Elements of Stochastic Methods

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PREFACE

When AIP Publishing invited me to contribute a book to their new eBook series, I knew at once that I should write a brief book on stochastic methods. The fields in which stochastic methods are relevant are very diverse—population dynamics, infectious diseases, genetic diversity, financial modeling, hydrology, psychology, physics, chemistry, and many others. The compact format of the AIP series suited my belief that a *short* book would be appreciated, something in which the main aim is to give practitioners an idea of how to formulate their current problem stochastically, and how to get an idea of what are the predictions of such modeling. And all of this should be contained in one compact volume.

I expect typical readers of this book to be people with a need to model practical problems in their own field. They are not mathematicians, and while they are interested in why things are true, they have little interest in rigorous proof or abstract mathematical formulations. They want answers.

The result is this book. It should be viewed as both a distillation of my earlier book *Stochastic Methods*,[1] keeping only the essentials, and as an extension, widening the scope to include examples from many fields. Mostly, readers who want more detail or rigor will find explicit references to the appropriate sections of *Stochastic Methods*. As much as possible, notation and terminology are the same in both books, so that it should be easy to use both books simultaneously without undue strain.

To achieve compactness, I have had to be quite selective. There are no rigorous mathematical proofs, although I do try to explain why results are true. But I do cover most of the basic techniques used in practice. I restrict the coverage to Markov processes, and processes expressible in terms of Markov processes. This is in practice not a serious restriction.

Since it is now recognized that they play a major role in many natural phenomena, I have included heavy-tailed distributions and processes, not as an afterthought, but as an integral part of the development of the book. This means that my discussion of probability distributions distinguishes between:

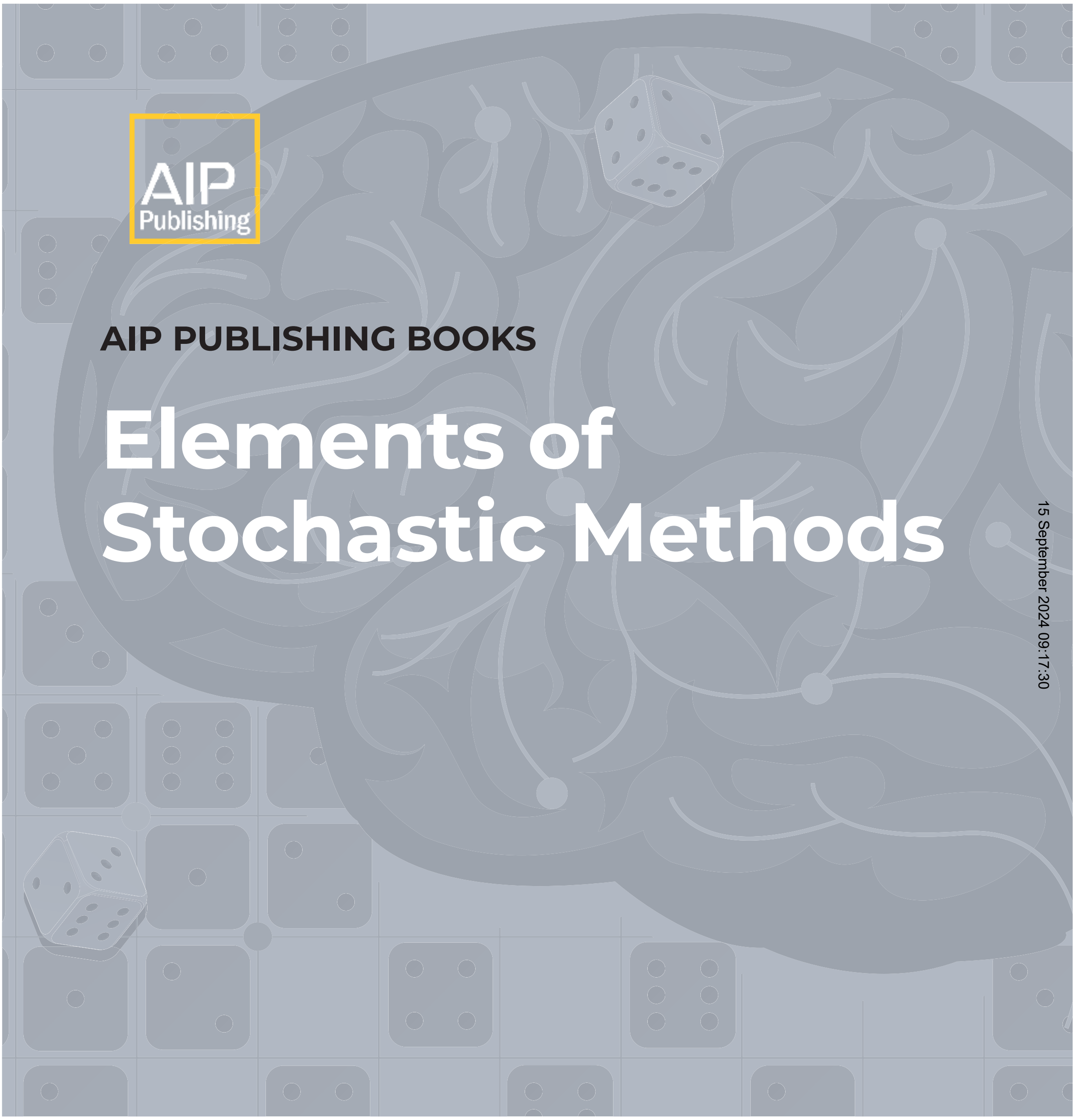
[1] C. W. Gardiner, *Stochastic Methods* (Springer, Heidelberg, Berlin; 1st ed. 1985, 2nd ed. 1997, 3rd ed. 2004, 4th ed. 2009)—I want to thank Springer Verlag for permission to include some material from *Stochastic Methods* in this book.

i) Those which have a well-defined variance, for which the central limit theorem is therefore valid,

ii) Heavy-tailed distributions, which do not have a well-defined variance, and for which the central limit theorem is not valid. For these, the central limit theorem is replaced by the concept of stable distributions as the appropriate limit distributions of sums of independent variables.

As well, I have also included *Mandelbrot's* fractional Brownian motion. This is not a Markov process, but is Gaussian, and easily expressed in terms of Markov processes. Its significance in the understanding of the long-time correlations that appear to be an essential part of many natural phenomena is unchallenged, even though it does not provide all of the answers.

The influence of my late colleagues, *Dan Walls* (1942–1999) and *Nico van Kampen* (1921–2013) on this book and much of my other work has been profound, and I would like to dedicate this book to their memory.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Stochastic Methods

TABLE OF CONTENTS

PART I: FOUNDATIONS

Chapter 1 **Introduction** **1-1**
1.1 Events and their Probabilities 1-1
1.2 Joint and Conditional Probabilities 1-4
1.3 Probability Notations 1-6
1.4 Mean Values of Random Variables 1-7
1.5 The Characteristic Function 1-10
References for Chapter 1 1-12

Chapter 2 **Probability Distributions** **2-1**
2.1 The Uniform Distribution 2-1
2.2 The Gaussian Distribution 2-3
2.3 The Central Limit Theorem 2-7
2.4 Discrete Distributions 2-10
2.A Appendix: Some Commonly Used Distributions 2-13
References for Chapter 2 2-14

Chapter 3 **Heavy-Tailed Distributions** **3-1**
3.1 Pareto's Law 3-1
3.2 Ranked Data and Zipf's Law 3-5
3.3 The Log-Normal Distribution 3-7
3.4 Stable Distributions 3-10
3.A Appendix: Behavior of Paretian Distributions when $\beta = 1$ 3-17
References for Chapter 3 3-18

PART II: STOCHASTIC PROCESSES

Chapter 4 **Markov Processes** **4-1**

4.1 The Markov Postulate 4-1

4.2 The Differential Chapman–Kolmogorov Equation 4-3

4.3 Algorithm for Simulating a Markov Process 4-9

References for Chapter 4 4-10

Chapter 5 **Stationary Processes** **5-1**

5.1 Stationary Markov Processes 5-1

5.2 Measurement in Fluctuating Systems 5-3

5.3 The Regression Theorem 5-5

5.4 Spectrum and Autocorrelation Function 5-5

5.5 Fourier Analysis of Fluctuating Functions 5-7

5.6 White Noise 5-8

References for Chapter 5 5-9

PART III: DIFFUSION PROCESSES

Chapter 6 **The Wiener Process and Brownian Motion** **6-1**

6.1 Physical Brownian Motion 6-1

6.2 The Wiener Process—Mathematical Brownian Motion 6-7

6.3 Interpolating a Wiener Process—the Brownian Bridge 6-12

References for Chapter 6 6-13

Chapter 7 **Stochastic Differential Equations** **7-1**

7.1 Ito Stochastic Differential Equation 7-1

7.2 The Fokker–Planck Equation 7-5

7.3 The Stratonovich Stochastic Differential Equation 7-6

7.4 Systems with Many Variables 7-9

7.5 Numerical Simulation of Stochastic Differential Equations 7-12

References for Chapter 7 7-14

Chapter 8 The Fokker–Planck Equation 8-1

8.1 Fokker–Planck Equation in One Dimension 8-1
8.2 Boundary Conditions 8-4
8.3 Eigenfunction Methods for Homogeneous Processes 8-6
8.4 Many-Variable Fokker–Planck Equations 8-8
References for Chapter 8 8-11

Chapter 9 Elementary Diffusion Processes 9-1

9.1 Single-Variable Processes 9-1
9.2 Processes with Many Variables 9-6
9.3 Complex Variable Oscillator Processes 9-12
References for Chapter 9 9-15

Chapter 10 Approximating Diffusion Processes 10-1

10.1 The Limit of Small Noise 10-1
10.2 Small Noise in Multivariable Processes 10-6
10.3 Adiabatic Elimination of Fast Variables 10-11
10.4 Gaussian Approximations 10-15
References for Chapter 10 10-18

Chapter 11 Escape and Extinction 11-1

11.1 Exit Times for Diffusion Processes 11-2
11.2 Applications 11-5
References for Chapter 11 11-10

PART IV: JUMP PROCESSES

Chapter 12 Elementary Jump Processes 12-1

12.1 The Master Equation 12-1
12.2 The Poisson Process 12-2
12.3 The Random Telegraph Process 12-4
12.4 The Continuous Time Random Walk 12-8
12.5 The Kramers–Moyal Expansion 12-10
References for Chapter 12 12-13

Chapter 13 Population Processes 13-1
13.1 Birth–Death Master Equations—One Variable 13-1
13.2 Birth-Death Systems with Many Variables 13-6
13.3 The System Size Expansion for Population Processes 13-9
13.4 The Gillespie Algorithm 13-12
References for Chapter 13 13-14

Chapter 14 Modeling Population Processes 14-1
14.1 Infectious Diseases 14-1
14.2 The Kermack–McKendrick Epidemic Model 14-3
14.3 Cell and Systems Biology 14-15
References for Chapter 14 14-20

PART V: HEAVY-TAILED PROCESSES

Chapter 15 Lévy Processes 15-1
15.1 Definition of Lévy Processes 15-1
15.2 Lévy Processes with Infinite Intensity 15-4
15.3 The Paretian Processes 15-7
15.4 Using the Paretian Processes 15-13
15.5 Lévy Stochastic Differential Equations 15-14
15.6 Applying Lévy Processes 15-17
References for Chapter 15 15-19

Chapter 16 Fractional Brownian Motion 16-1
16.1 The Work of H. E. Hurst on River Flows 16-1
16.2 Fractional Brownian Motion 16-7
16.3 Modeling Fractional Brownian Motion 16-11
16.4 Conclusion 16-14
References for Chapter 16 16-15

Author Index I-1

Subject Index I-3