



**THE ART OF INSIGHT
IN SCIENCE AND ENGINEERING**

Mastering Complexity

SANJOY MAHAJAN

The Art of Insight in Science and Engineering

The Art of Insight in Science and Engineering

Mastering Complexity

Sanjoy Mahajan

The MIT Press
Cambridge, Massachusetts
London, England

© 2014 Sanjoy Mahajan

The Art of Insight in Science and Engineering: Mastering Complexity by Sanjoy Mahajan (author) and MIT Press (publisher) is licensed under the Creative Commons Attribution–Noncommercial–ShareAlike 4.0 International License. A copy of the license is available at creativecommons.org/licenses/by-nc-sa/4.0/



MIT Press books may be purchased at special quantity discounts for business or sales promotional use. For information, please email special_sales@mitpress.mit.edu.

Typeset by the author in 10.5/13.3 Palatino and Computer Modern Sans using ConT_EXt and LuaT_EX.

LIBRARY OF CONGRESS CATALOGING-IN-PUBLICATION DATA

Mahajan, Sanjoy, 1969- author.

The art of insight in science and engineering : mastering complexity / Sanjoy Mahajan.

pages cm

Includes bibliographical references and index.

ISBN 978-0-262-52654-8 (pbk. : alk. paper) 1. Statistical physics.

2. Estimation theory. 3. Hypothesis. 4. Problem solving. I. Title.

QC174.85.E88M34 2014

501'.9-dc23

2014003652

Printed and bound in the United States of America

10 9 8 7 6 5 4 3 2 1

For my teachers, who showed me the way

Peter Goldreich
Carver Mead
Sterl Phinney

And for my students, one of whom said

I used to be curious, naively curious. Now I am fearlessly curious. I feel ready to attack any problem that comes at me, and at least get a feel for why things happen ... roughly.

Brief contents

	Preface	xiii
	Values for backs of envelopes	xvii
<i>Part I</i>	<i>Organizing complexity</i>	1
	1 Divide and conquer	3
	2 Abstraction	27
<i>Part II</i>	<i>Discarding complexity without losing information</i>	55
	3 Symmetry and conservation	57
	4 Proportional reasoning	103
	5 Dimensions	137
<i>Part III</i>	<i>Discarding complexity with loss of information</i>	197
	6 Lumping	199
	7 Probabilistic reasoning	235
	8 Easy cases	279
	9 Spring models	317
<hr/>		
	Bon voyage: Long-lasting learning	357
	Bibliography	359
	Index	363

Contents

	Preface	xiii
	Values for backs of envelopes	xvii
<i>Part I</i>	<i>Organizing complexity</i>	1
	1 Divide and conquer	3
	1.1 Warming up	3
	1.2 Rails versus roads	6
	1.3 Tree representations	7
	1.4 Demand-side estimates	10
	1.5 Multiple estimates for the same quantity	16
	1.6 Talking to your gut	17
	1.7 Physical estimates	20
	1.8 Summary and further problems	25
	2 Abstraction	27
	2.1 Energy from burning hydrocarbons	28
	2.2 Coin-flip game	31
	2.3 Purpose of abstraction	34
	2.4 Analogies	36
	2.5 Summary and further problems	53
<i>Part II</i>	<i>Discarding complexity without losing information</i>	55
	3 Symmetry and conservation	57
	3.1 Invariants	57
	3.2 From invariant to symmetry operation	66
	3.3 Physical symmetry	73
	3.4 Box models and conservation	75
	3.5 Drag using conservation of energy	84
	3.6 Lift using conservation of momentum	93
	3.7 Summary and further problems	99

4	Proportional reasoning	103
4.1	Population scaling	103
4.2	Finding scaling exponents	105
4.3	Scaling exponents in fluid mechanics	117
4.4	Scaling exponents in mathematics	123
4.5	Logarithmic scales in two dimensions	126
4.6	Optimizing flight speed	128
4.7	Summary and further problems	135
5	Dimensions	137
5.1	Dimensionless groups	139
5.2	One dimensionless group	147
5.3	More dimensionless groups	152
5.4	Temperature and charge	165
5.5	Atoms, molecules, and materials	175
5.6	Summary and further problems	192
<i>Part III</i>	<i>Discarding complexity with loss of information</i>	197
6	Lumping	199
6.1	Approximate!	199
6.2	Rounding on a logarithmic scale	200
6.3	Typical or characteristic values	203
6.4	Applying lumping to shapes	212
6.5	Quantum mechanics	229
6.6	Summary and further problems	234
7	Probabilistic reasoning	235
7.1	Probability as degree of belief: Bayesian probability	235
7.2	Plausible ranges: Why divide and conquer works	239
7.3	Random walks: Viscosity and heat flow	249
7.4	Transport by random walks	263
7.5	Summary and further problems	276
8	Easy cases	279
8.1	Warming up	279
8.2	Two regimes	281
8.3	Three regimes	291
8.4	Two dimensionless quantities	308
8.5	Summary and further problems	312

9 Spring models	317
9.1 Bond springs	317
9.2 Energy reasoning	321
9.3 Generating sound, light, and gravitational radiation	331
9.4 Effect of radiation: Blue skies and red sunsets	345
9.5 Summary and further problems	353
<hr/>	
Bon voyage: Long-lasting learning	357
Bibliography	359
Index	363

Preface

Science and engineering, our modern ways of understanding and altering the world, are said to be about accuracy and precision. Yet we best master the complexity of our world by cultivating insight rather than precision.

We need insight because our minds are but a small part of the world. An insight unifies fragments of knowledge into a compact picture that fits in our minds. But precision can overflow our mental registers, washing away the understanding brought by insight. This book shows you how to build insight and understanding first, so that you do not drown in complexity.

LESS
RIGOR Therefore, our approach will not be rigorous—for rigor easily becomes *rigor mortis* or paralysis by analysis. Forgoing rigor, we'll study the natural and human-created worlds—the worlds of science and engineering. So you'll need some—but not extensive!—knowledge of physics concepts such as force, power, energy, charge, and field. We'll use as little mathematics as possible—algebra and geometry mostly, trigonometry sometimes, and calculus rarely—so that the mathematics promotes rather than hinders insight, understanding, and flexible problem solving. The goal is to help you master complexity; then no problem can intimidate you.

Like all important parts of our lives, whether spouses or careers, I came to this approach mostly unplanned. As a graduate student, I gave my first scientific talk on the chemical reactions in the retinal rod. I could make sense of the chemical chaos only by approximating. In that same year, my friend Carlos Brody wondered about the distribution of twin primes—prime pairs separated by 2, such as 3 and 5 or 11 and 13. Nobody knows the distribution for sure. As a lazy physicist, I approximately answered Carlos's question with a probabilistic model of being prime [32]. Approximations, I saw again, foster understanding.

As a physics graduate student, I needed to prepare for the graduate qualifying exams. I also became a teaching assistant for the "Order-of-Magnitude Physics" course. In three months, preparing for the qualifying exams and learning the course material to stay a day ahead of the students, I learned

more physics than I had in the years of my undergraduate degree. Physics teaching and learning had much room for improvement—and approximation and insight could fill the gap.

DEDICATION In gratitude to my teachers, I dedicate this book to Carver Mead for irreplaceable guidance and faith; and to Peter Goldreich and Sterl Phinney, who developed the “Order-of-Magnitude Physics” course at Caltech. From them I learned the courage to simplify and gain insight—the courage that I look forward to teaching you.

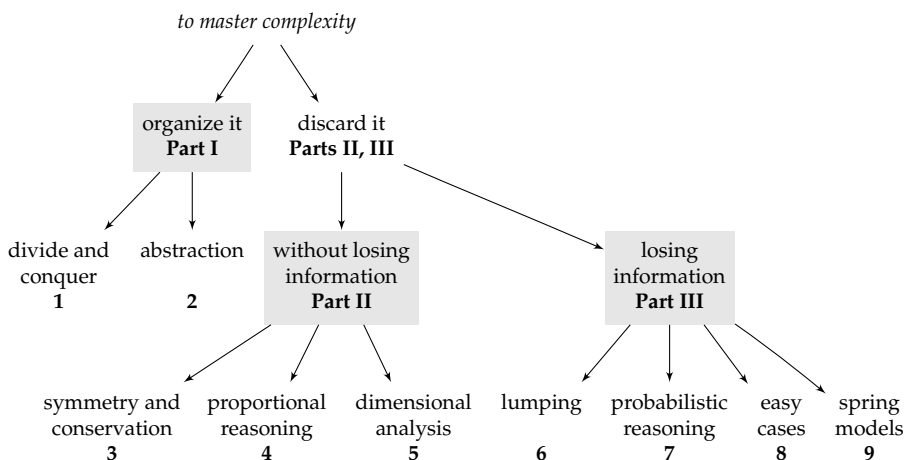
ORGANIZATION For many years, at the University of Cambridge and at MIT, I taught a course on the “Art of Approximation” organized by topics in physics and engineering. This organization limited the material’s generality: Unless you become a specialist in general relativity, you may not study gravitation again. Yet estimating how much gravity deflects starlight (Section 5.3.1) teaches reasoning tools that you can use far beyond that example. Tools are more general and useful than topics.

Therefore, I redesigned the course around the reasoning tools. This organization, which I have used at MIT and Olin College of Engineering, is reflected in this book—which teaches you one tool per chapter, each selected to help you build insight and master complexity.

There are the two broad ways to master complexity: organize the complexity or discard it. Organizing complexity, the subject of Part I, is taught through two tools: divide-and-conquer reasoning (Chapter 1) and making abstractions (Chapter 2).

Discarding complexity (Parts II and III) illustrates that “the art of being wise is the art of knowing what to overlook” (William James [24, p. 369]). In Part II, complexity is discarded *without* losing information. This part teaches three reasoning tools: symmetry and conservation (Chapter 3), proportional reasoning (Chapter 4), and dimensional analysis (Chapter 5). In Part III, complexity is discarded while losing information. This part teaches our final tools: lumping (Chapter 6), probabilistic reasoning (Chapter 7), easy cases (Chapter 8), and spring models (Chapter 9).

FINDING MEANING Using these tools, we will explore the natural and human-made worlds. We will estimate the flight range of birds and planes, the strength of chemical bonds, and the angle that the Sun deflects starlight; understand the physics of pianos, xylophones, and speakers; and explain why skies are blue and sunsets are red. Our tools weave these and many other examples into a tapestry of meaning spanning science and engineering.



SHARING THIS WORK Like my earlier *Street-Fighting Mathematics* [33], this book is licensed under a Creative Commons Attribution–Noncommercial–Share Alike license. MIT Press and I hope that you will improve and share the work noncommercially, and we would gladly receive corrections and suggestions.

INTER-SPERSED QUESTIONS The most effective teacher is a skilled tutor [2]. A tutor asks many questions, because questioning, wondering, and discussing promote learning. Questions of two types are interspersed through the book. *Questions marked with a ► in the margin*, which a tutor would pose during a tutorial, ask you to develop the next steps of an argument. They are answered in the subsequent text, where you can check your thinking. *Numbered problems*, marked with a shaded background, which a tutor would give you to take home, ask you to practice the tool, to extend an example, to use several tools, and even to resolve an occasional paradox. Merely watching workout videos produces little fitness! So, try many questions of both types.

IMPROVE OUR WORLD Through your effort, mastery will come—and with a broad benefit. As the physicist Edwin Jaynes said of teaching [25]:

[T]he goal should be, not to implant in the students' mind every fact that the teacher knows now; but rather to implant a way of thinking that enables the student, in the future, to learn in one year what the teacher learned in two years. Only in that way can we continue to advance from one generation to the next.

May the tools in this book help you advance our world beyond the state in which my generation has left it.

Acknowledgments

In addition to the dedication, I would like to thank the following people and organizations for their generosity.

For encouragement, forbearance, and motivation: my family—Juliet Jacobsen, Else Mahajan, and Sabine Mahajan.

For a sweeping review of the manuscript and improvements to every page: Tadashi Tokieda and David MacKay. Any remaining mistakes were contributed by me subsequently!

For advice on the process of writing: Larry Cohen, Hillary Rettig, Mary Carroll Moore, and Kenneth Atchity (author of *A Writer's Time* [1]).

For editorial guidance over many years: Robert Prior.

For valuable suggestions and discussions: Dap Hartmann, Shehu Abdussalam, Matthew Rush, Jason Manuel, Robin Oswald, David Hogg, John Hopfield, Elisabeth Moyer, R. David Middlebrook, Dennis Freeman, Michael Gottlieb, Edwin Taylor, Mark Warner, and many students throughout the years.

For the free software used for typesetting: Hans Hagen, Taco Hoekwater, and the Con_TE_Xt user community (Con_TE_Xt and Lua_TE_X); Donald Knuth (T_EX); Taco Hoekwater and John Hobby (MetaPost); John Bowman, Andy Hammerlindl, and Tom Prince (Asymptote); Matt Mackall (Mercurial); Richard Stallman (Emacs); and the Debian GNU/Linux project.

For the NB document-annotation system: Sacha Zyto and David Karger.

For being a wonderful place for a graduate student to think, explore, and learn: the California Institute of Technology.

For supporting my work in science and mathematics education: the Whitaker Foundation in Biomedical Engineering; the Hertz Foundation; the Gatsby Charitable Foundation; the Master and Fellows of Corpus Christi College, Cambridge; Olin College of Engineering and its Intellectual Vitality program; and the Office of Digital Learning and the Department of Electrical Engineering and Computer Science at MIT.

Values for backs of envelopes

π	pi	3	
G	Newton's constant	7×10^{-11}	$\text{kg}^{-1} \text{m}^3 \text{s}^{-2}$
c	speed of light	3×10^8	m s^{-1}
hc	h shortcut	200	eV nm
$m_e c^2$	electron rest energy	0.5	MeV
k_B	Boltzmann's constant	10^{-4}	eV K^{-1}
N_A	Avogadro's number	6×10^{23}	mol^{-1}
R	universal gas constant $k_B N_A$	8	$\text{J mol}^{-1} \text{K}^{-1}$
e	electron charge	1.6×10^{-19}	C
$e^2/4\pi\epsilon_0$	electrostatic combination	2.3×10^{-28}	$\text{kg m}^3 \text{s}^{-2}$
$(e^2/4\pi\epsilon_0)/hc$	fine-structure constant α	0.7×10^{-2}	
σ	Stefan-Boltzmann constant	6×10^{-8}	$\text{W m}^{-2} \text{K}^{-4}$
M_{Sun}	solar mass	2×10^{30}	kg
m_{Earth}	Earth's mass	6×10^{24}	kg
R_{Earth}	Earth's radius	6×10^6	m
AU	Earth-Sun distance	1.5×10^{11}	m
$\theta_{\text{Moon or Sun}}$	angular diameter of Moon or Sun	10^{-2}	rad
day	length of a day	10^5	s
year	length of a year	$\pi \times 10^7$	s
t_0	age of the universe	1.4×10^{10}	yr
F	solar constant	1.3	kW m^{-2}
p_0	atmospheric pressure at sea level	10^5	Pa
ρ_{air}	air density	1	kg m^{-3}
ρ_{rock}	rock density	2.5	g cm^{-3}
$L_{\text{vap}}^{\text{water}}$	heat of vaporization of water	2	MJ kg^{-1}
γ_{water}	surface tension of water	7×10^{-2}	N m^{-1}
P_{basal}	human basal metabolic rate	100	W
a_0	Bohr radius	0.5	Å
a	typical interatomic spacing	3	Å
E_{bond}	typical bond energy	4	eV
\mathcal{E}_{fat}	combustion energy density	9	kcal g^{-1}
ν_{air}	kinematic viscosity of air	1.5×10^{-5}	$\text{m}^2 \text{s}^{-1}$
ν_{water}	kinematic viscosity of water	10^{-6}	$\text{m}^2 \text{s}^{-1}$
K_{air}	thermal conductivity of air	2×10^{-2}	$\text{W m}^{-1} \text{K}^{-1}$
K	... of nonmetallic solids/liquids	2	$\text{W m}^{-1} \text{K}^{-1}$
K_{metal}	... of metals	2×10^2	$\text{W m}^{-1} \text{K}^{-1}$
$c_{\text{p}}^{\text{air}}$	specific heat of air	1	$\text{J g}^{-1} \text{K}^{-1}$
c_{p}	... of solids/liquids	25	$\text{J mol}^{-1} \text{K}^{-1}$

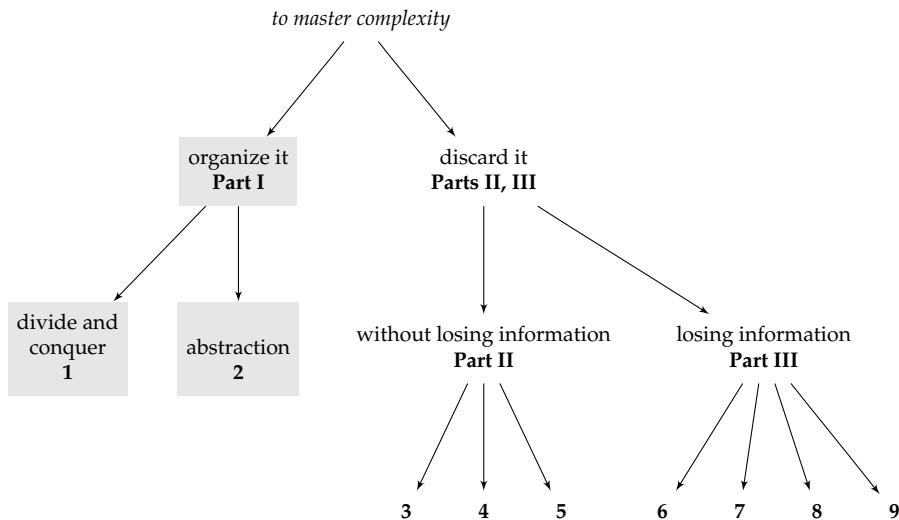
Part I

Organizing complexity

We cannot find much insight staring at a mess. We need to organize it. As an everyday example, when I look at my kitchen after a dinner party, I feel overwhelmed. It's late, I'm tired, and I dread that I will not get enough sleep. If I clean up in that scattered state of mind, I pick up a spoon here and a pot there, making little progress. However, when I remember that a large problem can be broken into smaller ones, calm and efficiency return. I begin at one corner of the kitchen, clear its mess, and move to neighboring areas until the project is done. I divide and conquer (Chapter 1).

Once the dishes are clean, I resist the temptation to dump them into one big box. I separate pots from the silverware and, within the silverware, the forks from the spoons. These groupings, or abstractions (Chapter 2), make the kitchen easy to understand and use.

In problem solving, we organize complexity by using divide-and-conquer reasoning and by making abstractions. In Part I, you'll learn how.



Bibliography

- [1] Kenneth John Atchity. *A Writer's Time: Making the Time to Write*. W. W. Norton & Company, New York, revised and expanded edition, 1995.
- [2] Benjamin S. Bloom. The 2 sigma problem: The search for methods of group instruction as effective as one-to-one tutoring. *Educational Researcher*, 13(6):4–16, 1984.
- [3] Carl H. Brans and Robert H. Dicke. Mach's principle and a relativistic theory of gravitation. *Physical Review*, 124:925-935, 1961.
- [4] Edgar Buckingham. On physically similar systems. *Physical Review*, 4(4):345–376, 1914.
- [5] Adam S. Burrows and Jeremiah P. Ostriker. Astronomical reach of fundamental physics. *Proceedings of the National Academy of Sciences of the USA*, 111(7):2409-16, 2014.
- [6] Robert A. Caro. *The Power Broker: Robert Moses and the Fall of New York*. Vintage Books, New York, 1975.
- [7] Thomas P. Carpenter, Mary M. Lindquist, Westina Matthews and Edward A. Silver. Results of the third NAEP assessment: Secondary school. *Mathematics Teacher*, 76:652–659, 1983.
- [8] Michael A Day. The no-slip condition of fluid dynamics. *Erkenntnis*, 33(3):285–296, 1990.
- [9] Stanislas Dehaene. *The Number Sense: How the Mind Creates Mathematics*. Oxford University Press, New York, revised and updated edition, 2011.
- [10] Persi Diaconis and Frederick Mosteller. Methods for studying coincidences. *Journal of the American Statistical Association*, 84(408):853–861, 1989.
- [11] Peter G. Doyle and Laurie Snell. *Random Walks and Electric Networks*. Mathematical Association of America, Washington, DC, 1984.
- [12] Arthur Engel. *Problem-Solving Strategies*. Springer, New York, 1998.
- [13] William Feller. *An Introduction to Probability Theory and Its Applications*, volume 1. Wiley, New York, 3rd edition, 1968.
- [14] Richard P. Feynman, Robert B. Leighton and Matthew L. Sands. *The Feynman Lectures on Physics*. Addison-Wesley, Reading, MA, 1963. A "New Millenium" edition of these famous lectures, with corrections accumulated over the years, was published in 2011 by Basic Books.
- [15] Neville H. Fletcher and Thomas D. Rossing. *The Physics of Musical Instruments*. Springer, New York, 2nd edition, 1988.

- [16] Edward M. Forster. *Howard's End*. A. A. Knopf, New York, 1921.
- [17] Mike Gancarz. *The UNIX Philosophy*. Digital Press, Boston, 1995.
- [18] Mike Gancarz. *Linux and the Unix Philosophy*. Digital Press, Boston, 2003.
- [19] Robert E. Gill, T. Lee Tibbitts, David C. Douglas, Colleen M. Handel, Daniel M. Mulcahy, Jon C. Gottschalck, Nils Warnock, Brian J. McCaffery, Philip F. Battley and Theunis Piersma. Extreme endurance flights by landbirds crossing the Pacific Ocean: Ecological corridor rather than barrier? *Proceedings of the Royal Society B: Biological Sciences*, 276(1656):447-457, 2009.
- [20] Simon Gindikin. *Tales of Mathematicians and Physicists*. Springer, New York, 2007.
- [21] Geoffrey Grimmett. *Percolation*. Springer, Berlin, 2nd edition, 1999.
- [22] John Harte. *Consider a Spherical Cow: A Course in Environmental Problem Solving*. University Science Books, Mill Valley, CA, 1988.
- [23] Sighard F. Hoerner. *Fluid-Dynamic Drag: Practical Information on Aerodynamic Drag and Hydrodynamic Resistance*. Hoerner Fluid Dynamics, Bakersfield, CA, 1965.
- [24] Williams James. *The Principles of Psychology*, volume 2. Henry Holt, New York, 1890.
- [25] Edwin T. Jaynes. A backward look into the future. In W. T. Grandy Jr. and P. W. Milonni, editors, *Physics and Probability: Essays in Honor of Edwin T. Jaynes*. Cambridge University Press, Cambridge, UK, 1993.
- [26] Edwin T. Jaynes. *Probability Theory: The Logic of Science*. Cambridge University Press, Cambridge, UK, 2003.
- [27] Thomas B. Greenslade Jr.. Atwood's machine. *The Physics Teacher*, 23(1):24–28, 1985.
- [28] Anatoly A. Karatsuba. The complexity of computations. *Proceedings of the Steklov Institute of Mathematics*, 211:169–183, 1995.
- [29] Anatoly A. Karatsuba and Yuri Ofman. Multiplication of many-digit numbers by automatic computers. *Doklady Akad. Nauk SSSR*, 145:293–294, 1962. English translation in *Physics-Doklady* 7:595–596 (1963).
- [30] Doug King. Design masterclass 2: Thermal response. *CIBSE Journal*, pages 47–49, August 2010.
- [31] Rodger Kram, Antoinette Domingo and Daniel P. Ferris. Effect of reduced gravity on the preferred walk-run transition speed. *Journal of Experimental Biology*, 200(4):821–826, 1997.
- [32] Sanjoy Mahajan. *Order of Magnitude Physics: A Textbook with Applications to the Retinal Rod and to the Density of Prime Numbers*. PhD thesis, California Institute of Technology, 1998.
- [33] Sanjoy Mahajan. *Street-Fighting Mathematics: The Art of Educated Guessing and Opportunistic Problem Solving*. MIT Press, Cambridge, MA, 2010.
- [34] Ned Mayo. Ocean waves—their energy and power. *The Physics Teacher*, 35(6):352-356, 1997.
- [35] Karen McComb, Craig Packer and Anne Pusey. Roaring and numerical assessment in contests between groups of female lions, *Panthera leo*. *Animal Behaviour*, 47(2):379–387, 1994.

- [36] George Pólya. Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Strassennetz. *Mathematische Annalen*, 84(1):149–160, 1921.
- [37] George Pólya. Let us teach guessing: A demonstration with George Pólya [video-recording]. Mathematical Association of America, Washington, DC, 1966
- [38] George Pólya. *How to Solve It: A New Aspect of Mathematical Method*. Princeton University Press, Princeton, NJ, 2004.
- [39] Edward M. Purcell. Life at low Reynolds number. *American Journal of Physics*, 45:3–11, 1977.
- [40] Kenneth A. Ross and Donald E. Knuth. A programming and problem solving seminar. Technical Report, Stanford University, Stanford, CA, 1989. STAN-CS-89-1269.
- [41] Knut Schmid-Nielsen. *Scaling: Why Animal Size is So Important*. Cambridge University Press, Cambridge, UK, 1984.
- [42] Gilbert Strang. *Linear Algebra and its Applications*. Thomson, Belmont, CA, 2006.
- [43] David Tabor. *Gases, Liquids and Solids and Other States of Matter*. Cambridge University Press, Cambridge, UK, 3rd edition, 1990.
- [44] Geoffrey I. Taylor. The formation of a blast wave by a very intense explosion. II. The atomic explosion of 1945. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical*, 201(1065):175–186, 1950.
- [45] John R. Taylor. *Classical Mechanics*. University Science Books, Sausalito, CA, 2005.
- [46] David J. Tritton. *Physical Fluid Dynamics*. Oxford University Press, Oxford, UK, 1988.
- [47] Lawrence Weinstein. *Guesstimation 2.0: Solving Today's Problems on the Back of a Napkin*. Princeton University Press, Princeton, NJ, 2012.
- [48] Lawrence Weinstein and John A. Adam. *Guesstimation: Solving the World's Problems on the Back of a Cocktail Napkin*. Princeton University Press, Princeton, NJ, 2009.
- [49] Kurt Wiesenfeld. Resource letter: ScL-1: Scaling laws. *American Journal of Physics*, 69(9):938-942, 2001.
- [50] Michael M. Woolfson. *Everyday Probability and Statistics: Health, Elections, Gambling and War*. Imperial College Press, London, 2nd edition, 2012.

