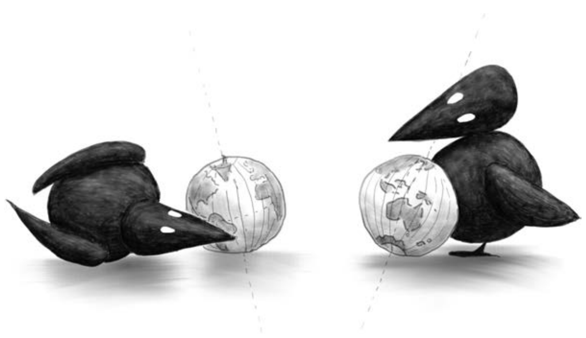
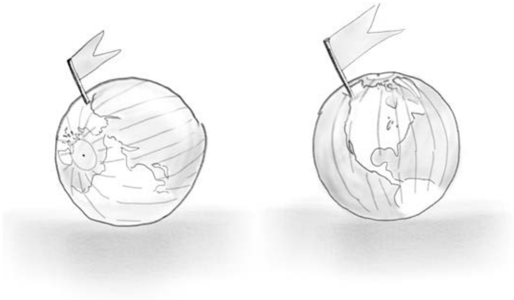
7 THE EARTH AND AN EIGENVALUE

7.1 THE GAME

Number of players:	2 or more
You will need:	2 (inflatable) world globes



Two contestants are each given an inflatable globe. They are asked to place the globe on the floor. The globe may be in any rotation that they choose. Once they have placed the globes, the goal is to find a spot on the globe, such as a city, that has the same position on both globes. Being in the same position means that if 2 tiny people would stand on that place on each of the inflatable globes and would point a laser pointer into the sky, perpendicular to the surface, the 2 laser beams would be parallel. The place can be anywhere on the globe, and it can be a city, a point in a desert, or in the ocean. As an example, if both contestants chose to place the South Pole on the floor, then the North Pole would be such a point: on both globes, it points vertically into the sky (the South Pole would be another correct answer, of course). But the contestants can choose any rotation, so in general, the correct point does not need to be the North Pole, and it does not need to point vertically upward. The figure below shows another example.



The task is now to find the place on the globe that has exactly the same position on both globes. In this case, it is a location close to Seattle, marked by a flag on both globes in the figure (which is just for illustration; the flag would not be present in the actual game).

The person who is first to announce such a spot on the globe (or, alternatively, declaring that no such place exists) wins the game. But only, of course, if the answer is correct. If it is incorrect, the other person wins. Even though the game is about being fast, we do not discuss strategies on how to find such a point. Instead, we turn to a more fundamental question.

Does such a point always exist? And if so, how can we prove that it exists?

We can view the left globe as the reference globe and the right one as a rotated version. The question becomes whether there is a point on the right globe that has exactly the same position as on the reference (left) globe.

Mathematically, the globes can be described as sets of points in 3-dimensional space. And a rotation R is a map: Every point on the left globe is mapped to another point on the globe (to obtain the right globe).

Let us start by describing the left globe mathematically. Its placement can be arbitrary, but for simplicity we assume that it is placed such that the North Pole is pointing upward and the prime meridian "longitude 0°" (this is the line on which Greenwich, UK, lies and that also passes through Ghana and Togo) is in the front (see the picture on the next page). We describe the center of the left globe as

$$\text{center of globe:} \quad v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

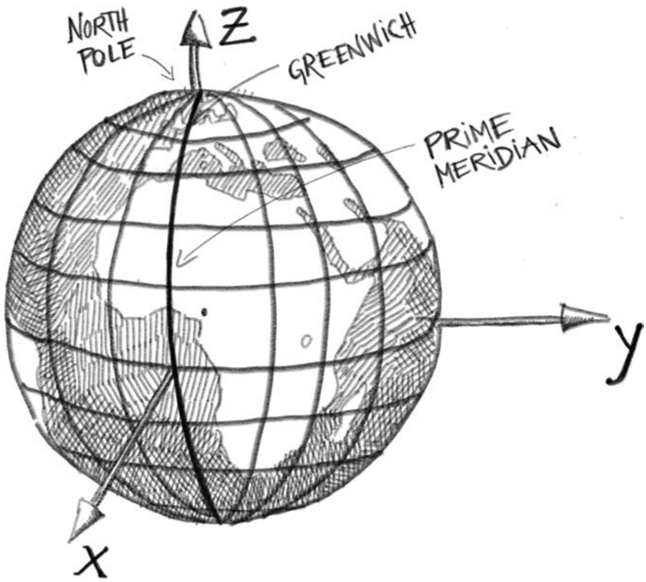
For simplicity, let us further say that each of the 2 globes is perfectly round, and that it has radius 1 (e.g., 1 meter). Each point on the globe can then be represented as a 3-dimensional vector:

$$\text{point on globe surface:} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ with } v_1^2 + v_2^2 + v_3^2 = 1,$$

where the latter condition ensures that v has distance 1 from the center. Considering the above orientation of the left globe, the North Pole can be described by the coordinates

$$\text{North Pole (left globe):}\quad v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and all points on the prime meridian, such as Greenwich have a 0 y-coordinate (see the figure below).



Having a model for the globe, we can now concentrate on the map R. If the rotation R (which can be a sequence of rotations) is such that the North Pole is mapped to the position where the South Pole was, then

$$R\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Alternatively, if the rotation is such that Zurich is mapped to where Copenhagen was before, we would have

$$R(\text{Zurich}) = \text{Copenhagen},$$

that is, roughly,[1]

$$R\begin{pmatrix} 0.6722088 \\ 0.1010620 \\ 0.7334315 \end{pmatrix} = \begin{pmatrix} 0.5528803 \\ 0.1232626 \\ 0.8240932 \end{pmatrix}.$$

In this game, we are looking for a location v that, after the rotation, does not change its position:

$$R(v) = v.$$

We encourage the reader to pause here and think a bit more about the problem and whether such points exist.

7.2 SOLUTION

The surprising aspect of the game is that there are always at least 2 points on the globe that share the same position on the 2 globes. More precisely, either there are exactly 2 positions on opposite sides of the globe that share the same position, or the 2 globes are in exactly the same position, so all points are identical. It is thus always wrong to declare that there is no place with identical positions on the 2 globes, and the players have to keep looking until they spot the correct places (which can be surprisingly difficult if playing for the first time).

Mathematically, no matter what R is, there are at least 2 points v that satisfy $R(v) = v$. And they lie on opposite sides of the globe: if v remains fixed, then $-v$ does so, too. More precisely, there are either 2 such points, or we have $R(v) = v$ *for all* points v. In the remainder of this chapter, we will prove the above statement. To do so, we need to study properties of rotations.

1. These coordinates are sometimes called "ECEF" coordinates: earth-centered, earth-fixed; the system considers a y-axis pointing to the right.

Consider first a *basic rotation* of the globe in 3-dimensional space: It consists of first choosing an axis of rotation and, second, rotating the object by a certain angle. Angles can be measured in different ways. For example, as a number between 0° and 360° or between 0 and 2π. We will follow the preference of many mathematicians and use the second notation. The angle is then equal to the distance one would have to walk on the circumference of a circle with radius 1 to complete the rotation. A *rotation* we define as a concatenation of such basic rotations.

The key argument for solving this game is based on the following fact (that was first proved by Leonhard Euler more than 200 years ago):

> *Any rotation of the globe can be described by a single basic rotation around a single axis.*

This holds, no matter how many basic rotations we concatenate in the movement. We can now describe the difference between the placement of the right globe, relative to the left one, by a basic rotation. This basic rotation yields an axis of rotation. Taking the intersection of that axis with the globe yields the 2 points that have the same position on both globes. We are next going to prove the above observation using linear algebra. We believe that if you have not heard about linear algebra before, this chapter might be challenging to follow at first reading. If this is the case, we suggest that you do what we usually do in such cases: Put the book aside for a few days and give it another try later.

7.3 SOME MATHEMATICS: LINEAR ALGEBRA

We argued above that a rotation R can be described as a map that maps any point v on the globe to another point $R(v)$ on the globe; for example, $R(\text{Zurich}) = \text{Copenhagen}$. Here and below, v

is always a 3-dimensional vector, so we write

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

A few examples of such vectors can be found on page 113. We will see below that our maps R take a very special form. In fact, they can always be written as

$$R\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} a_1v_1 + a_2v_2 + a_3v_3 \\ b_1v_1 + b_2v_2 + b_3v_3 \\ c_1v_1 + c_2v_2 + c_3v_3 \end{pmatrix}, \tag{7.1}$$

for some choice of real numbers $a_1, a_2, a_3,\ b_1, b_2, b_3,$ and c_1, c_2, c_3. If we turn the globe upside-down around the y-axis, for example, so $R(\text{North Pole}) = \text{South Pole}$, then we have

$$R\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_1 \\ v_2 \\ -v_3 \end{pmatrix},$$

which, for suitable constants $a_1, a_2, a_3, b_1, \ldots,$ has the form of equation (7.1). Can you find a point v on the globe with $R(v) = v$?

Maps of this form are sometimes called "linear maps"—hence the expression "linear algebra". And the question whether there exists a v such that

$$\begin{pmatrix} a_1v_1 + a_2v_2 + a_3v_3 \\ b_1v_1 + b_2v_2 + b_3v_3 \\ c_1v_1 + c_2v_2 + c_3v_3 \end{pmatrix} = 1 \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{7.2}$$

is an instance of a well-studied question: the so-called "eigenvalue" problem. The rotation R is characterized by the constants $a_1, a_2, a_3, \ldots$. If there is a v that satisfies equation (7.2), we say that 1 is an *eigenvalue* of R, and v is an *eigenvector* with eigenvalue 1.

We now argue that there has to be an eigenvalue of 1. Following this part requires some background knowledge in linear algebra. We do our best to explain all the relevant terms, but following some of the arguments might be difficult, and you can also choose to skip to section 7.4.

The Eigenvalue Problem

We first simplify the notation. Equation (7.1) can be written as a matrix-vector multiplication (see appendix B.7). More precisely, defining the matrix

$$R := \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

then any point v maps to Rv, where Rv is a matrix-vector multiplication. That is, we use the same letter R for the map R and for the matrix R.

So far, we have considered a map R that maps any point on the globe to another point on the globe. It will, however, be easier if we consider the map on the whole 3-dimensional space, that is, on $\mathbb{R}^3$. In fact, we even consider a larger space, namely, $\mathbb{C}^3$. The real numbers are a subset of a larger collection of numbers, the "complex numbers" $\mathbb{C}$, which we introduce in appendix B.8. In short, the space of complex numbers contains even (weird) numbers, such as $\sqrt{-1}$. As for $\mathbb{R}^3$, the set $\mathbb{C}^3$ contains vectors of the form

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

but now each of the components does not have to be real, but can be complex, that is, $v_1, v_2, v_3 \in \mathbb{C}$. We have $\mathbb{R} \subset \mathbb{C}$, and $\mathbb{R}^3 \subset \mathbb{C}^3$.

We call $\lambda \in \mathbb{C}$ an *eigenvalue* of R if there exists a vector $v \in \mathbb{C}^3$ such that

$$Rv = \lambda v,$$

that is, R leaves the vector v unchanged except for multiplication with the eigenvalue. Such a vector v is called an *eigenvector* of R with eigenvalue λ. As we argued above, it suffices to show that the matrix R that describes the rotation of the globe has an eigenvalue $\lambda = 1$. Then, there will be a real[2] eigenvector $v \in \mathbb{R}^3$ such that $Rv = v$, which means that the vector (place on the globe) v is unchanged under the movement R. If v is an eigenvector with eigenvalue 1, then so is $-v$, which means that the location $-v$ on the opposite side of the globe will then also remain unchanged in its position.

Below, we will establish that

(a) there are 3 (not necessarily distinct) eigenvalues of R, whose product equals 1,

(b) all 3 eigenvalues of R have absolute value 1, and

(c) the existence of any complex eigenvalue implies that its complex conjugate is an eigenvalue, too.

These three properties imply that 1 is an eigenvalue of R.

Basic Rotations

Let us first consider basic rotations (i.e., rotations around a single axis), which have a particularly easy form. A basic rotation by an angle $\phi \in [0, 2\pi]$ around the z-axis, for example, will transport the vector v on the sphere to a vector $R_{z,\phi}v$, where $R_{z,\phi}$ is the matrix

$$R_{z,\phi} = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It leaves the value v_3 along the z-axis unchanged and rotates the coordinates (v_1, v_2) in the x-y-plane. Now, any basic rotation can

2. If $Rv = v$ for a complex, nonreal v, then $Rv = v$ still holds if you set the imaginary parts of v to 0.

be written in the more general form

$$R_{A,\phi} = A \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} A^\top, \tag{7.3}$$

where $A \in \mathbb{R}^{3\times 3}$ with $A^\top A = AA^\top = \mathrm{Id}$ is a matrix that defines a basis transformation ($A^\top$ is the transpose of A and Id is the identity matrix, which contains ones on the diagonal and zeros everywhere else). After the transformation by $A^\top$, the rotation is then just a rotation around the new z-axis by angle ϕ, and the final multiplication with A rotates back into the original coordinate system.

There are two important observations about a basic rotation, that is, a rotation that can be written in the form of equation (7.3):

1. First, the rotation is orthogonal in the sense that for all A and ϕ,

$$R_{A,\phi}^\top R_{A,\phi} = \mathrm{Id},$$

as $R_{A,\phi}^\top R_{A,\phi}$ equals

$$A \begin{pmatrix} \cos^2(\phi) + \sin^2(\phi) & 0 & 0 \\ 0 & \cos^2(\phi) + \sin^2(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} A^\top = AA^\top.$$

2. Second, the determinant of $R_{A,\phi}$ equals 1 (i.e., $\det(R_{A,\phi}) = 1$). This follows because the determinant is multiplicative and hence

$$\begin{aligned} \det(R_{A,\phi}) &= \det(AR_{z,\phi}A^\top) = \det(A)\det(R_{z,\phi})\det(A^\top) \\ &= 1 \cdot \det(R_{z,\phi}) \cdot 1 = \cos^2(\phi) + \sin^2(\phi) = 1. \end{aligned}$$

Arbitrary Rotations

The same two properties now carry over if we concatenate several rotations. Consider the rotations $R_{A,\phi}$, $R_{B,\psi}$, and their product $R := R_{A,\phi} R_{B,\psi}$. The properties described above still hold:

$$R^\top R = R_{B,\psi}^\top R_{A,\phi}^\top R_{A,\phi} R_{B,\psi} = \mathrm{Id}, \tag{7.4}$$

and

$$\det(R) = \det(R_{A,\phi})\det(R_{B,\psi}) = 1. \tag{7.5}$$

By induction, the same holds, of course, for a product of arbitrarily many rotations.

As the determinant equals the product of all eigenvalues, this establishes property (a). Furthermore, eigenvalues of matrices with property (7.4) necessarily have an absolute value of 1:

$$\|v\|^2 = v^\top v = v^\top R^\top R v = \|Rv\|^2 = \|\lambda v\|^2 = |\lambda|^2 \|v\|^2,$$

implying that $|\lambda| = 1$, and therefore establishing property (b). Finally, we are now ready to argue property (c): that complex eigenvalues always come in pairs. One typically writes $\overline{z}$ for the complex conjugate of z, where the imaginary part switches sign. If $z = a + ib$ for $a, b \in \mathbb{R}$, then $\overline{z} = a - ib$. Now, if R has an eigenvalue $\lambda \in \mathbb{C}$ with eigenvector $v \in \mathbb{C}^3$, then $\overline{\lambda}$ is an eigenvalue of $\overline{v}$:

$$R\overline{v} = \overline{R}\overline{v} = \overline{Rv} = \overline{\lambda v} = \overline{\lambda}\overline{v}.$$

We are now able to put together these observations. Consider the three eigenvalues λ_1, λ_2, and λ_3, which are not necessarily distinct. We know that not only their absolute values but also their product equals 1. If two of them are not real, but complex, then the third eigenvalue must be 1 (this is because $\lambda\overline{\lambda} = |\lambda|^2 = 1$). The same follows if all eigenvalues are real. This proves that one of the eigenvalues equals 1.

All matrices $R \in \mathbb{R}^{3\times 3}$ with $R^\top R = \mathrm{Id}$ and $\det(R) = 1$ form a group that is often denoted by $SO(3)$, called the *special orthogonal group*. They all correspond to a basic rotation for a suitable basis transformation A and a rotation angle ϕ. The rotation angle can be easily inferred. The trace of a matrix, trace(R), is the sum of the diagonal entries and is invariant under permuting products, which here implies that $\mathrm{trace}(R) = \mathrm{trace}(AR_{z,\phi}A^\top) =$

$\mathrm{trace}(A^\top A R_{z,\phi}) = \mathrm{trace}(R_{z,\phi})$. The trace of $R_{z,\phi}$ equals $1 + 2\cos(\phi)$, and the sum of the diagonal entries of R hence determines the rotation angle.

7.4 SHORT HISTORY

Many textbooks on linear algebra cover the necessary material [e.g., Fischer, 2002], and they usually use arguments from linear algebra. Instead of considering two globes, one sometimes considers a soccer ball during a soccer match at the beginning of the first half and at the beginning of the second half. The mathematical statement itself was first proved by Leonhard Euler using elementary geometry [Euler, 1776].

7.5 PRACTICAL ADVICE

It is, of course, impossible to describe the location of the invariant points with very high precision. Instead, the players can be asked to name a country after 5 seconds. Then one checks which country is closest to 1 of the 2 invariant points, looking for the closest distance between the invariant point and any point in the country (the invariant point might lie in an ocean, in which case one would try to name a country that is as close as possible). Using countries (instead of cities) also adds an interesting aspect to the game since, say, Russia is more likely to be right a priori than Monaco. So players who do not see right away where the invariant point is can still win by choosing a country with a large area.

The game can in principle be played with objects that are not round, such as a cello. The above arguments still hold: Any sequence of rotations can be described by a single rotation with a single axis of rotation. On this axis all points are invariant. That is, if we describe the right cello as the rotation of the left cello, the intersection of the axis of rotation and the cello yields two points that have the same position on both cellos. Using two spheres, however, has the advantage that there is a natural center that the laser beams point away from (in general, one needs to agree on such a center). We found that in general, nonspherical objects make the game more difficult.

Rather than using globes, the game is perhaps easier to play with two printouts of random rotations of the globe, as seen from a large distance. The figure on page 110 is an example of two such printouts, with Seattle being the stationary point. Of course, this way we might miss places that are on the "backside," that is, on the part of the globe that is not visible. However, this is not really a restriction, since one of the two points of interest will always be visible.

Since the two solutions on the globe will be antipodes (that is, directly opposite on the globe), then one of these two points will be visible to the observer (the one on whichever half of the globe we are looking at). We will thus have exactly one point that is common on the visible part of the globe (assuming that not *all* points are in the same place and that we can really see half of the globe, which is only approximately true if we take a snapshot from space). In the example, Seattle was in a common location on both globes. The antipode is southwest of South Africa, close to Île de la Possession. This second fixed location will become visible as soon as Seattle is rotated out of the visible part of the globe. With the same reasoning, we can consider the following slight variation of the game: One takes a picture from the globe, rotates it, takes another picture, and then asks the audience to spot the place on the globe (e.g., the city) that has the same position in both pictures. To verify, one can then overlay the two pictures. This variation of the game requires that the position in space from which one takes the pictures is fixed.