CHAPTER

1 REVIEW OF LOGIC, SET THEORY, AND ISOMORPHISM

1.1 LOGIC

Logic is the process of derivation/deduction of properties/propositions. Within mathematical logic, we have *propositional algebra* and *predicate logic*, which we will review in the subsections below. For more detailed and thorough discussions of logic, the author directs the reader toward the book *Logic for Physicists* (Pereyra, 2018).

1.1.1 Propositional algebra

Propositional algebra is the sub-branch of mathematical logic that studies propositions and *logical operators*. A proposition is any statement that can clearly be assigned a unique value of either “True” (T) or “False” (F). Propositions satisfy:

- the *Law of Dichotomy*: that is, a proposition must have a logical value of either true (T) or false (F), and
- the *Law of Excluded Middle*: that is, a proposition cannot be simultaneously true (T) and false (F).

In propositional algebra, two propositions are said to be equal if and only if they have the same logical value.

Following the discussion in *Logic for Physicists* (Pereyra, 2018), in studying logical operators, we can begin with the BOTH-FALSE operator “$\otimes$.” Given two propositions A and B, the new proposition “$A \otimes B$” is true (T) if and only if both A and B are false (F).

In turn, the NOT operator “$\neg$,” the OR operator “$\vee$,” the AND operator “$\wedge$,” the IMPLIES operator “$\Longrightarrow$,” and the EQUIVALENT operator “ $\Longleftrightarrow$ ” may then be defined through the equations

$$\neg A \equiv A \otimes A, \tag{1.1}$$
$$A \vee B \equiv \neg(A \otimes B), \tag{1.2}$$
$$A \wedge B \equiv \neg((\neg A) \vee (\neg B)), \tag{1.3}$$
$$A \Longrightarrow B \equiv \neg(A \wedge (\neg B)), \tag{1.4}$$
$$A \iff B \equiv (A \Longrightarrow B) \wedge (B \Longrightarrow A). \tag{1.5}$$

From the above definitions, one can derive the truth tables for the different logical operators, and one finds

A	B	$A \otimes B$
T	T	F
T	F	F
F	T	F
F	F	T

;

A	$\neg A$
T	F
F	T

;

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

;

Real Exponential, Logarithmic, and Trigonometric Functions for Physicists, Nicolas A. Pereyra

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

;

A	B	$A \Longrightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

;

A	B	$A \Longleftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

.

To simplify the propositional expressions, one may introduce notational priority to reduce the necessity of parentheses and thus simplify the written expressions. From highest to lowest priority, for the five standard logical operators, we have

$$\neg, \wedge, \vee, \Longrightarrow, \Longleftrightarrow \ .$$

For example,

$$A \vee \neg B \wedge C = A \vee ((\neg B) \wedge C), \tag{1.6}$$

$$\neg(A \vee B) \wedge C = (\neg(A \vee B)) \wedge C, \tag{1.7}$$

$$A \iff B \wedge C \Longrightarrow D \vee E = A \iff ((B \wedge C) \Longrightarrow (D \vee E)), \tag{1.8}$$

$$A \iff \neg B \wedge C \Longrightarrow D \vee E = A \iff (((\neg B) \wedge C) \Longrightarrow (D \vee E)). \tag{1.9}$$

Note that the BOTH-FALSE operator $\otimes$ is not a standard logical operator, and, thus, any propositional expression that includes this operator (except for the simple expression "$A \otimes B$") should include parentheses.

Another notation that may be used is the omitting of the symbol "$\wedge$" when implementing the AND operation. That is, when two propositions are placed next to each other without an explicit logical operator in between them, the $\wedge$ operator is assumed. For example,

$$AB = A \wedge B, \tag{1.10}$$

$$A \vee BC = A \vee (B \wedge C), \tag{1.11}$$

$$\neg BC = (\neg B) \wedge C. \tag{1.12}$$

We may also introduce two additional symbols: T and F (i.e., the letters T and F in italic). Above, the roman letter "T" has been used to denote the logical value of true. In turn, the roman letter "F" has been used to denote the logical value of false. The italic letter "T" will be used to denote any proposition that always has a logical value of true and such a proposition will be said to be *identically true*. Similarly, the italic letter "F" will be used to denote any proposition that always has a logical value of false and such a proposition will be said to be *identically false*.

For example, consider the proposition

$$A \vee \neg A.$$

Deriving the truth table for $A \vee \neg A$, one finds

A	$A \vee \neg A$
T	T
F	T

.

Thus, the proposition "$A \vee \neg A$" always has a logical value of true, regardless of whether A is true or false. We may then state that

$$\text{"}A \vee \neg A\text{" is identically true} \tag{1.13}$$

or, equivalently,

$$A \vee \neg A = T. \tag{1.14}$$

Some useful properties of propositional algebra that can be derived from the above discussions are

$$\neg(\neg A) = A, \tag{1.15}$$
$$\neg T = F, \tag{1.16}$$
$$\neg F = T, \tag{1.17}$$
$$A \vee B = B \vee A, \tag{1.18}$$
$$A \vee (B \vee C) = (A \vee B) \vee C, \tag{1.19}$$
$$A \vee A = A, \tag{1.20}$$
$$A \vee T = T, \tag{1.21}$$
$$A \vee F = A, \tag{1.22}$$
$$A \wedge B = B \wedge A, \tag{1.23}$$
$$A \wedge (B \wedge C) = (A \wedge B) \wedge C, \tag{1.24}$$
$$A \wedge A = A, \tag{1.25}$$
$$A \wedge T = A, \tag{1.26}$$
$$A \wedge F = F, \tag{1.27}$$
$$A \vee \neg A = T, \tag{1.28}$$
$$A \wedge \neg A = F, \tag{1.29}$$
$$\neg(A \vee B) = \neg A \wedge \neg B, \tag{1.30}$$
$$\neg(A \wedge B) = \neg A \vee \neg B, \tag{1.31}$$
$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C), \tag{1.32}$$
$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C), \tag{1.33}$$
$$A \wedge (A \vee B) = A, \tag{1.34}$$
$$A \vee (A \wedge B) = A, \tag{1.35}$$
$$A \vee (\neg A \wedge B) = A \vee B, \tag{1.36}$$
$$A \wedge (\neg A \vee B) = A \wedge B, \tag{1.37}$$
$$\{A \implies A\} = T, \tag{1.38}$$
$$\{A \implies T\} = T, \tag{1.39}$$
$$\{F \implies A\} = T, \tag{1.40}$$
$$\{T \implies F\} = F, \tag{1.41}$$
$$\{A \implies B\} = \{\neg B \implies \neg A\}. \tag{1.42}$$

1.1.2 Predicate logic

Mathematics is based on the derivations of properties or propositions with respect to given objects or elements belonging to a given set. In turn, predicate logic is the sub-branch of mathematical logic that studies propositions that depend on the elements of a given set. Predicate logic includes all of propositional algebra and the logical symbols $\forall$ ("For-All") and $\exists$ ("Exists").

Predicate logic considers propositions "$P(x)$" that depend on the element "x" of a given set. That is, $P(x)$ can be either "True" (T) or "False" (F) depending on the value of x. The For-All symbol $\forall$ indicates that the given proposition is always True (T), regardless of the specific value that we may assign to the given variable. The symbol $\exists$ indicates that there exists *at least* one value of the variable such that the given proposition has a logical value of True (T).

The For-All symbol ∀ and the Exists symbol ∃ can be defined in the following manner:

$$\forall_x P(x) \equiv \text{“For all possible values of } x, P(x) \text{ is always True”}, \tag{1.43}$$

$$\exists_x P(x) \equiv \neg[\forall_x \neg P(x)]. \tag{1.44}$$

Some general properties of predicate logic that can be derived from the above discussions are

$$\{\forall_x P(x)\} \implies P(a), \tag{1.45}$$

$$P(a) \implies \{\exists_x P(x)\}, \tag{1.46}$$

$$\{\neg \forall_x P(x)\} \iff \{\exists_x \neg P(x)\}, \tag{1.47}$$

$$\{\neg \exists_x P(x)\} \iff \{\forall_x \neg P(x)\}, \tag{1.48}$$

$$\{\forall_x P(x)\} \vee \{\forall_x Q(x)\} \implies \{\forall_x [P(x) \vee Q(x)]\}, \tag{1.49}$$

$$\{\forall_x P(x)\} \wedge \{\forall_x Q(x)\} \iff \{\forall_x [P(x) \wedge Q(x)]\}, \tag{1.50}$$

$$\{\exists_x P(x)\} \vee \{\exists_x Q(x)\} \iff \{\exists_x [P(x) \vee Q(x)]\}, \tag{1.51}$$

$$\{\exists_x [P(x) \wedge Q(x)]\} \implies \{\exists_x P(x)\} \wedge \{\exists_x Q(x)\}. \tag{1.52}$$

1.2 SET THEORY

Set theory is the study of the general properties of elements and sets. We present here a review of the general properties of sets, set operators, relations and functions, and equivalence relations and classes. For more detailed and thorough discussions of set theory, the author directs the reader toward the book *Set Theory for Physicists* (Pereyra, 2019).

1.2.1 General properties of sets

A set is a well-defined group of elements. To state that a given element x belongs to a given set A, one may use the symbol “∈,”

$$x \in A \equiv \text{“}x \text{ is an element of set } A,\text{”} \tag{1.53}$$

that is, “x belongs to A” or “x is in A.”

Given two sets A and B, they are stated to be equal if and only if it holds that any element x that belongs to A also belongs to B and any element x that belongs to B also belongs to A, that is,

$$A = B \iff (x \in A \iff x \in B). \tag{1.54}$$

One can define a set that contains elements that are sets themselves, that is, a set can be an element of another set. However, a set *cannot* contain itself, that is, for any set A, it will always hold that

$$A \notin A. \tag{1.55}$$

A set can be such that it contains no elements. Such a set is said to be *empty* or to be a *null* set and is denoted by the symbol ∅. That is, for any element x, it always holds that

$$x \notin \emptyset. \tag{1.56}$$

Given two sets A and B, it will be stated that set A is a subset of set B if and only if any element in A is also in B, that is,

$$A \subset B \iff (x \in A \implies x \in B). \tag{1.57}$$

Two properties that can be derived from the discussion above are

$$A \subset A, \tag{1.58}$$
$$\emptyset \subset A, \tag{1.59}$$

that is, any given set A is a subset of itself, and the empty set $\emptyset$ is a subset of any set.

1.2.2 Set operators

Given two sets A and B, the union of A and B is denoted by "$A \cup B$," and defined by the condition

$$x \in A \cup B \iff (x \in A) \vee (x \in B). \tag{1.60}$$

In turn, given two sets A and B, the intersection of A and B is denoted by "$A \cap B$," and defined by the condition

$$x \in A \cap B \iff (x \in A) \wedge (x \in B). \tag{1.61}$$

Also, given two sets A and B, the set subtraction of A and B is denoted by "$A \backslash B$," and defined by the condition

$$x \in A \backslash B \iff (x \in A) \wedge (x \notin B). \tag{1.62}$$

A set operator that commonly appears implicitly or explicitly in many natural science applications is the Cartesian product. Given two sets A and B, the Cartesian product of A and B is denoted by "$A \times B$," and it is a set of *ordered pairs* defined by the condition

$$(x, y) \in A \times B \iff (x \in A) \wedge (y \in B). \tag{1.63}$$

Some general properties of set theory that can be derived from the above discussions are

$$A \cup A = A, \tag{1.64}$$
$$A \cup B = B \cup A, \tag{1.65}$$
$$A \cup (B \cup C) = (A \cup B) \cup C, \tag{1.66}$$
$$A \cup \emptyset = A, \tag{1.67}$$
$$A \cap A = A, \tag{1.68}$$
$$A \cap B = B \cap A, \tag{1.69}$$
$$A \cap (B \cap C) = (A \cap B) \cap C, \tag{1.70}$$
$$A \cap \emptyset = \emptyset, \tag{1.71}$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \tag{1.72}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \tag{1.73}$$
$$A \backslash A = \emptyset, \tag{1.74}$$
$$A \backslash \emptyset = A, \tag{1.75}$$
$$\emptyset \backslash A = \emptyset, \tag{1.76}$$
$$C \backslash (A \cup B) = (C \backslash A) \cap (C \backslash B), \tag{1.77}$$
$$C \backslash (A \cap B) = (C \backslash A) \cup (C \backslash B), \tag{1.78}$$
$$C \backslash (B \backslash A) = (C \cap A) \cup (C \backslash B), \tag{1.79}$$
$$C \backslash (C \backslash A) = C \cap A, \tag{1.80}$$
$$(B \backslash A) \cap C = (B \cap C) \backslash A, \tag{1.81}$$
$$(B \cap C) \backslash A = B \cap (C \backslash A), \tag{1.82}$$
$$(B \backslash A) \cup C = (B \cup C) \backslash (A \backslash C). \tag{1.83}$$

1.2.3 Relations and functions

Given two sets A and B, a relation R between A and B is a subset of the Cartesian product $A \times B$, that is,

$$R \subset A \times B. \tag{1.84}$$

Let "a" be an element of set A and let "b" be an element of set B, one states that "a" is related through R to "b" ("aRb") if and only if the pair (a, b) belongs to relation R, that is,

$$aRb \iff (a, b) \in R. \tag{1.85}$$

The domain of R is the subset of A whose elements are related to at least one element of set B, that is,

$$a \in \text{domain of } R \iff \exists_b\, aRb. \tag{1.86}$$

In turn, the range of R is the subset of B whose elements are related to at least one element of set A, that is,

$$b \in \text{range of } R \iff \exists_a aRb. \tag{1.87}$$

Also, a relation R is said to be a "one-to-one relation" if and only if every element of set A is related to one and only one element of set B and in turn every element of set B is related to one and only one element of set A.

Given two sets A and B, a function f from A to B is a relation R_f between A and B such that every element of A is related to one and only one element of B. Standard notations for denoting a function f from A to B are

$$f : A \to B, \tag{1.88}$$

$$A \xrightarrow{f} B, \tag{1.89}$$

$$b = f(a). \tag{1.90}$$

If a function f from A to B is also a one-to-one relation between A and B, then we can define the inverse function f^{-1} from B to A such that

$$bR_{f^{-1}}a \iff aR_f b. \tag{1.91}$$

1.2.4 Equivalence relations and classes

Given a set A, an equivalence relation $\sim$ in A is a relation between A and A such that the following three conditions hold:

$$a \sim a \qquad \text{(Reflexivity)}, \tag{1.92}$$

$$a \sim b \iff b \sim a \qquad \text{(Symmetry)}, \tag{1.93}$$

$$(a \sim b) \wedge (b \sim c) \Longrightarrow (a \sim c) \qquad \text{(Transitivity)}. \tag{1.94}$$

Given a set A and an equivalence relation $\sim$ in A, the equivalence relation will "partition" the set A into a series of subsets of A such that each subset contains elements of A that are equivalent to each other. These subsets, containing equivalent elements, are called *equivalent classes*.

Following the notation of the book *Set Theory for Physicists* (Pereyra, 2019), we will denote an equivalence class with an element contained in that class in between square brackets,

$$[a]. \tag{1.95}$$

Given elements x and y of a set A, and an equivalence relation $\sim$ in A, it follows that

$$[x] \subset A \wedge [y] \subset A, \tag{1.96}$$

$$x \sim y \iff [x] = [y], \tag{1.97}$$
$$[x] \cap [y] = \emptyset \iff \neg(x \sim y). \tag{1.98}$$

1.3 ISOMORPHISM

Two sets are considered isomorphic if one can establish a one-to-one relation between the two sets such that the corresponding operators and relations within each set are equivalent.

1.3.1 Isomorphic sets

Given a set A with a unary operator "$-$," a binary operator "$+$," and a relation R_* between A and A, and also a second set B with a unary operator "$\ominus$," a binary operator "$\oplus$," and a relation $R_\circledast$ between B and B, we will state that

$$(A, -, +, R_*) \text{ is isomorphic to } (B, \ominus, \oplus, R_\circledast) \tag{1.99}$$

if there exists a one-to-one relation R_f between A and B such that

$$aR_f b \iff b = f(a), \tag{1.100}$$
$$a_2 = -a_1 \iff f(a_2) = \ominus f(a_1), \tag{1.101}$$
$$a_3 = a_1 + a_2 \iff f(a_3) = f(a_1) \oplus f(a_2), \tag{1.102}$$
$$a_1 R_* a_2 \iff f(a_1) R_\circledast f(a_2). \tag{1.103}$$

Also, note that once we prove that two sets with their corresponding operators are isomorphic, it will follow, in general, that the properties that we derive for one set (that are based on the isomorphic operators) will also correspondingly apply to the second isomorphic set.

1.4 EXAMPLES

1.4.1 Set theory

Set theory is found implicitly and explicitly throughout the natural sciences. One example where set theory explicitly appears and is applied is the description of the movement of an object. We can, for example, consider a relatively simple case of an object that moves along a straight line. To describe the movement of the object, we are led to consider the position "x" of the object on the line at different times "t." In particular, the movement of the object is considered to be completely described if one can determine the position x for anytime t. In other words, the movement of the object is "solved" if we can determine the function "f" that relates t to x, that is,

$$x = f(t). \tag{1.104}$$

Note that "solving" the problem in this case means obtaining a function. Thus, the concept of function, that is within set theory, becomes a fundamental aspect of physics.

That said, most physicists would tend to write the previous equation in the form

$$x = x(t) \tag{1.105}$$

to emphasize that the function "$x(t)$" is returning the position "x" given a time "t." Note that in Eq. (1.105), we are using the symbol "x" to represent two different things: (1) the position of the object (left-hand side of the equation) and (2) the function that gives the position of the object for a given time (right-hand side of the equation). Thus, some mathematicians may tend to use the expression "$x = f(t)$" instead [Eq. (1.104); rather than Eq. (1.105)] to distinguish with different symbols the position "x" and the function "f" that returns the position. Note that x and f represent two different mathematical objects: x is a real number while f is a real function,

$$x \in \mathbb{R}, \tag{1.106}$$

$$f : \mathbb{R} \to \mathbb{R}. \tag{1.107}$$

In studying the movement of an object moving along a straight line, we can use either Eq. (1.104) or (1.105) (once again, they are the same equation, but using different notations). That said, what may be key in analyzing the physical problem is realizing that the equation involves three distinct mathematical objects: the position x (real number), the time t (real number), and the function $x(t)$ (real function). Therefore, set theory may be a key aspect in analyzing physical problems.

1.4.2 Isomorphism

In Sec. 1.4.1, we presented the study of an object moving along a straight line as an example of an application of set theory. The movement of an object along a straight line may also be seen as an example of the application of isomorphism.

As discussed in Sec. 1.4.1, the study of a movement of an object along a straight line requires the use of real numbers in order to be able to represent any position on that line with a number [note that, as discussed in *Real and Complex Numbers for Physicists* (Pereyra, 2020), rational numbers cannot represent all the points on the line, thus giving a strong motivation for constructing real numbers]. As discussed in *Real and Complex Numbers for Physicists* (Pereyra, 2020), real numbers can be constructed and their general properties derived from rational numbers and the general properties of rational numbers. Also, as discussed in *Real and Complex Numbers for Physicists* (Pereyra, 2020), in the definition and study of the general properties of real numbers, integers can be seen as a subset of rational numbers through isomorphism, significantly simplifying the study of properties of real numbers. In turn, as discussed in *Real and Complex Numbers for Physicists* (Pereyra, 2020), rational numbers can also be seen as a subset of real numbers, through isomorphism. Thus, a position of "$x = 1/2$" is a valid position of an object in a straight line, even though "1/2" is a rational number, because all rational numbers can be seen as real numbers.

Therefore, isomorphism, in playing a key role in the study of real numbers, may also be key in studying the movement of objects in a straight line, and in physics problems in general. It should also be noted at this point that isomorphism can be applied to various physical systems. In particular, when one finds two distinct (and apparently unrelated) physical systems that have mathematical descriptions that are isomorphic, then the mathematical properties that have already been proven for the first system will in general also apply to the second system, significantly increasing the efficiency of study of the second physical system.

1.4.3 Logic

Logic is found implicitly and explicitly throughout the natural sciences. In Secs. 1.4.1 and 1.4.2, we presented the study of an object moving along a straight line as an example of an application of set theory and isomorphism, respectively. As we will see, the movement of an object along a straight line may also be seen as an example of the application of logic.

Logic explicitly appears and is applied in the definition and application of limits. Many fundamental physical quantities (e.g., velocity, acceleration, etc.) are defined as limits. For example, for an object moving in a straight line, we may wish to calculate the average velocity v_{avg} of the object during times t_1 and t_2 and, thus, apply the equation

$$v_{avg} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}. \tag{1.108}$$

Taking $\Delta t = t_2 - t_1$, we can rewrite the previous equation in the form

$$v_{avg} = \frac{x(t_1 + \Delta t) - x(t_1)}{\Delta t}. \tag{1.109}$$

If now we wish to calculate the velocity v_1 at t_1 (i.e., the instantaneous velocity at t_1, rather than the average velocity in a time interval that includes t_1), then we would apply the equation

$$v_1 = \lim_{\Delta t \to 0} \frac{x(t_1 + \Delta t) - x(t_1)}{\Delta t}. \tag{1.110}$$

In order for v_1 to be a well-defined physical quantity, the limit must result in a well-defined real number that we can compare with observations. In turn, this means that the limit operation must be well-defined. Applying the definition of limit, we have that

$$\forall_{\epsilon > 0}\, \exists_{\delta > 0}\, \forall_{\Delta t \neq 0}\, |\Delta t| < \delta \Longrightarrow \left| \left(\frac{x(t_1 + \Delta t) - x(t_1)}{\Delta t} \right) - v_1 \right| < \epsilon. \tag{1.111}$$

Thus, "v_1" is the velocity at "t_1" if and only if the value of "v_1" satisfies Eq. (1.111). It is not obvious, but mathematical analysis shows that given a real function "$x(t)$," if the real number "v_1" exists, it is unique. That is, in general, the velocity of a given object at a given time t is a well-defined physical quantity.

Note that in order to calculate the velocity v_1, we have to solve Eq. (1.111). In turn, in order to solve Eq. (1.111), we need to apply the elements of logic that we reviewed in Sec. 1.1.

Therefore, logic, in playing a key role in the application of limits and, therefore, in the definition of many fundamental physical quantities, may also be key in studying the movement of objects in a straight line, and in physical problems in general.

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