8 THE FALLEN PICTURE AND ALGEBRAIC TOPOLOGY

Number of players:	1 or more
You will need:	a picture with a long string attached to it; some nails in the wall on which the picture can be hung using the string; it should be possible to remove the nails easily (for an alternative setup: a wooden structure with removable bars; rope)

8.1 THE FALLEN PICTURE

Suppose your friend has changed careers and has decided to become a painter. Since you had advised her to follow her interests, she proudly announces that you are receiving her first completed artwork. She considers her debut work a big success and wants to ensure that it hangs safely on the wall. She therefore watches you hammering 5 nails into the wall that will secure the painting. You attach a string to the picture's frame (top left and top right) that is supposed to wind around the nails. Unfortunately, you do not share your friend's enthusiasm about the painting. You would not be upset if the painting fell down and broke, but since you do not want to hurt your friend's feelings, you do not say anything.

> *Is there a way to wind the string around the nails, such that the painting hangs on the wall but falls down as soon as one of the nails is pulled out of the wall?*

You are looking for a solution that ensures the painting falls down no matter which nail is pulled out.

It is easiest to start with the challenge for 2 nails. The picture should hang on the wall and fall down if either the left or the right nail is pulled out. In most households, pictures are hung by putting the string over two nails at the same time. This standard solution is clearly insufficient for our case: If we pull out the left nail, for example, the picture may lose its horizontal alignment, but it will still hang on the wall due to the right nail. Thus, this configuration cannot be the solution we are looking for—the requirement was that the picture falls down as soon as either one of the 2 nails is pulled out.

What happens if we, instead, hang the picture on the left nail and ignore the right one? The picture would then fall down if we remove the left nail; but it would not if we remove the right nail. Thus, again, this is not the correct solution. The trick must therefore be to wind the string around the 2 nails in a sufficiently complicated manner. We encourage you to take a piece of paper and try to find the solution for 2 nails by trial and error. (There is a solution!)

If you have found the solution, you can take some time to think about the problem a bit further: Is there a way to modify the solution such that it still works? What does the problem have to do with mathematics? And what about more than 2 nails?

8.2 SOLUTION FOR 2 NAILS

The figure on the next page reveals a solution for 2 nails. Convince yourself that the solution works: If we pull out the left nail, the picture will fall down. If we leave the left nail in place and pull out the right nail, the picture also falls down.

Is there a more systematic way to come up with a solution than trial and error? And is there a solution if we have more than 2 nails (e.g., 5, 6, or even 411950)? We first rewrite the game as a problem of lining up dancers at dancing school. We then argue that both problems have the same structure, that is, solving one is as good as solving the other. Perhaps surprisingly, discovering a systematic way of constructing solutions is easier for the dancing problem than for the picture-hanging problem.

8.3 DANCING

Consider the following situation at a dancing school. Two dancing classes (Tango Argentino and Viennese Waltz) are about to start, and all dancers are assembled in one of the classrooms. They are standing in a line, the ones dancing the male part (“males”), those taking the female part (“females”), tango dancers, and waltz dancers mixed. The instructor wants to give some general comments about dancing, but he faces the following difficulty: The dancers are highly motivated. When they are standing next to a possible dancing partner, they immediately move away from the line, start to dance, and stop listening. The remaining dancers close the gaps. New pairs might form and they, too, would start dancing until there are no more matches between immediate neighbors. In this problem, we assume that people dance only with a person who prefers the same dance and who takes the other part (i.e., tango males with tango females, waltz females with waltz males, etc.). For example, the line

(tango-female, waltz-male, waltz-female,
tango-male, tango-female, tango-male)

would dissolve: First, the inner two pairs vanish (waltz-male and waltz-female, and tango-male and tango-female), and then the two dancers on the outside (tango-female and tango-male) become neighbors and start dancing, too. The instructor is now trying to order the dancers such that the following two constraints are satisfied:

1. There is no matching pair in the group, and therefore nobody starts dancing, listening instead to the instructor.
2. As soon as one of the two groups is called out of the room (either the tango dancers or the waltz dancers), the remaining people in the line start dancing, because now (or

after a short time) they find themselves next to a suitable partner.

Suppose there is a group of 4 people: a tango-male, a tango-female, a waltz-male, and a waltz-female. We challenge you to stop reading for a moment, grab a piece of paper, and find a solution for the instructor. The key idea here is to separate the waltz from the tango couple. One possibility is to line them up in the following order:

$$\text{(tango-male, waltz-female, tango-female, waltz-male).} \qquad (8.1)$$

There is no matching pair at the moment, but if the waltz couple leaves the room, the tango people start dancing, and, similarly, if the tango couple leaves, the waltz dance can begin.

How does this dancing problem relate to the falling picture? These two problems are the same! We simply identify tango with the left nail, waltz with the right nail, men with clockwise circles, and women with counterclockwise circles. The line up of dancers described in equation (8.1) corresponds exactly to the solution shown on page 126.

The dancing problem captures the essential structure of the problem. We will see that this is the so-called *topology* of the problem. In section 8.4, we summarize some concepts of algebraic topology. We tried to write the section in a way that is easy to follow, even if you have never encountered topology before. If some parts are not clear after the first reading, try putting the book aside and giving it another try later—for us, this often helps.

8.4 SOME MATHEMATICS: ALGEBRAIC TOPOLOGY

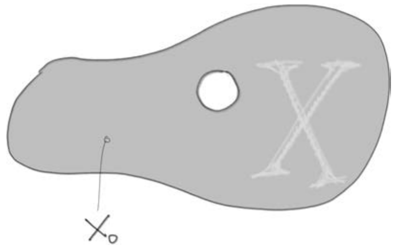
When we study properties of mathematical objects, such as sets, numbers, or groups, it is often helpful to state which properties we are *not* interested in. Objects that differ only in those neglected properties are then considered to be equivalent. In chapter 4, for example, we did not want to distinguish between (the sums) 2 and 5, since they corresponded to the same box. Therefore, we simply defined these numbers to be the same. Formally, we constructed the set $\mathbb{Z}/3\mathbb{Z}$.

In *topology*, we do not distinguish between two sets that can be transformed into each other by stretching, pulling, or poking. A topologist does not distinguish, for example, between two footballs that differ only in the amount of air inside. Similarly, two rubber bands still have the same topological properties, even if one of them is extended to twice its unstretched length. Cutting, however, usually changes the topological properties: If one of the rubber bands rips apart, it becomes topologically different from a nonbroken version. Mathematically, nonripping transformations are usually described by continuous maps (see appendix C.5 for details).

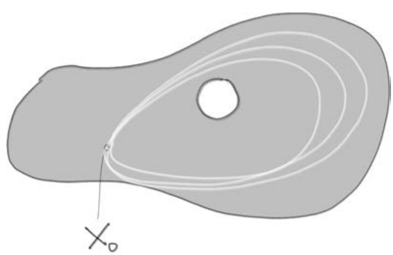
In *algebraic topology*, we can describe an object by a group (see section 4.3). We will first discuss the fundamental group of a set X. This group is denoted by $\Pi_1(X)$ and captures important (topological) properties of the set X. A flat and an inflated football, for example, is each assigned to the same fundamental group. The torn and nontorn rubber bands, on the other hand, are not. The concept of fundamental groups will help us find solutions to the picture-hanging problem for an arbitrary number of nails. We try to explain all the steps. For those of you who have a bit more mathematical training, we provide more precise details in appendix C.5.

Fundamental Group

Consider a space X and a point x_0 that lies in this set. For now, it is easiest to think about X as being a subset of $\mathbb{R}^2$, such as the shaded area in the following figure. Since you can "move around" in X, we decided to call it a space rather than a set.



The basic building blocks for the fundamental group of X are loops. These are "walks" in X that start at x_0 and end at x_0. Mathematically, they are continuous maps from $[0,1]$ into X. As topologists, we do not want to distinguish between loops that are just slightly stretched version of each other. We therefore call all these loops equivalent, or *homotopic*, and regard them as the same thing. This is similar to what we saw in chapter 4, where we identified the sums 2, 5, and 8. We believe that the following figure may help clarify the idea. All three loops in the figure are homotopic to one another. You can think of them as rubber bands that you can stretch.



Note that all of them loop around the hole in the middle. It may be helpful to think about the hole as being the trunk of a very large tree growing out of this page. The tree is an obstacle when transforming the rubber bands. Without ripping them apart, the rubber bands will always make exactly one loop around the tree. With the equivalence, or *homotopy*, of loops in mind, when speaking about loops, we will really mean the loop's equivalence class (that is, the loop together with all of its equivalent partner loops).

Loops seem very different from integers, say, but surprisingly, we can still perform computations on loops that are similar to adding numbers. As an example, consider two (different) loops ℓ and m. Similarly to taking two numbers 4 and 5, and computing its sum $4+5$, we can construct a new loop from the pair ℓ and m: the product loop $\ell \circ m$. This new loop is described by first running on the path ℓ, and then around the loop m (we have to run the whole path in double speed, though). This new loop $\ell \circ m$ also starts and ends at x_0. We call $\circ$ an *operation* on the set of loops. In fact, the set of loops with operation $\circ$ forms a *group*. The title "group" can be awarded, because the following four properties hold. Do they ring a bell?

1. The product loop $\ell \circ m$ is again a loop.
2. The loop action is *associative*: For all loops k, ℓ, and m, we have
$$k \circ (\ell \circ m) = (k \circ \ell) \circ m.$$
(We will not prove this property, but, instead, we ask you to trust us that this is indeed the case.)
3. There exists a neutral element e, such that for all loops ℓ, we have $\ell \circ e = \ell$ and $e \circ \ell = \ell$. These properties are satisfied by a not very exciting loop e: the one that starts in x_0 and … stays there. Let us call this the *neutral loop*.
4. Finally, for each loop ℓ, there exists another loop, usually called the *inverse loop* ℓ^{-1}, that satisfies $\ell^{-1} \circ \ell = \ell \circ \ell^{-1} = e$. Indeed, for a given loop ℓ, the inverse loop can be defined by running ℓ backward. The concatenated walk is then topologically equivalent (homotopic) to having stayed at x_0 in the first place—even though doing nothing may be less exhausting.

Done—we have a group structure! So far, all loops we have considered start and finish in x_0, so it seems as if the group seems to depend on x_0. For many spaces X, however, the starting point does not have any influence on the group. In particular, this is the case for "path-connected" spaces, that is, spaces in which any 2 points can be connected by a (continuous) path. We can therefore define the *fundamental group* of X as the group of homotopy classes of loops around an arbitrarily chosen starting point $x_0 \in X$. The last sentence sounds complicated, but hopefully we clarified what it means. Topologists usually denote the fundamental group by $\Pi_1(X)$.

So far, the fundamental group appears abstract and not very practical to work with. We will show that for many spaces X, the group actually takes an easy form.

Fundamental Group of a One-Tree Meadow

How can we describe the fundamental group of the example on page 129—the space, let us call it W, with one tree in the middle? Not surprisingly, the tree plays a crucial role here. Any loop that does not circuit the tree is equivalent to the neutral loop e: We can make the loop smaller and smaller until we obtain a loop that barely leaves our starting point x_0. But a loop that circuits the tree (say, clockwise, and exactly one time) is different: There is no way to transform it to the neutral loop without ripping it into pieces. Let us call such a loop a. The figure on page 129 shows some members of this equivalence class. What about the element $a \circ a$? This circuits the tree twice and is different from both a and e. For notational convenience, we do not write $a \circ a$, but rather a^2. Similarly, the loops $a^3 := a \circ a \circ a$ and $a^4 := a \circ a \circ a \circ a$ circuit the tree 3 and 4 times, respectively, and are also not equivalent to the previous loops. Taking the product of a^3 and a^4 yields a loop that circuits the tree 7 times clockwise: $a^3 \circ a^4 = a^7$. What about inverse elements? We can "neutralize" the loop a^7, for example, by taking the product with a loop that walks around the tree seven times in the opposite direction (that is, counterclockwise). We can, of course, denote this new loop however we like, but it is very convenient to call it a^{-7}. The minus in the exponent says that the circuits are performed counterclockwise. Finally, what is the result if we concatenate 7 counterclockwise circuits with 9 clockwise circuits? Exactly 2 clockwise circuits. Summarizing this paragraph, we have the equations

$$a^4 = a \circ a \circ a \circ a, \quad a^3 \circ a^4 = a^7, \quad (a^7)^{-1} = a^{-7}, \quad (a^7)^{-1} \circ a^9 = a^2. \tag{8.2}$$

These computations should remind us of a group that we have seen before: the integers $\mathbb{Z}$ with the addition $+$, but clearly, the notation is somewhat different. It turns out that there exists a one-to-one correspondence between our fundamental group of the one-tree meadow and $\mathbb{Z}$. We say that they are "isomorphic" as groups and write

$$\Pi_1(W) \cong \mathbb{Z}. \tag{8.3}$$

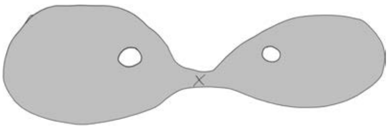
For example, the equations in $\mathbb{Z}$ that correspond to the ones in equation (8.2) read as follows:

$$4 = 1 + 1 + 1 + 1, \quad 3 + 4 = 7, \quad -(+7) = -7, \quad -7 + 9 = 2.$$

A formal proof of equation (8.3) can be found in a book by Allen Hatcher, for example (see section 8.6). It is a famous result in algebraic topology and can be used as a building block for computing the fundamental group of more complex spaces, too. The following theorem, for example, shows us how to compute the fundamental group when connecting two meadows with a single tree each to a large meadow with two trees.

The Seifert—van Kampen Theorem and a Two-Tree Meadow

Different operations allow us to combine several spaces into more complex structures. One of them is called the "gluing operation," usually referred to as the wedge sum. One takes two spaces X and Y and glues them together by a small bridge. The example of the space $W \vee W$ (W "wedge" W), obtained by taking the wedge sum of the one-tree meadow W with itself, is shown in the figure below and should suffice to understand the concept.



As a topologist, we would ask immediately, of course, what the fundamental group of the new structure looks like. The following theorem by van Kampen and Seifert states that it is straightforward to compute the fundamental group of the wedge sum if the individual fundamental groups are known. The result involves the free product of groups, a concept that we explain below. Let us first state the result. Consider spaces $X_1, X_2, \ldots, X_n$. Then the fundamental group of the wedge sum can be computed from the individual fundamental groups:

$$\Pi_1(X_1 \vee X_2 \vee \cdots \vee X_n) \cong \Pi_1(X_1) * \Pi_1(X_2) * \cdots * \Pi_1(X_n). \quad (8.4)$$

Here, $*$ denotes the free product of groups. As a direct consequence, we have that

$$\Pi_1(W \vee W) \cong \mathbb{Z} * \mathbb{Z}. \quad (8.5)$$

So, how is the free product of two groups defined? The first factor in $\mathbb{Z} * \mathbb{Z}$ corresponds to the fundamental group connected to the first tree. Recall that we have written a^4 instead of 4. Following this convention, let us write b^4 instead of 4 for the second factor in $\mathbb{Z} * \mathbb{Z}$. (The loop b^4 will correspond to making

Solution to the figures at the end of section 3.4. The correct cards are the ten of spades (first example) and ten of clubs (second example). The score in the latter example is 57.

4 clockwise circuits around the second tree.) The free product $\mathbb{Z} * \mathbb{Z}$ now contains "words" that concatenate elements of both groups, such as

$$ab^{-4}b^3b^2a^{-1}a^3b^{-1}ba. \tag{8.6}$$

It was a good idea to write a^4 and b^4 instead of 4, since otherwise this expression would read $1-4+3+2-1+3-1+1+1$ and we would not be able to see which part of the word comes from the first and which part from the second version of $\mathbb{Z}$. The word (8.6) is an element of the fundamental group $\Pi_1(W \vee W)$, and as such, it must correspond to a loop. The loop is constructed by circuiting the left tree once (clockwise), then circuiting the right tree four times (counterclockwise), then three times the right tree (clockwise), and so forth. This operation can be simplified to

$$ab^{-4}b^3b^2a^{-1}a^3b^{-1}ba = a(b^{-4}b^3b^2)(a^{-1}a^3)(b^{-1}b)a = aba^3.$$

As before, the group action is the concatenation of such words:

$$b^5aba^2 \circ a^{-1}b^{-6} = b^5abab^{-6}.$$

Certainly, you can find the inverse element of this word.[1] We have learned to perform computations using loops in relatively complex spaces. This allows us to solve the original problem for 2 nails in a systematic way, which can easily be extended to more than 2 nails.

8.5 SOLUTION, CONTINUED

The key observation is that the wall corresponds to the space $W \vee W$, with the nails taking the role of the trees. The string winding around the nails represents a homotopy class of loops.

1. The inverse element of b^5abab^{-6} is $b^6a^{-1}b^{-1}a^{-1}b^{-5}$ as their product equals $b^5abab^{-6} \circ b^6a^{-1}b^{-1}a^{-1}b^{-5} = b^5aba(b^{-6}b^6)a^{-1}b^{-1}a^{-1}b^{-5} = \cdots = e$.

That is, each candidate solution corresponds to an element in the fundamental group $\Pi_1(W \vee W)$. We now need to translate the problem description of the picture-hanging problem into properties of the solution element in $\Pi_1(W \vee W)$. The picture hangs on the wall if and only if the element is not equivalent to the neutral loop. But how can we translate the property that removing any of the nails results in a falling picture? To do so, we need to understand how to mathematically describe the removal of a nail. Consider the word $ab^2a^{-2}b^3a^2$, for example. Removing the second nail lets all the loops around the second nail collapse—the second tree in the meadow is gone. Mathematically, this corresponds to removing all b components from the word, so that $ab^2a^{-2}b^3a^2$ becomes $aa^{-2}a^2 = a$. The key question therefore becomes:

> *Is there an element in* $\Pi_1(\mathrm{W} \vee \mathrm{W})$*, different from the neutral loop, such that removing either all* a *components or all* b *components yields the neutral loop* e*?*

The answer is yes. It is achieved by the word

$$s(a,b) := ab^{-1}a^{-1}b, \tag{8.7}$$

for example. Please try to draw this solution. It should yield the same figure as the one on page 126. (Hint: Recall that a represents a clockwise wind around the left nail; also, your drawing becomes easier to interpret if you make sure that there are no more than 2 lines that cross at a single point.)

Equation (8.7) shows the solution for 2 nails. Surprisingly, we can take this as a starting point to construct a solution for an arbitrary number of nails. Can you figure out how? The proof goes by induction. Suppose that we have a solution s (that is, a word) for $n \geq 2$ nails. We can then construct a solution for $n+1$ nails by $szs^{-1}z^{-1}$, where z denotes the loop around the new $(n+1)$th nail. The solutions for 3 and 4 nails become

$$s(a,b,c) := ab^{-1}a^{-1}bcb^{-1}aba^{-1}c^{-1} \tag{8.8}$$

$$s(a,b,c,d) := ab^{-1}a^{-1}bcb^{-1}aba^{-1}c^{-1}dcab^{-1}a^{-1}bc^{-1}b^{-1}aba^{-1}d^{-1}. \tag{8.9}$$

We challenge you to draw these solutions for the falling-picture problem. A representation for 3 different dance styles (indicated by colors) is shown below:

If you remove, for example, the blue dancers that are dancing a swing, then the remaining dancers will also naturally form pairs and the whole line will dissolve. The outcome is identical if you remove the green tango dancers or the red waltz dancers.

Solution Length

The length of the solution presented earlier grows exponentially with the number of nails: An induction argument shows that the length of the above solution for n nails equals

$$2^n + 2^{n-1} - 2.$$

These solutions very quickly become impractical. Researchers have therefore tried to find shorter solutions to the problem.

Some solutions grow "only" polynomially with the number of nails. The following table shows what difference this makes in practice. We can save a lot of string!

number of nails	exponential solution length	polynomial solution length
1	1	1
2	4	4
3	10	10
4	22	16
5	46	28
6	94	40
7	190	52
8	382	64
9	766	88
10	1534	112
⋮	⋮	⋮
50	1688849860263934	2752

8.6 SHORT HISTORY

Equation (8.3) can be found as theorem 1.7 in Hatcher [2002], for example. The Seifert–van Kampen theorem dates back to Van Kampen [1933] and Seifert [1931]. The solution that grows only polynomially in the number of nails is from Pegg [2002] and is presented in Demaine et al. [2012], for example. The prisoner idea (see section 8.7) we developed jointly with Anders Tolver.

8.7 PRACTICAL ADVICE

Before playing this game, you should practice drawing the solutions a couple of times. In particular, intersections of different lines can be hard to interpret. We therefore suggest always avoiding intersections of more than 2 lines.

The inductive construction also provides a hint for drawing the solutions. Start with a solution for 2 nails, say, add a new nail with a clockwise circuit, follow the original solution backward, and then add one counterclockwise circuit around the new nail. This solution, however, is not always the easiest when doing a visual check.

When dealing with 4 or 5 nails, you might want to consider using the polynomial solution rather than the exponential one.

There is one more subtle aspect that you need to keep in mind: The solution may not work anymore if you produce knots when winding the string around the nails. One possibility to prevent this from happening is to start with a closed string and to always place the string loops on top of each other.

Alternative Setup

There is an alternative setup of the problem, in which the audience takes a slightly more active role. In that game, some participants are asked to help "arrest" one member of the audience (for some made-up reason). The prison is supposedly high security, and the convicted person has to be tied to 5 bars. The participants are friends of the convicted person and know that the prison hosts one friendly prison guard who usually removes one of the bars, but they do not know in advance which bar will be removed. They therefore want to tie up their friend to the bars such that the friend will be able to escape as soon as one arbitrarily chosen bar is removed. This setup requires a rope and some wooden structure with several bars that can be removed.