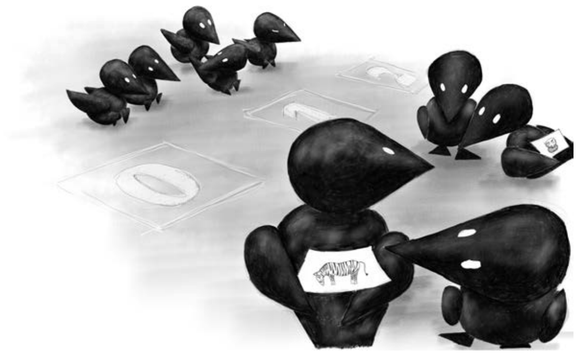
4 ANIMAL STICKERS AND CYCLIC GROUPS

4.1 THE GAME

Number of players:	4–10
You will need:	1 animal sticker per player, each featuring a picture of an owl, tiger, or zebra; 3 boxes, lined up on the ground

The audience attaches 1 animal sticker to the back of each player, so that no player can see his or her own animal. In principle, every player could have the same animal sticker. As in previous chapters, the players can only win if they work as a team. The players are free to walk around the room and look at the stickers of other players but are not allowed to look at their own sticker. Three boxes have been placed in a line on the ground. After 1 minute, each player is asked to line up behind 1 of the 3 boxes. The group wins if and only if the players in each of the 3 groups all have the same animal. For example, if all players behind box 1 have a tiger on their back, all players behind box 2 have a zebra, and no one stands behind box 3, the game is won. But if behind one of the boxes there is both a player with a zebra and a player with an owl, the game is lost.



What is the best strategy for the group? If the players use that strategy, how often will they win?

In general, there will be p animals, p boxes, and n players. But first, let us look at the example above, where we have $p = 3$ animals (zebra, owl, and tiger) and $n = 4$ players: Anwar, Bella, Charlie, and Deepti. We assume first that the audience member is choosing the animal for each player randomly from the set of 3 animals. A possible strategy for the players would be as follows: they prearrange that Anwar and Bella are going to line up behind the first box, Charlie will take the second and Deepti the third. What is their chance of success with this strategy? Suppose that the audience member chose a zebra for Anwar. Then Bella also needs to have a zebra for the group to win, and neither Charlie nor Deepti can have a zebra. Also, the animals for Charlie and Deepti need to be different. This leaves two options for the animals of Bella, Charlie, and Deepti: either zebra, owl, tiger or zebra, tiger, owl. Both of these sequences have a probability of $1/3 \cdot 1/3 \cdot 1/3 = 1/3^3 = 1/27$ if the animals are chosen independently for each player, so that the overall chance of winning is 2/27, which is below 8%. If Anwar gets a different animal, the same argument applies, so the overall chance of winning is 2/27.

Thus, lining up behind the boxes in prearranged configurations will not be a good strategy, and the probability of winning will fall further for more players.

Choosing a different configuration will not help either. For example, if all 4 players line up behind the same box, the chance is only 1/27, as Bella, Charlie, and Deepti then all need to have the same animal as Anwar, whereas spreading out across more boxes leaves more options for success.

The key to success thus is to exploit what players are observing about the animals of other players. At first, this seems unlikely to be helpful, as the animals of Anwar, Charlie, and Deepti do not hold any information about the animal of Bella if animals are chosen randomly by the audience. So what can Bella learn by her observations? Remember that Bella does not have to guess her animal in this game, she just has to line up behind the same box as other players with her animal. So what does Bella have in common with the players who have the same animal? What about the general case, where there are p animals, p boxes, and n players? We encourage the reader to pause here and think a bit more about this game.

4.2 SOLUTION FOR 3 ANIMALS

Let us revisit the example including Anwar, Bella, Charlie, and Deepti. The key insight is the following. If Bella observes 2 zebras and 1 tiger on the other players, say, then she will know that any other player with the same animal as hers (if there are any) will also observe 2 zebras and 1 tiger, irrespective of which animal is hers. And players with a different animal will see a different configuration of animals. For example, if she had a zebra on her back, and so did Anwar, then Anwar would also see 1 tiger and 2 zebras (instead of his own zebra, he will see Bella's). If she has an owl on her back, then nobody else will see 2 zebras and a tiger, as all of them will see her owl among the 3 animals they observe. The players thus need to be able to translate the collection of animals they observe into a choice of the box, so that players seeing the same collection of animals pick the same box, and players seeing a different collection of animals do not pick the same box. This can be done using numbers and so-called "modulo" computations.

First, we can attribute to each animal a value in $\{0, 1, 2\}$:

zebra is Zero, owl is One, tiger is Two.

Taking the previous example, the situation is as follows:

$$\text{Anwar has a zebra} \equiv 0,$$
$$\text{Bella has an owl} \equiv 1,$$
$$\text{Charlie has a tiger} \equiv 2,$$
$$\text{Deepti has a zebra} \equiv 0.$$

Having seen the animals on the other players, every player lines up behind 1 of the 3 boxes. To do so, they pretend that the boxes are numbered 0, 1, and 2 (left to right from the players viewpoint). Anwar can see that Bella has an owl on her back, Charlie has a tiger, and Deepti has a zebra. His choice of which box to stand behind now depends on the animals he sees on the other players. But how can Anwar choose a box so that everybody else with a zebra (Deepti in the example) will choose the same box as him, and all the players with a different animal will choose a different box?

As mentioned above, the key is that players with the same animal will see the same collection of animals. As a strategy, the players can, for example, sum up the numeric value of all the animals they are seeing. For Anwar, this means

$$1\,(\text{Bella's owl}) + 2\,(\text{Charlie's tiger}) + 0\,(\text{Deepti's zebra}) = 3.$$

The computations performed by the 4 players are summarized in the following table:

	Anwar	Bella	Charlie	Deepti	Sum
Anwar can see	✗	1 (owl)	2 (tiger)	0 (zebra)	3
Bella can see	0 (zebra)	✗	2 (tiger)	0 (zebra)	2
Charlie can see	0 (zebra)	1 (owl)	✗	0 (zebra)	1
Deepti can see	0 (zebra)	1 (owl)	2 (tiger)	✗	3

Anwar and Deepti will both arrive at a numeric value of 3. They now have to translate this numeric value into a choice of 1 of the 3 boxes, making sure that players who see a different collection of animals (different by, at most, only 1 animal) will choose a different box. Here, the observed sums could range between value 0 (only seeing zebras on all other players) to 6 (all tigers). So, how can these numbers be distributed among 3 boxes?

Let us say that 2 numbers are equivalent if they are equal modulo 3, which means that their difference is a multiple of 3. The numbers −1, 2, 5, 8, 14, and 602 are therefore equivalent, for example, and are said to be in the same equivalence class. Looking only at nonnegative integer values, the equivalence classes in our example are

$$[0] = \{0, 3, 6, \ldots\},$$

$$[1] = \{1, 4, 7, \ldots\},$$

$$[2] = \{2, 5, 8, \ldots\}.$$

Section 4.3 contains a further, more precise definition of equivalence classes. For the solution, we now associate box 0 with equivalence class [0], box 1 with equivalence class [1], and box 2 with equivalence class [2]. Players have to line up behind the box whose equivalence class contains the sum of the animal values they are seeing on all other players, excluding themselves.

In the example, Anwar translates the sum of 3 into equivalence class [0] and lines up behind box 0. Bella obtains a sum of 2 and hence lines up behind box 2. Charlie will see a sum of 1 and lines up behind box 1, and Deepti lines up behind box 0 (since her sum equals 3). All players are grouped correctly.

This is not a coincidence. If they use this strategy, the team will win every time, no matter how the animal stickers are distributed. To prove this, we need to show that (a) players with different animals choose different boxes, and (b) players with the same animals choose the same box. The second part is easy to see: players with the same animal will see the same collection of other animals and therefore will arrive at the same sum and hence the same box.

So why do players with different animals always choose different boxes? What if they arrive at different sums but the boxes associated with these sums are identical? To see that this cannot happen, consider the difference between the sums of 2 players. It is largest when one carries an animal with a value of 0 and the other with a value of 2. The difference between the two sums is then 2 at most. Two different numbers are only equivalent, however, if they differ by a multiple of 3. Thus, the two players cannot arrive at the same box if they carry different animals.

For the more general case, we have p boxes and p animals and equivalence classes

$$\begin{aligned}
\text{box } 0 \text{ associated with } [0] &= \{0, p, 2p, \ldots\},\\
\text{box } 1 \text{ associated with } [1] &= \{1, p+1, 2p+1, \ldots\},\\
&\vdots\\
\text{box } p-1 \text{ associated with } [p-1] &= \{p-1, 2p-1, 3p-1, \ldots\},
\end{aligned}$$

and, as we discuss below, all the above arguments and strategies still apply.

We will now introduce some mathematics that will help us solve the general case of p animals and n players. If you are instead interested in a different variation of the game, feel free to skip the following section and jump directly to section 4.4.

4.3 SOME MATHEMATICS: CYCLIC GROUPS

Equivalence Classes

In the example above, we did not distinguish between the sums 1, 4, or 7, because all of them implied that the players lined up in front of the second box. Mathematically, we have created an equivalence relation on the integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. We say that $\sim$ is an *equivalence relation* on $\mathbb{Z}$ if it satisfies the following three properties:

1. *Reflexivity*: $a \sim a$ for all $a \in \mathbb{Z}$.
2. *Symmetry*: if $a \sim b$, then also $b \sim a$ for all $a, b \in \mathbb{Z}$.
3. *Transitivity*: if $a \sim b$ and $b \sim c$, then also $a \sim c$ for all $a, b, c \in \mathbb{Z}$.

These three properties are clearly fulfilled if we say two numbers a and b are equivalent if their difference $a - b$ is divisible by 3. Instead of writing $3 \sim 0$ and $11 \sim 2$, it is common to write

$$3 \equiv 0 \bmod 3, \text{ and } 11 \equiv 2 \bmod 3,$$

which is also called taking numbers *modulo* 3. All integers that are equivalent to a form the equivalence class

$$[a] := \{b \in \mathbb{Z} : b \sim a\}.$$

In our example, the equivalence class of [0] is $[0] = \{\ldots, -3, 0, 3, 6, \ldots\}$, and the equivalence classes of 2 and 11 are identical: $[11] = [2]$. It is possible to perform calculations with equivalence classes, just like we can calculate with numbers: they form a group.

Groups

A set G and an operation $\oplus$ form a *group* if the following holds:

1. *Closure*: for all $x, y \in G$, we have $x \oplus y \in G$.
2. *Identity*: there exists an identity element $1 \in G$ such that for all $x \in G$, it holds $x \oplus 1 = 1 \oplus x = 1$.

3. *Associativity*: for all $x, y, z \in G$, we have $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.
4. *Inverse*: for all $x \in G$, there exists an element x^{-1} with $x \oplus x^{-1} = x^{-1} \oplus x = 1$.

For our example, the relevant set G is the set of all distinct equivalence classes if using the modulo 3 equivalence relation. These equivalence classes are [0], [1], and [2]. The relation $\oplus$ in this case is defined by

$$[a] \oplus [b] := [a+b], \tag{4.1}$$

for example, $[2] \oplus [2] = [2+2] = [4] = [1]$. All four properties are then satisfied.[1] The identity element is [0], and the inverse of $[a]$ is $[-a]$. Our group G is often denoted by $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}_3$, and is usually referred to as "$\mathbb{Z}$ modulo $3\mathbb{Z}$".

Cyclic Groups

A *cyclic group* G is a group that can be generated by a single element. In our example, all members of the set can be generated by the element [1], as $[0] = [3] = [1] \oplus [1] \oplus [1]$ and $[2] = [1] \oplus [1]$.

Revisiting the Solution

We are now able to rewrite the solution above with the new notation. First, let us denote the animals of Anwar, Bella, Charlie and Deepti by

$$C_{Anwar}, C_{Bella}, C_{Charlie}, \text{ and } C_{Deepti},$$

respectively. Then, each player calculates the sum of the other players' animals as follows:

$$\begin{aligned} S_{Anwar} &= \quad C_{Bella} + C_{Charlie} + C_{Deepti} \\ &= \quad 1 \text{ (owl)} + 2 \text{ (tiger)} + 0 \text{ (zebra)} = 3, \end{aligned}$$

1. Strictly speaking, we would also need to show that our operation is well defined: if $[a] = [\tilde{a}]$ and $[b] = [\tilde{b}]$, then $[a+b] = [\tilde{a}+\tilde{b}]$. Otherwise, our definition (4.1) would not make any sense. Please have a go at proving this statement.

$$\begin{aligned}
S_{Bella} &= C_{Anwar} && + C_{Charlie} + C_{Deepti} \\
&= 0 \text{ (zebra)} && + 2 \text{ (tiger)} + 0 \text{ (zebra)} = 2, \\
S_{Charlie} &= C_{Anwar} + C_{Bella} && + C_{Deepti} \\
&= 0 \text{ (zebra)} + 1 \text{ (owl)} && + 0 \text{ (zebra)} = 1, \\
S_{Deepti} &= C_{Anwar} + C_{Bella} + C_{Charlie} && \\
&= 0 \text{ (zebra)} + 1 \text{ (owl)} + 2 \text{ (tiger)} && = 3.
\end{aligned}$$

Player k then chooses a box $b \in \{0, 1, 2\}$ such that $[b] = [S_k]$. So, $b = 0$ for Deepti, for example, since $[0] = [S_{Deepti}] = [3]$.

Solution to the General Case

Somewhat surprisingly, the game still works for p animals, p boxes, and n players, no matter how large p and n are. The solution can then be formulated as follows. Let $C_k \in \{0, \ldots, p-1\}$ be the value of the animal on player k. Let further

$$S_k := \sum_{\ell=1;\ell \neq k}^{n} C_\ell = \sum_{\ell=1}^{n} C_\ell - C_k$$

be the sum of the animal values on all players other than player k. The box $b \in \{0, \ldots, p-1\}$ chosen by player k is the box for which the equivalence classes agree, that is $[b] = [S_k]$. Instead of $\mathbb{Z}/3\mathbb{Z}$, we here consider the group $\mathbb{Z}/p\mathbb{Z}$, that is, two numbers are equivalent if their difference can be divided by p. If $p = 7$, for example, and a player computes the sum 18, she should then choose box number 4, since

$$[4] = [18]$$

(because $18 - 4$ can be divided by 7). We then have the equivalences

$$[S_k] = [S_{k'}] \quad \Leftrightarrow \quad S_k = S_{k'} \quad \Leftrightarrow \quad C_k = C_{k'},$$

so that the equivalence classes of two players match if and only if they have the same animal on their backs. $[S_k] = [S_{k'}]$ implies $S_k = S_{k'}$, because the values of two different sums S_k and $S_{k'}$ can differ by at most $p - 1$. Two players will hence choose the same box if and only if their animals match.

4.4 VARIATION: COLORED HATS IN A LINE

Number of players:	4–10
You will need:	colored hats in 3 (or more) unique colors

All n players form a line and members of the audience choose a colored hat for each player; several players may receive the same color. There are p distinct hat colors, and everybody knows what the color options are.

As illustrated above, each player can see the color of the hats in front of them. They can neither see the color of their own hat nor the colors of any of the other players behind them. The person at the front of the line cannot see any of the hats. Each player guesses their color, starting with the person at the back. All the players can hear all the answers. The group can agree together on a strategy beforehand to make sure that as many people as possible will guess their color correctly.

What is the best strategy, and how many colors can the group guess correctly? Is there a way to make sure that a certain number of colors are always guessed correctly every time the game is played?

How Well Can a Strategy Work?

The general idea is similar to the animal sticker game. However, it is clear that, generally speaking, not all players will be able to guess their colors correctly. The player at the back of the line is in a tricky situation. She has to make her guess first and can base her decision only on the colors of the hats in front of her. If we assume that the colors in front of her are chosen independently of her own hat color, then the observed colors will not contain any information about her own color. Even worse, if the audience guesses the team's strategy, they can make the player in the back answer incorrectly all the time. If the player forgets about the group and thinks only about herself, all she can do is guess randomly. In this case, her choice will be wrong with probability $1-1/p$ for p distinct colors. We thus have a bound on the expected number[2] of mistakes:

$$\mathbf{E}(\text{mistakes}) \geq 1 - \frac{1}{p}.$$

As the number of correct guesses is equal to n minus the number of mistakes, we get an upper bound on the number of correct guesses:

$$\mathbf{E}(\text{correct guesses}) \leq n - 1 + \frac{1}{p}.$$

Can we find a strategy that achieves this bound in that there will be on average $n-1+1/p$ correct guesses among all n players?

Solution

Let us start by discussing the solution for the case where three hat colors (zucchini, orange, and turquoise) are chosen randomly for four players (Anwar, Bella, Charlie, and Deepti). We first assign a numerical value to a color in the following way:

2. See also "What Is …an Expectation?" (appendix B.6).

zucchini is Zero, orange is One, and turquoise is Two,

just as we did in the animal sticker game. If you have not heard of the color "zucchini" before, you can imagine it as being green (it is unfortunately the only color we could think of that starts with a "z"). The distribution of hats is:

Anwar	Bella	Charlie	Deepti
zucchini	orange	turquoise	zucchini

Every player now has to guess their color, starting with Deepti, the player at the back.

The argument above implies that the first player to guess her color, Deepti, has no better chance to get her color right than if she were randomly guessing her color. One strategy for her would indeed be to just guess a color at random and hope that the guess is correct. There would be a high probability that she would be wrong, namely $1 - 1/p$. Even worse, Charlie in front of her then again has no information about his own color and is in the same clueless state that Deepti is in. If Charlie and subsequent players then repeat the random guessing strategy, we can hardly expect many correct guesses.

So far, we have not taken advantage of the fact that there is a key difference between Charlie and Deepti. Charlie can base his guess on both the colors of Anwar's and Bella's hats in front of him and also on the guess of Deepti. The colors of Anwar's and Bella's hats in front of him were chosen independently of Charlie's color and therefore contain no information about the color of Charlie's when considered on their own. Therefore, the key has to be to create a dependence between Deepti's guess and Charlie's hat. The dependence is only relevant if Deepti's guess is based on the color of Charlie's hat.

How can we get Deepti to make a guess that will help Charlie work out the color of his hat? One possibility is that Deepti just announces the color of Charlie's hat in front of him instead of trying to guess his own color. Then Charlie can guess his own color correctly, as he just has to repeat the color that Deepti just said. However, a downside of this strategy is then that Bella again has no information about the color of her hat. She will have to repeat the procedure by announcing the color of Anwar's hat in front of her, which Anwar can then repeat and guess his color correctly. Following this strategy, we guarantee that at least half of the guesses will be correct (Anwar's and Charlie's). Bella's and Deepti's guesses will be correct only if by chance their hat color matches Anwar's and Charlie's color, respectively.

It turns out that we can can do better than that and, in fact, we can match the previous bound by making sure that everybody except possibly Deepti can guess their color correctly!

To avoid Bella having to start from scratch, we clearly have to use a different strategy, and we can use one similar to that used in the animal sticker game. The difference is now that the players only have partial information. Consider the table of colors that the players can see:

	Anwar	Bella	Charlie	Deepti	Sum
Deepti can see	0	1	2	✗	$3 = S_{Deepti}$
Charlie can see	0	1	✗	✗	$1 = S_{Charlie}$
Bella can see	0	✗	✗	✗	$0 = S_{Bella}$
Anwar can see	✗	✗	✗	✗	$0 = S_{Anwar}$

Here and below, we denote the sum of all observed colors for player k by S_k and the player k's true color by C_k. The associated colors are again the equivalence classes

$$\text{zucchini color: } [0] = \{\ldots, -3, 0, 3, 6, 9, 12, \ldots, \},$$

$$\text{orange color: } [1] = \{\ldots, -2, 1, 4, 7, 10, 13, \ldots, \},$$

$$\text{turquoise color: } [2] = \{\ldots, -1, 2, 5, 8, 11, 14, \ldots\}.$$

The first player to announce, Deepti, now announces the color of her observed sum

$$\text{Deepti announces: } [S_{Deepti}] = [3] = [0] = \text{zucchini}.$$

Perhaps surprisingly, the next players can now calculate their colors. First, the difference between S_{Deepti} and $S_{Charlie}$ is all down to the color of Charlie's hat. Thus

$$\text{Charlie announces: } [S_{Deepti} - S_{Charlie}] = [0 - 1] = [2] = \text{turquoise},$$

and he gets his color correctly. Here, it does not matter that he substitutes 0 for the color zucchini that Deepti announced instead of the value 3 that Deepti had in mind, because both belong to the same equivalence class. In fact, he could take any member of the equivalence class of color zucchini instead of the value 0 and still arrive at the correct solution.

Continuing like this, Bella can announce her color correctly as

$$\text{Bella announces: } [S_{Deepti} - C_{Charlie} - S_{Bella}]$$
$$= [0 - 2 - 0] = [-2] = [0] = \text{ orange}.$$

Note that she has to subtract the guessed color of Charlie, not the sum Charlie has been seeing. Why is the guess correct for Bella? The sum that Deepti sees is

$$S_{Deepti} = C_{Charlie} + C_{Bella} + C_{Anwar} = C_{Charlie} + C_{Bella} + S_{Bella},$$

and thus, substracting $C_{Charlie}$ and S_{Bella} on both sides,

$$C_{Bella} = S_{Deepti} - C_{Charlie} - S_{Bella}.$$

The equivalence class (color) of C_{Bella} is therefore equal to the equivalence class of the right-hand side, where we replace the correct values, S_{Deepti} and $C_{Charlie}$, by any arbitrarily chosen member of their respective equivalence classes. The color $[S_{Deepti}]$ is announced by Deepti, and the color $[C_{Charlie}]$ is announced by Charlie. To arrive at her own color, Bella is thus effectively subtracting the color that Charlie announced and the sum of colors she has seen from the color that Deepti announced first.

Finally, Anwar can guess his color correctly as

$$\text{Anwar announces: } [S_{Deepti} - C_{Charlie} - C_{Bella} - S_{Anwar}]$$
$$= [0 - 2 - 1 - 0] = [-3] = [0] = \text{zucchini.}$$

Anwar has to subtract the colors $[C_{Charlie}]$ and $[C_{Bella}]$ that Charlie and Bella announced and the sum of colors he has seen from the color $[S_{Deepti}]$ announced by Deepti.

The scheme can also be used with a general number $p \in \mathbb{N}$ of different colors and $n \in \mathbb{N}$ players. The sums of the observed colors of player k is defined as above:

$$S_k := C_{k-1} + C_{k-2} + \cdots + C_1.$$

Player n in the back announces the equivalence class of S_n, and subsequently, player k announces the equivalence class of

$$S_n - C_{n-1} - C_{n-2} - \cdots - C_{k+1} - S_k.$$

Because

$$S_n = C_{n-1} + C_{n-2} + \cdots + C_{k+1} + C_k + S_k,$$

this will always be the correct color.

All players will thus announce their color correctly except, possibly, for the player at the back of the line. This guarantees that out of n players, at least $n-1$ players will guess their color correctly! The player in the back can be right, too, if his color matches the color of the sum of all colors in front of him (that is, if $[S_n] = [C_n]$). If the color of the hats are chosen independently, this will happen with probability $1/p$. The expected number of successes with this scheme is based on the assumption of independently chosen colors $n - 1 + 1/p$, which matches the upper bound we derived in section 4.4. Thus, the scheme cannot be improved.

A (not very practical but mathematically interesting) version for infinitely many players can be found in appendix C.2.

Adversarial Audience

If the audience expects the team to use the above strategy, it can of course provide a color to the player in the back that is not in the equivalence class of S_n. This way, the last player will certainly answer incorrectly, and the expected number of successes is down to $n-1$. There is, however, a way out for the team: The players first agree on a random number R between 0 and $m-1$. Player n in the back then announces S_n+R instead of S_n, and all players adjust accordingly. This way, the expected number of successes is back to $n-1+1/p$, independently of how the audience distributes the colors.

4.5 SHORT HISTORY

The origin of the hats-in-a-line game could not clearly be traced, but both games are mentioned, for example, in two papers in the *Electronic Journal of Combinatorics*, namely, Aravamuthan and Lodha [2006] and Paterson and Stinson [2010]. The latter also introduce a new game that sits somewhere between the hats-in-a-line game and the hat-color game of section 4.1. The idea for solving the infinite hat game presented in section C.2 can be found in the article by Hardin and Taylor [2008], which also contains further references on infinite hat problems.

4.6 PRACTICAL ADVICE

We can, of course, use colored hats instead of animal stickers for the animal sticker game (and vice versa).

Variations on the Animal Sticker Game

In the version described, the players line up behind the boxes jointly and are able to see where everybody else is lining up. In a seemingly more difficult version, the players sit down and then close their eyes (or the other way around, as they like) and are called one after another to choose a box, without being able to see or hear the choices of other players. They can, of course, use the same strategy here, and the game is thus not more difficult to play.

An additional variation is as follows: The first player to choose a box can look at her animal after having chosen a box. She then has to mark all three boxes with a zebra, tiger, or owl, so that only the audience but not the players can see the markings. The remaining players then have to line up behind the boxes that are marked with their animal.

She is, of course, marking the box she chose with her animal and the others in reverse order of their numeric values. In other words, after having put her own animal on her chosen box, the box with a numeric value 1 *lower* than her own box (first box if she had the second, second box if her own was the third, third box if her own box was the first) is marked with the the animal with a numeric value 1 *higher* (so an owl if her own animal is the zebra, a tiger if her own animal is an owl, and a zebra if her own animal is a tiger). Analogously, then the box with a numeric value *lower* by 2 (or equivalently, higher by 1) is paired with the animal with numerical value *higher* by 2 (or equivalently, lower by 1). The remaining players will play just as in the original game and will end up behind the box with the correct label.

All of the above versions assume that the boxes are either numbered 0, 1, and 2, or have a particular order (from left to right, say). There is a variation that works without having such an order. If any animal can be assumed to appear at least once, one can perform a slight variation of the game. One player with a 0 can take a look at her sticker (cheating) and say: “I have got a …” (announcing the animal he finds). Then all players with a zero could follow: “we do, too!” Then the group of players with a 1 follows, and so on. That is, each player can announce her own animal—without having to line up behind any box.

Variation on the Colored Hat Game

The first player to announce (at the back of the line) is generally wrong (unless he is lucky), while all other players get their colors right. The first player to announce can, of course, say something along the lines of "I would have guessed 'turquoise' but I am quite sure it is wrong," in this way signaling the color turquoise to the other players while acknowledging that he is probably wrong.

Faster Modulo Operation in Both Games

Taking the modulo operation can take some time if the sums involved grow very large. This can be problematic for a large number of players. It becomes easier to play the games if we also use negative values, such as replacing

$$\text{zebra} \equiv 0, \quad \text{owl} \equiv 1, \quad \text{tiger} \equiv 2$$

with

$$\text{zebra} \equiv 0, \quad \text{owl} \equiv 1, \quad \text{tiger} \equiv -1.$$

or, to make it memorable,

$$\text{zebra} \equiv 0 \text{ (zero)}, \quad \text{owl} \equiv 1 \text{ (one)}, \quad \text{mole} \equiv -1 \text{ (minus one)}.$$

Then the players only have to count how many more owls than moles they can see (and take the result modulo 3).