

\[ F'_{M}(z) = -2\pi z f(z). \]

**Type 2**: \( f = f(\sqrt{x^2 + y^2 + z^2}, \sqrt{x^2 + y^2}) \)

We introduce the function

\[ F_{M}(z) = \left\{ \begin{array}{ll} 0, & (z \geq M) \\ \pi \int_{z^2}^{M^2} du f(\sqrt{u},\sqrt{u-z^2}), & (z < M) \end{array} \right. \]

Then we obtain Eq. (A·7) again for \( R \). The first derivative of \( F_{M}(z) \) is given by

\[ F'_{M}(z) = \pi \left\{ -2zf(z,0) + \int_{z^2}^{M^2} du \frac{\partial f}{\partial z}(\sqrt{u},\sqrt{u-z^2}) \right\}, \quad (A·8) \]

\[ \frac{\partial f}{\partial z}(\sqrt{u},\sqrt{u-z^2}) = \frac{-z}{\sqrt{u-z^2}} \frac{\partial f}{\partial t}(\sqrt{u},t) \bigg|_{t=\sqrt{u-z^2}}. \]

Note that the second term of \( F'_{M} \) depends on \( M \) and therefore we cannot take \( M \to \infty \). However, in the cases that appear in the present study, the second term vanishes.

**References**

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