6 ANIMAL MATCHING AND PROJECTIVE GEOMETRY

6.1 THE GAME

Number of players:	7 (see section 6.6 on advice for using different numbers of players)
You will need:	7 different kinds of animal toys, 3 of each animal

The 21 different toy animals are arranged on a table; 3 lions, 3 mice, 3 snakes, 3 koalas, 3 cats, 3 elephants, and 3 giraffes. The players enter the room one after another, and each player takes 3 animals. Players are not allowed to see what the other players have chosen. The players can hide their choices behind their backs or in their pockets, so neither the other players nor the audience can see which animals they have taken.

The audience is now asked to select a pair of players. The players' challenge is to produce a common animal from the 3 they have behind their back. Say the first pair of players are as shown on the previous page. Once the pair of players is chosen, they present all the animals they have taken and see whether they find a match between them. They have both taken a giraffe and will each show the giraffe to the audience to answer the challenge. If they do not have an animal in common, all players lose. If they do produce a match, the audience can select a new pair of players, challenging them to produce a matching pair of animals. The audience can ask 3 times, and the players only win if all 3 pairs of players can show the same animal.

To make the game more difficult, the audience can, for each player, choose the first of the 3 animals that the player has, but they must make sure that they choose a different animal for each person.

What is the best strategy for the players? What is the best strategy for the audience? How likely are the players to succeed? What about other numbers of players and animals?

We can check the success rate of some simple strategies. The simplest strategy is perhaps that all players take animals at random from the ones remaining on the table. In the case of 7 players, the chance of success of the team will be just over 27%.

There is an obvious way to improve the purely random strategy. For a given player, it is clearly not optimal to have 2 animals of the same kind in her hand. A simple modification is hence that each player makes sure that she takes three distinct animals from the ones remaining (at least as long as possible; the players toward the end of the selection process might no longer have the option of taking 3 distinct animals). This simple modification already improves the success rate considerably. Let us first look at 2 randomly chosen players out of the 7. If both players take 3 animals at random from those available, the chance of a match between them will be around 64%. If they take 3 distinct animals at random each, then they have a chance of just over 88% of having at least 1 animal in common. If all the players make sure that they each take 3 different animals from those remaining, for as long as it is possible to take 3 different animals, their chances of success will already be almost 63% instead of the 27% with the previous strategy. Can the team beat the 63% success rate?

Another way to think about the problem is to first just look at the koalas. There are 3 koalas in the game. Let us assume that the 3 koalas are distributed among 3 different players. Then each of the 3 possible pairs of these 3 players will be "linked," as they possess an animal in common (the koala). So the koala is introducing 3 links in the game, where a *link* is a pair of players that can successfully show an animal in common. Each of the other animals is also introducing 3 links into the game (and fewer than 3 if players start having more than 1 of each kind of animal). The 7 animals in total thus introduce 21 links. How many links would we need to make sure that each pair of players has an animal in common? We would need as many links as there are pairs of players. For 7 players, we have $7 \cdot 6/2 = 21$ unique pairs. So the number of available links just about matches the number of links we need to connect all players! Therefore, it seems just about possible that the game could be played with a 100% success rate. But we cannot waste any link if we want to achieve this. If one pair of players has 2 links in common (they share more than one pair of animals), then, as a consequence, there would have to be another pair of players with no animal in common. We believe that now is a good time for the reader to pause and think about a possible solution.

6.2 SOLUTION

There is one solution that will never fail, because every pair of players the audience selects is guaranteed to have an animal in common (that is, the success rate equals 100%). If the players choose their 3 animals following a pattern equivalent to the one set out below:

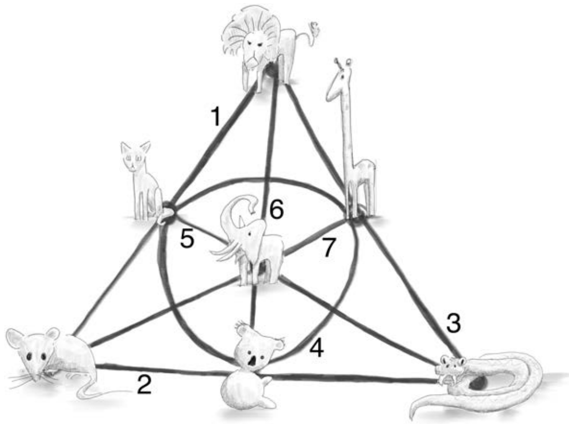
Animal assigned by audience	Other animals taken
lion	cat, mouse
mouse	koala, snake
snake	giraffe, lion
koala	giraffe, cat
cat	elephant, snake
elephant	lion, koala
giraffe	elephant, mouse

If you take any pair of players with this choice of animals, they will always have 1 animal in common.

Is it also possible to construct such a solution if we have 8 players and 8 different kinds of animal? In general, it turns out that solutions are known if p is a prime number (or a power of a prime number) and there are p^2+p+1 players and p^2+p+1 unique animals, with each player taking $p+1$ different animals. In our example, $p=2$ and therefore, we have 7 players and 7 unique animals, with each player taking 3 animals. For $p=3$, we play with 13 players and 13 unique animals, with each player taking 4 unique animals.

6.3 FANO PLANES

Let us look at the case of $p=2$ and 7 animals. Each of the 7 players takes 3 animals.



The solution can be characterized by the so-called Fano plane. Each point corresponds to 1 of 7 animals. In the above figure, each of the lines marked 1–7 corresponds to 1 player. Here, we call line 4 a line, even though it is bent. Each line (player) passes through exactly 3 points (animals). Each point (animal) is contained in 3 lines (players).

The way the animals are shared among the players can be recorded in a so-called incidence matrix A that contains only 0s and 1s. Each row corresponds to a player, and each column corresponds to an animal. In our example, the matrix A has dimension 7×7 for $p = 2$, as we have 7 players and 7 animals. A value of 1 in position (k, j) indicates that player k has taken animal j.

	Lion	Mouse	Snake	Koala	Cat	Elephant	Giraffe
Player 1	1	1	0	0	1	0	0
Player 2	0	1	1	1	0	0	0
Player 3	1	0	1	0	0	0	1
Player 4	0	0	0	1	1	0	1
Player 5	0	0	1	0	1	1	0
Player 6	1	0	0	1	0	1	0
Player 7	0	1	0	0	0	1	1

$= A.$

The first row of A indicates that player 1 takes a lion, a mouse, and a cat. This corresponds to line 1 in the Fano plane connecting the lower left with the top corner of the triangle. The second player (second row in A) takes a mouse, a snake, and a koala, corresponding to line 2 in the Fano plane.

We show in appendix C.4 how the desired properties of the solution (every pair of players has an animal in common) relate to properties of the incidence matrix. The relation will also help us understand why every player has to hold exactly 3 animals; there is no valid solution where one player, for example, holds only 2 animals and another players holds 4.

So, the question is how to construct general incidence matrices that can also be applied to other numbers of players and animals.

6.4 SOME MATHEMATICS: PROJECTIVE GEOMETRY

In section 6.3, we have identified lines in a plane with players and points on the plane as animals. In mathematics, the relation between points and lines is studied in the field of geometry. Formally, a line is just a set of points; we say that the point is *incident* with the line and the line is *incident* with the point if the point lies on the line.

For a standard 2-dimensional Euclidean plane (think about drawing points and straight lines on this page), the following two statements are valid:

I. Given any 2 distinct points m and m', there is exactly 1 line ℓ incident with both of them.

II. Given any line ℓ and any point m not incident with ℓ, there is exactly 1 line incident with m that does not meet ℓ.

Property I states that any two points can be connected by exactly one line. The line in question in property II is, of course, a line that is parallel to ℓ, as parallel lines do not intersect in standard Euclidean geometry. If we associate lines with players and points with animals, then property II can be problematic, as it means that 2 players who correspond to parallel lines have no animal in common.

Projective geometry is different from the Euclidean geometry that we are used to. In a *projective plane*, every pair of lines will intersect at exactly 1 point (which can, perhaps unintuitively, be achieved by adding a point at infinity for each pair of parallel lines), so that the following statements hold:

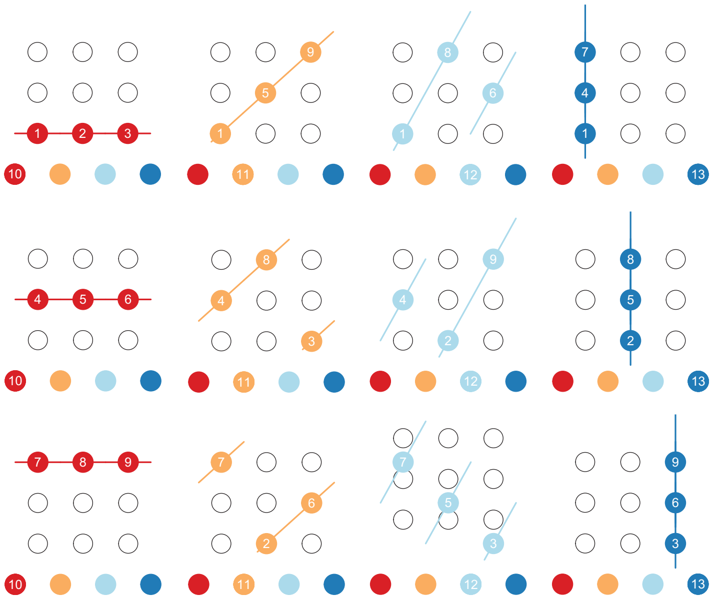
i. Given any 2 distinct points, there is exactly 1 line incident with both of them.

ii. Given any 2 distinct lines, there is exactly 1 point incident with both of them.

These are the properties we are looking for. Assume that we have just a finite number of points. Then we will have a finite number of lines, and the properties i and ii are exactly what we desire: Each pair of animals will appear for exactly 1 player, and as desired in the game, each pair of players will have exactly 1 animal in common.

A Simple Construction of the Solution

An illustration of a solution for $p = 3$ with $p^2 + p + 1 = 13$ distinct animals and 13 players is shown on the next page. The animal types are identified with the numbers 1, 2, …, 13. The numbers 1 to $p^2 = 9$ are arranged in a square, while the remaining numbers 10, 11, 12, and 13 indicate different slopes of lines passing through this grid of points.



Number 10, for example, corresponds to the slope of all horizontal red lines and can be interpreted as the "point at infinity." In this geometric construction, the nodes in both dimensions of the $p \times p$ grid are defined modulo p (see chapter 4). The modulo operation can be visualized by multiple copies of the original $p \times p$ grid, arranged to the left, right, top, and bottom of the original $p \times p$ grid. The lines shown above in the original grid are straight lines in this extended version.

A player corresponds again to a line; she takes the animals of the 3 numbers that her line passes through and also the animal that corresponds to the slope indicator of her line. The top-left figure panel corresponds to the first player; she holds animals $1, 2, 3$ and also animal 10 as an indicator for the horizontal line that is used to connect 1, 2, and 3. The players in the left column of panels in the figure all have a horizontal slope and hence all have the slope indicator 10 in common. Lines with a different slope will intersect in exactly 1 animal on the square grid. We have p^2 animals on the grid and $p+1$ slopes, so there are p^2+p+1 distinct animals. The figure on the previous page shows $p(p+1)$ combinations of points on the grid and slopes. The configuration for the last player (not shown) consists of all the slope indicators $10, 11, 12, 13$. This last configuration will also have exactly 1 animal in common with all of the configurations shown above. The slope indicators force parallel lines to intersect in the sense of properties i and ii. The construction works only if p is a prime (or power of a prime), as lines with different slopes would otherwise not necessarily intersect.[1]

A More Formal Construction

The role of the prime numbers will become more apparent in a more abstract construction, which will lend itself more easily to actual implementation. Say we start with a prime number p and the set of integers $\{0, 1, \ldots, p-1\}$. For $p=2$, the set is $\{0, 1\}$, for example. Then we can define addition and multiplication by taking the results modulo p (this defines a so-called *finite field*; see chapter 4 for more discussion of the modulo operation and cyclic groups). If we now look at a 3-dimensional vector space over this field, we get for $p=2$ the following 8 points:

$$(0,0,0), (0,0,1), (0,1,0), (1,0,0), (0,1,1), (1,1,0), (1,0,1), (1,1,1)$$

as the 8 corners of a unit cube. Taking the results of addition and multiplication modulo 2 means that, for example,

$$(0,0,1)+(0,1,1)=(0,1,0).$$

Now one can look at the set of all lines in the 3-dimensional vector space that pass through the origin and one of the other

1. You can try this, for example, on a 6×6 grid.

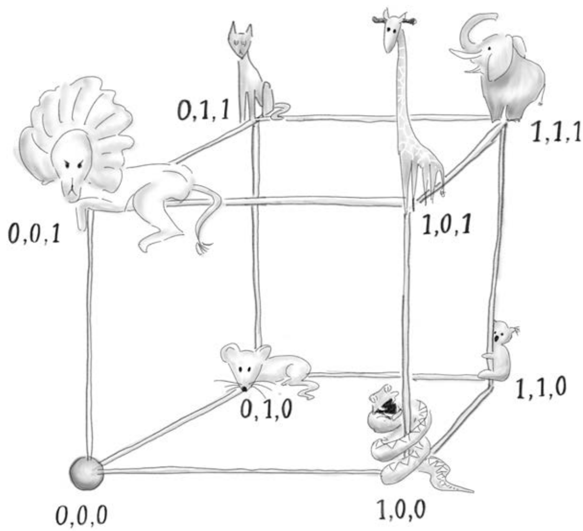
points. For $p=2$, there will be 7 distinct lines, each passing through the origin and one of these points:

$$(0,0,1),(0,1,0),(1,0,0),(0,1,1),(1,1,0),(1,0,1),(1,1,1).$$

Now we can identify each line passing through the origin with an animal:

(1) lion:	$(0,0,0)-(0,0,1)$
(2) mouse:	$(0,0,0)-(0,1,0)$
(3) snake:	$(0,0,0)-(1,0,0)$
(4) koala:	$(0,0,0)-(1,1,0)$
(5) cat:	$(0,0,0)-(0,1,1)$
(6) elephant:	$(0,0,0)-(1,1,1)$
(7) giraffe:	$(0,0,0)-(1,0,1)$

The line passing through $(0,0,0)$ and $(0,0,1)$ thus corresponds to a lion and the result is illustrated below.



Therefore, the lines in the 3-dimensional vector space are the animals, (i.e., the Fano points), but what are the players (i.e., the Fano lines)? These will be the planes in the 3-dimensional space!

These planes are obtained by taking two lines and then considering all the sums of all the points on such lines. If we know two of the animals already, we will be able to determine what the third animal will be. For example, the plane that contains both lion and cat is passing through the points

$$(0,0,0),(0,0,1), \text{ and } (0,1,1).$$

Since $(0,0,1)+(0,1,1)=(0,1,0)$, the plane will also contain the line $(0,0,0)-(0,1,0)$ and hence a mouse. In this sense we can write

$$\text{lion} + \text{cat} = \text{mouse},$$

and the player, holding a lion and a cat, will also have to hold a mouse.

There are 7 distinct planes in the 3-dimensional vector space and we identify them with players in the game (or Fano lines). We have

$$\text{player 1: plane through } \{(0,0,0),\ (0,0,1),\ (0,1,0),\ (0,1,1)\}.$$

As we have seen, the plane of player 1 contains the animals lion, mouse, and cat. Note that any combination of the points on this plane will again be one of these four points, as we have to take the result modulo 2, so that, for example,

$$\begin{aligned} 2\cdot(0,0,1)+(0,1,0)+(0,1,1) &= (0,0,0)+(0,1,0)+(0,1,1) \\ &= (0,0,1), \end{aligned}$$

which can be expressed as

$$2\cdot\text{lion} + \text{mouse} + \text{cat} = \text{lion}.$$

For player 2, we have

$$\text{player 2: plane through } \{(0,0,0),\ (0,1,0),\ (1,0,0),\ (1,1,0)\},$$

which contains the animals mouse, snake, and koala. Next is

$$\text{player 3: plane through } \{(0,0,0),\ (0,0,1),\ (1,0,0),\ (1,0,1)\},$$

corresponding to the lion, snake, and giraffe, and so on.[2]

For $p=3$, we start with the set $\{0,1,2\}$, instead of $\{0,1\}$ for $p=2$, and now take results of additions and multiplications modulo 3. The 3-dimensional vector space will then have 3^3 unique points:

$$(0,0,0), (0,0,1), (0,0,2), (0,1,0), (0,1,1), (0,1,2), (0,2,0), \ldots.$$

We can now enumerate the lines passing through the origin. This can be done, for example, by considering all points whose first nonzero coordinate is a 1:

$$(0,0,1), (0,1,0), (0,1,1), (0,1,2), (1,0,0), (1,0,1), (1,0,2),$$
$$(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2).$$

The corresponding lines are constructed by taking multiples of such points; for example, $(0,0,0)$, $(0,1,2)$, and $(0,2,1)$ form one line. These 13 points generate all unique 13 lines through the origin.[3]

Planes are generated by linear combinations of two lines. A plane, for example, that contains the origin, $(0,0,1)$, and

2. Note that when considering the cube as a subset of Euclidean space, not all 2-dimensional planes correspond to a player. For example, (koala, giraffe, cat) corresponds to a player as koala + giraffe = cat according to the calculus we have described, but (lion, snake, mouse) does not correspond to a player as lion + snake = giraffe.

3. This can be seen as follows: The lines are all distinct, since any multiple of any of the listed points cannot result in another point in the list. Furthermore, for any nonzero point v whose first nonzero coordinate is not a 1, there exists a $\lambda \in \{1,2\}$, such that λv is contained in the list; because $v = \lambda^{-1}\lambda v$, the point v lies on the line generated by λv.

$(0,1,0)$, will also contain $(0,1,1)$ and $(0,1,2)$. It will also contain the line through the origin and $(0,0,2)$, for example, but this line is identical to the line through $(0,0,1)$. The construction is then analogous to the case of $p=2$ by identifying planes with players and lines with animals, although these roles can also be reversed.

In general, as long as p is a prime number, this 3-dimensional embedding yields p^2+p+1 distinct lines (using the same enumeration of lines as above) corresponding to animals and p^2+p+1 unique planes corresponding to players. Each plane will contain $p+1$ lines, and each line is contained in $p+1$ distinct planes. Formal proofs that this construction does indeed yield a projective plane with the required number of lines and points can be found in [Kåhrström, 2002], for example.

The construction is also possible for prime powers (for example, if p equals $4=2^2$ and $8=2^3$ and $9=3^3$) but this is a bit more demanding, as we cannot use the integers modulo p as a field any longer, and we have to construct suitable polynomials instead. Hence the values $p=2,3,5,7,11,\ldots$ are all covered by the construction above, and by an extended argument, we can also cover the prime powers $p=4,8,9,16,\ldots$. The four smallest numbers in $\mathbb{Z}$ that are neither prime nor prime powers are 6, 10, 12, and 14. The Bruck-Ryser theorem states that there is no solution if p is such that it is congruent to 1 or 2 modulo 4 and is not a sum of two squares. Thus, there is no solution for $p=6$ and $p=14$, for example. The theorem does not say anything about the cases $p=10$ and $p=12$, however. While it has been shown (using a computationally very demanding proof) that there is no solution for $p=10$, what happens for $p=12$ or other numbers not covered by the above characterizations is still open.

6.5 SHORT HISTORY

The game "Dobble" (also called "Spot It!") is an example of already-constructed incidence matrices for $p=7$. There are 55 cards, showing 8 out of 57 unique objects. Each player is given a pile of cards and a starting card is placed in the middle.[4] Looking at the top card on their respective piles, each player can put it on the central pile as soon as he spots the object in common between his topmost card and the topmost card on the central pile. The first player to get rid of all of his cards wins the game. So, the aim of the game is to spot the common unique object. A "Mini Dobble" game with $p=2$ (that is, 7 cards with 3 objects each) was created by Maxime Bourrigan. A nice overview article appeared in *Math Horizons* in 2015 [Polster, 2015]. The simple construction of the first solution in section 6.4 is based on an answer by Sven Zwei on stackoverflow.com. The Bruck-Ryser theorem can be found in Bruck and Ryser [1949]. Lam [1991] showed that there is no projective plane for $p=10$. Projective planes are also used in experimental design; see, for example, Hughes and Piper [1985].

6.6 PRACTICAL ADVICE

There is a variation of the game that does not require the players to have such a good memory of the Fano plane as in the version described earlier in this chapter. The game is for one less player than in the original version, that is, 6 players with 7 unique animals. The audience is allowed to remove 2 animals at the start. Then, each player can take 3 of the remaining animals. For

4. The card in the middle is the reason that the total number of cards is 55 and not $p^2+p+1=57$. With 55 cards in total, the remaining 54 cards can be distributed equally among either 2 or 3 players.

this variation, a viable strategy is that each player memorizes one line in the table below, and all players remember the last line of the table as a backup option.

player 1	lion	mouse	cat
player 2	mouse	snake	koala
player 3	lion	snake	giraffe
player 4	cat	giraffe	koala
player 5	cat	snake	elephant
player 6	lion	koala	elephant
backup option	mouse	giraffe	elephant

If a player sees that two of the animals in his primary line have been taken away by the audience, then he can switch to the backup option. Then all the pairs of players will once again have 1 animal in common, as each pair of animals uniquely identifies a line.

For 12 players and 13 unique animals, we can use the table below.

player 1	lion	mouse	cat	snake
player 2	lion	giraffe	koala	spider
player 3	lion	elephant	eagle	earthworm
player 4	lion	frog	dolphin	jellyfish
player 5	mouse	giraffe	elephant	dolphin
player 6	mouse	koala	eagle	frog
player 7	mouse	spider	jellyfish	earthworm
player 8	cat	giraffe	eagle	jellyfish
player 9	cat	koala	dolphin	earthworm
player 10	cat	elephant	frog	spider
player 11	snake	giraffe	frog	earthworm
player 12	snake	koala	elephant	jellyfish
backup option	snake	eagle	spider	dolphin

The game can also be turned around, and the audience could be allowed to ask for any pair of animals, to see whether there is a player who has taken this particular combination. This would also always be true under these strategies, and the players would cover all possible animal-pairs among themselves.

The types of the animals can, of course, be varied or replaced by fruits, stickers, or children's names.