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David R. Rowland

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Understanding nonlinear effects on wave shapes: Comment on "An experimental analysis of a vibrating guitar string using high-speed photography" [Am. J. Phys. 82(2), 102–109 (2014)]

David R. Rowland[a)]
Student Services, The University of Queensland, Brisbane, Queensland 4072, Australia

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In a recent paper, Whitfield and Flesh found unusual bowing behavior in the waveform of a guitar string for large amplitude plucks. This Comment discusses the theory needed to understand this nonlinear effect, and it is shown that this theory provides reasonably good qualitative agreement with the observed wave form. This theory is interesting because: (i) it allows one to quantify the boundary between linear and nonlinear behavior in terms of key physical parameters; (ii) it reveals the importance of taking into account longitudinal displacements even when they are much smaller than the associated transverse displacements; and (iii) it reveals that dispersion due to tension changes and dispersion due to flexural rigidity have very similar functional forms, which leads to the question of when one effect can be neglected in comparison to the other.
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I. INTRODUCTION

Nonlinear effects in vibrating strings are a well-studied phenomenon, though primarily researchers have focused on the bending of the resonance curve when large amplitude transverse modes of vibration are excited,[1–4] and interesting transverse-transverse mode coupling effects.[5–9] Less well studied is the impact of nonlinear effects on the shape of a transverse wave. For example, according to linear wave theory a taut string fixed at each end and plucked at its center will evolve from an initial triangular shape into an expanding trapezoid, and this was in fact found to be the case for "small amplitude" plucks in the paper by Whitfield and Flesch.[10] However, for "large amplitude" plucks, a considerable amount of bowing in the center of the trapezoid was observed by these authors, as evidenced by their Fig. 7(c). Such bowing has been predicted in a previous numerical simulation of transverse waves on strings,[11] but this theoretical prediction does not seem to have been compared with experiment. Consequently, the observations in the Whitfield and Flesch[10] paper provide an opportunity to assess how well theory and experiment agree for this phenomenon.

In addition to the question of how the observed bowing in the transverse wave shape can be understood theoretically, another important question that the observation raises is: what determines the boundary between small amplitude plucks for which linear theory provides a good approximation and large amplitude plucks for which nonlinear effects become important? A further question arises from the fact that the authors of Ref. 10 found that dispersion from non-negligible flexural rigidity could also lead to similar bowing, and that is: what determines when flexural rigidity as well as nonlinearity should be considered? The purpose of this Comment is to provide answers to these questions. It is also hoped that the theoretical treatment here will facilitate further, more detailed empirical investigations aimed at better assessing the validity of the theory.

II. THE KIRCHHOFF-CARRIER NONLINEAR WAVE EQUATION

Large amplitude transverse waves are commonly studied using the Kirchhoff-Carrier equation of motion, which was first proposed by Kirchhoff (cited in Ref. 12) in the 1880s and rediscovered by Carrier[13] in 1945. In order to remove the complications arising from transverse-longitudinal mode coupling, this equation makes the simplifying approximation that tension remains uniform across the string during oscillations, and for transverse motion in one plane only is given by

$$\rho_0\ddot{\psi} \simeq \left[\tau_0 + \frac{YS}{L_0}\int_0^{L_0}\frac{1}{2}\psi'^2\,dX\right]\psi''$$
$$\simeq \tau_0\left[1 + \frac{1}{2e_0L_0}\int_0^{L_0}\psi'^2\,dX\right]\psi''. \quad (1)$$

In this equation, $\psi(X,t)$ denotes the transverse displacement of the string as a function of material coordinate[14] X at time t (i.e., relative to a set of Cartesian axes with one axis aligned along the quiescent string, the equilibrium coordinates of points in the string are given by $[X,0,0]$), ρ_0 denotes the mass per unit length of the string in its tensioned, equilibrium configuration, and τ_0 denotes the tension in the string when no waves are present. In addition, overdots denote differentiation with respect to time and primes denote differentiation with respect to X, while Y is the Young's modulus of the string, S its unstretched cross-sectional area, and L_0 its length when under tension τ_0 with no waves present. Finally, $e_0 = (L_0 - L_r)/L_r$ denotes the equilibrium strain, where L_r is the relaxed length of the string (i.e., the length of the string when under zero tension). For a monofilament obeying Hooke's law, we have $\tau_0 = YSe_0$ by definition. Note that the derivation of Eq. (1) assumes that the material of the string remains linearly elastic and consequently that the nonlinearity arises simply because the tension in the string differs non-negligibly from τ_0 for large amplitude transverse waves.

The uniform tension approximation is justified for metallic strings by the observation that for such strings the longitudinal wave speed c_L is many times greater than the transverse wave speed c_T, and arises from taking the limit as $c_L \to \infty$ of the longitudinal wave equation (see Refs. 4 and 15 for more detailed discussions, and Ref. 16, Sec. 7.5, for a clear and fairly careful derivation using perturbation theory). Readers are warned, however, that derivations of the Kirchhoff-Carrier wave equation and its extension to include nonplanar motion in studies of transverse-transverse mode coupling often include subtle errors. For example, "derivations" that neglect the effects of longitudinal displacements (e.g., Refs. 2, 3, and 17), even though these are small in comparison to transverse displacements, result in a nonlinear coefficient that is a factor of 3/2 too large. In addition, most authors, including Refs. 13 and 16 (p. 487), also without comment use the apparent rather than true strain in the definition of tension (see Refs. 4, p. 137 and 18, pp. 760–761 for a discussion of this issue), with Refs. 6 and 12 being rare exceptions in using true strain. While the use of apparent rather than true strain is not important for metallic strings, it becomes a problem for rubber-like strings for which c_T is not necessarily negligible in comparison to c_L.[19]

III. SOLVING THE KIRCHHOFF-CARRIER WAVE EQUATION WITH A MODAL ANALYSIS

As was the procedure in Ref. 10, Eq. (1) is readily solved numerically via a modal expansion. That is, letting

$$\psi(X,t) \simeq \sum_{n=1}^{N} a_n(t)\sin(n\pi X/L_0), \tag{2}$$

Eq. (1) becomes

$$\ddot{a}_n(t) \simeq -\left(\frac{n\pi c_T}{L_0}\right)^2\left[1+\frac{1}{4e_0}\sum_{p=1}^{N}p^2\pi^2\left(\frac{a_p(t)}{L_0}\right)^2\right]a_n(t), \tag{3}$$

for $n = 1, 2, \ldots, N$ and where $\tau_0/\rho_0 = c_T^2$ has been used. For a string displaced a distance $-A$ at its midpoint and released from rest, it follows from a Fourier series expansion of a triangular displacement that the coupled equations (3) need to be solved with the initial conditions

$$a_n(0) = -\frac{8}{n^2\pi^2}A\sin(n\pi/2); \qquad \dot{a}_n(0) = 0. \tag{4}$$

Note from Eq. (3) that the nonlinear term becomes important when $p^2(a_p/L_0)^2$ becomes non-negligible with respect to the equilibrium strain e_0 for some value of p, thus clarifying what constitutes a "large amplitude wave." This condition will be stated a bit more precisely below.

Equation (3) can be improved by adding to the right-hand side a phenomenological damping term of the form

$$F_d/\rho_0 = -\beta_n\dot{a}_n(t). \tag{5}$$

In Ref. 10, the damping factor was chosen to be a frequency independent constant so that the wave died out as $\exp(-Cf_0t)$, where $f_0 \equiv f_1$ is the frequency of the fundamental transverse mode when vibrating linearly (in this Comment frequencies are labeled by the mode number). In reality, the energy loss is likely to be more complicated than this. For example, Chaigne and Askenfelt[20] found a quadratic dependence of damping on frequency in experiments with piano wires. To relate Eq. (5) to Ref. 10, we note that Eq. (5) would lead that mode to die out as $\exp(-\beta_n t/2)$, and so for the damping to match for the fundamental mode of vibration requires

$$\beta_1 = Cc_T/L_0, \tag{6}$$

using $f_0 \equiv f_1 = c_T/(2L_0)$.

Equation (3) can also be improved by considering the effects of flexural rigidity. For a monofilament with a circular cross section of radius r_0, this effect can be modeled by adding to the transverse wave equation a force density term of the form[21]

$$F_b = -YS(r_0/2)^2\psi^{(4)}, \tag{7}$$

where $\psi^{(4)}$ denotes the fourth-order spatial derivative of ψ.

Using Eq. (5), and Eq. (2) to expand Eq. (7), to add damping and flexural rigidity to the Kirchhoff-Carrier model, Eq. (3) becomes

$$\ddot{a}_n(t) = -\omega_n^2\left[1+\frac{YS}{4\tau_0}n^2\pi^2\left(\frac{r_0}{L_0}\right)^2 + \frac{1}{4e_0}\sum_{p=1}^{N}p^2\pi^2\left(\frac{a_p(t)}{L_0}\right)^2\right]a_n(t) - \beta_n\dot{a}_n(t), \tag{8}$$

where $\omega_n = n\pi c_T/L_0$ is the angular frequency of free oscillations in the limit as both r_0 and the amplitude of oscillations go to zero. Note that for a monofilament, for which $\tau_0 = YSe_0$, $YS/(4\tau_0)$ in this expression would equal $1/(4e_0)$. We thus see that the perturbations to the linear wave equation due to flexural rigidity and changes in tension (assuming tension to be uniform across the string) are functionally almost identical, and from Eq. (8) we can see that the two terms become comparable when the amplitude of the mode is of the same order as the radius of the string. This result raises the question as to why inharmonicity experiments in pianos and guitars have been consistent with linear theory and no inharmonicity due to nonlinear effects seem to have been reported. Presumably, the answer is that for pulses arising from a struck string, the resulting modal amplitudes are, or quickly become, sufficiently small that nonlinear effects are small by comparison, but this perhaps deserves more careful investigation.

To get some understanding of the effects of nonlinearity due to tension changes, consider Eq. (8) without the loss or flexural rigidity terms. We are left with the equation of a simple harmonic oscillator with a "hardening" spring described by a cubic nonlinearity. Such problems have been thoroughly studied and analyzed using standard perturbation theory techniques. From Ref. 22, pp. 140–1, for example, it can be shown that the first-order perturbation theory correction to the angular frequency of free oscillations is

$$\frac{\omega_n'}{\omega_n} \approx 1 + \frac{3}{32}\frac{n^2\pi^2}{e_0}\left(\frac{a_n(0)}{L_0}\right)^2, \tag{9}$$

where $a_n(0)$ is the amplitude of the mode at $t = 0$. For the fundamental mode to behave linearly thus requires $(a_1(0)/L_0)^2/e_0 \ll 1$. Note also that although one loses linear superposition in nonlinear systems, the relative frequency shift of a single transverse mode is proportional to the order of the mode squared, and so one can see how introducing dispersion, as the authors of Ref. 10 did, could lead to the sort of bowing effect observed.

IV. ESTIMATING THE EQUILIBRIUM STRAIN OF THE STRING USED IN THE EXPERIMENTS

The above theory shows that the key parameter determining the strength of nonlinear effects is the equilibrium strain e_0, which can be determined from more typically given system parameters as follows. For a monofilament, $e_0 = \tau_0/(YS)$ by definition. Standard linear theory gives both $\tau_0 = \rho_0 c_T^2$ and $c_T = 2f_1L_0$. Consequently, for a monofilament we have

$$e_0 = 4\rho_0 L_0^2 f_1^2/(YS). \tag{10}$$

Unfortunately, the experiments in Ref. 10 used a wrapped string rather than a monofilament. Wrapping is done to add to the linear mass density of a string so as to achieve lower fundamental frequencies for a typical tension without adding much to the flexural rigidity of the core.[23] (A thick monofilament would introduce too much undesirable inharmonicity into the vibrations of a string.) If we assume that the wrapping contributes negligibly to the determination of e_0, then using[24] a core diameter of $d_{\text{core}} = 0.33$ mm, $S \approx 8.55 \times 10^{-5}$ m^2, $Y(\text{nickel}) \approx 214$ GPa,[25] $\rho_0 \approx 0.0094$ kg m^{-1}, $L_0 = 0.597$ m, and $f_1 \approx 75.8$ Hz, it follows from Eq. (10) that $e_0 \sim 0.0042$. Given the uncertainties in determining e_0, this calculated value for e_0 can only be considered a rough estimate.

V. COMPARING THEORY AND EXPERIMENT

To compare theory with experiment, the following additional experimental results[26] were used. For a triangular pluck with initial normalized amplitude $R \equiv A/L_0 = 0.0532$, the length of the first period of oscillation was estimated to be $T_{NL} \approx 10.4$ ms. (Note that a video rate of 1,200 frames per second equates to a time between frames of about 0.83 ms, which limits how precisely the length of a single period can be determined.) Thus, assuming a period for linear oscillations of $T_L = 1/f_1 \approx 13.2$ ms, we find $T_{NL}/T_L \approx 0.79$. In addition, the waveform shown in Fig. 7(c) of Ref. 10 was observed after about $0.67T_{NL} \approx 0.53T_L$. Finally, the quoted value of $C = 0.3$ was used in Eq. (6), and it was also confirmed that for the assumed value of e_0, linear behavior was observed for a "small amplitude" pluck with $R = 0.01$.

The coupled set of differential equations given by Eq. (8) were solved numerically using MATHEMATICA 8.0's numerical differential equation solver, initially with $r_0 = 0$ so that the effects of flexural rigidity were initially neglected. It was found that $N = 100$ terms achieved good convergence. Using $C = 0.3$, $\beta_n = \beta_1$ for all n, and $e_0 \simeq 0.00422$ (more precisely, $L_r/L_0 = 0.9958$)[27] led to a first period of length $T_{NL} \approx 0.80T_L$, very nearly what was estimated from the experiment. As can be seen from Fig. 1 though, agreement with Fig. 7(c) of Ref. 10 was less satisfactory. While a similar depth of bowing is seen in the numerical solution as was seen in the empirical observation at about $0.54T_L$, which is very near the empirically estimated time of $0.53T_L$, the amplitude is too low and the peaks are too far apart. The experimentally observed peak separation was seen in the numerical solution at the slightly earlier time of $0.48T_L$ (about one frame earlier at 1,200 fps), but at this time the depth of bowing is significantly less than what was observed empirically.

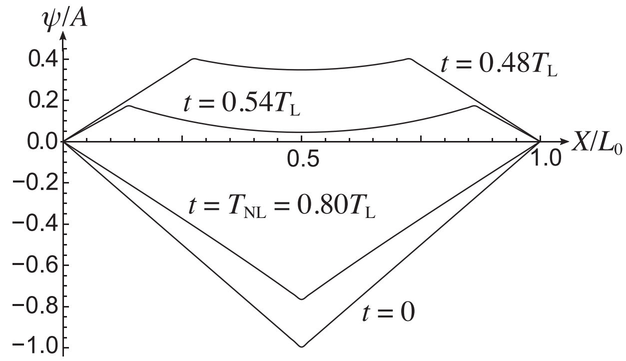

Fig. 1. Transverse displacement of the string for four different times during the first period as predicted by numerically solving Eq. (8) with string radius $r_0 = 0$, equilibrium strain $e_0 \simeq 0.00422$, and loss coefficient $C = 0.3$. The nonlinear period is very nearly that estimated from experiment. The time $t = 0.54T_L$ was chosen to obtain the relative depth of bowing seen in Fig. 7(c) of Ref. 10, while $t = 0.48T_L$ was chosen to obtain a similar peak separation as was observed in Fig. 7(c) of Ref. 10. At a video rate of 1,200 fps, these two times are about one frame apart.

Another discrepancy between Figs. 1 and 7(c) of Ref. 10 is that the peaks in Fig. 1 are too sharp. As dispersion due to a finite flexural rigidity or bending stiffness can be expected to smooth these peaks, one might wonder if adding flexural rigidity into the model might solve some of the discrepancies noted with Fig. 1. To test this possibility, Eq. (8) was solved numerically with $r_0 \neq 0$. As a first approximation, it was assumed that the wrapping contributed negligibly to the flexural rigidity of the string. Figure 2 shows the result of solving Eq. (8) with $r_0 = 0.165$ mm, the radius of the core, and with frequency independent damping terms with $C = 0.3$ in Eq. (6). As expected, adding flexural rigidity rounds out the corners in the wave form somewhat and also reduces the first period to $T_{NL} \simeq 0.79T_L$, as was estimated from the

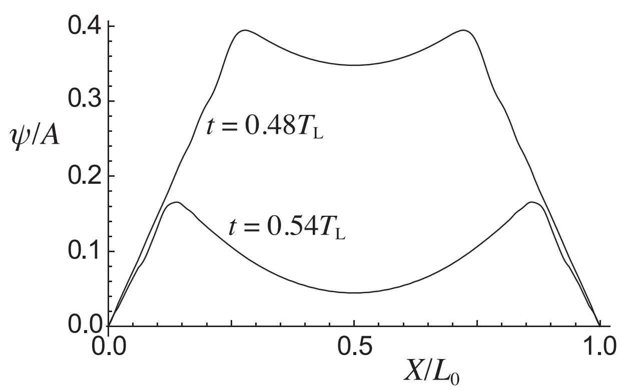

Fig. 2. Transverse displacement at $t = 0.48T_L$ and $0.54T_L$ with parameters as per Fig. 1, but now with the effects of flexural rigidity added. In Eq. (8), r_0 was set to be the radius of the core of the string.

experimental results. However, the other problems identified with Fig. 1 have not been resolved.

Note that for the physical parameters given above, the inharmonicity factor defined in Eq. (6) of Ref. 10 evaluates to $J = \pi^2 r_0^2/(8e_0 L_0^2) \simeq 2.24 \times 10^{-5}$. This value is well short of the value of $J = 0.001$ that Ref. 10 found was needed to get significant bowing. But this is as it needs to be, because inharmonicity due to flexural rigidity is an amplitude independent effect and negligible bowing was observed for small amplitude plucks in Ref. 10. (Note also that making r_0 equal to the full radius of the string only increases J to around 3×10^{-4}, although this is unlikely to be realistic given the purpose of wrapping is to reduce the inharmonicity that would occur for a monofilament of the same radius.)

One possible source of the discrepancies between Figs. 2 and 7(c) of Ref. 10 is that because e_0 was not measured directly, it could vary non-negligibly from the calculated value due to the influences of the wrapping and variations in Y due to impurities in the nickel.[28] In addition, the effective radius for modeling flexural rigidity might also be larger than the core radius due to some impact by the wrapping, and frequency-dependent damping might also influence the results. Furthermore, one further complication for comparing theory and experiment is as follows. At 1,200 fps the time lapse between frames is about 0.83 ms, which represents about $0.063T_L$. Consequently, T_{NL} and the peak displacement at that time had to be estimated as it occurred between two frames.[26] As the slope of the sides of the triangular displacement changes with time, estimating the peak amplitude by projection from a waveform at a time just before T_{NL} could lead to a significant overestimate of the magnitude of the peak at $t = T_{NL}$, as shown in Fig. 3, though the overestimate seems consistent with an empirical estimate of the peak being around −0.9A.

Given all the above possible uncertainties, the effects of variations in the various parameters in the model were explored. Unfortunately, these tend to work against each other. For example, while reducing e_0 would increase the amount of bowing seen at a given time, as does reducing damping for a given e_0, both these changes also reduced T_{NL} and the time at which a wave form similar to that shown in Fig. 7(c) of Ref. 10 was obtained. To counter the reductions in times due to decreasing e_0, damping could be increased, but increasing damping reduced the depth of bowing. Increasing the effects of flexural rigidity by increasing r_0 didn't affect the amount of bowing seen by much, but it did increase the amount of rippling seen in the wave forms. This rippling could be smoothed out by assuming frequency-dependent damping coefficients of the form

$$\beta_n = \beta_1[1 + b(n-1)^2], \tag{11}$$

consistent with Ref. 20, but apart from adding extra smoothing, using Eq. (11) in Eq. (8) did not otherwise seem to improve the model. Consequently, the numerical experiments conducted did not lead to substantially improved agreement between theory and experiment for any variations in the parameters tried.

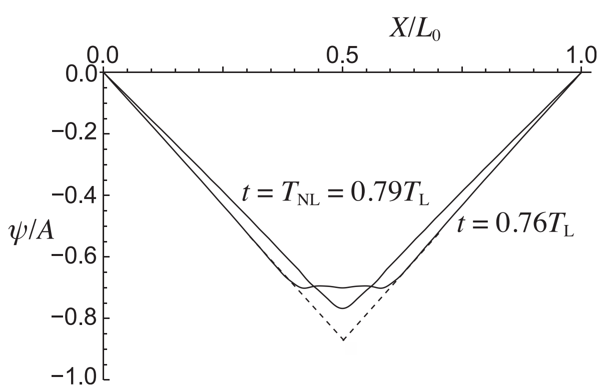

Fig. 3. Numerical solutions of Eq. (8) using the same parameters as given in Fig. 2. Estimating the peak amplitude at $T_{NL} = 0.79T_L$ by projecting along the sides of the waveform at $t = 0.76T_L$ (dashed lines) leads to a significant overestimate of the actual value.

Two further checks on the theory and numerics were made. First, it was confirmed that a small amplitude pluck with $R = A/L_0 = 0.01$ behaved fairly closely to what would be expected for linear behavior as required by the empirical results of Fig. 5 in Ref. 10. For the fundamental mode of a triangular pluck, $a_1(0)/L_0 = 8A/(\pi^2 L_0)$, and thus Eq. (9) leads to a frequency shift of about 2%, which reduced to about 1% as a consequence of damping. A very small amount of bowing was observed with this size pluck, but it disappeared when the damping was set to zero, indicating that damping also contributes a little to bowing behavior.

The second check was that because we have been looking at waveforms over the first cycle of oscillation, it is possible that some violation of the uniform tension approximation might be an issue. This latter possibility was tested by modeling the string as a chain of point masses joined by massless Hookean springs and then numerically solving the resulting set of coupled equations of motion (see Ref. 29 for more details of the model). However, for an initial triangular transverse displacement, this approach faced a significant problem. The wave equation for longitudinal displacements $\xi(X,t)$ from equilibrium (see Ref. 30, Sec. 14.3), taking into account coupling from the transverse displacements $\psi(X,t)$, is to lowest order given by[29]

$$\ddot{\xi} \simeq c_L^2 \xi'' + \frac{1}{2}\left(c_L^2 - c_T^2\right)\frac{\partial}{\partial X}\left(\psi'^2\right). \tag{12}$$

In the linear regime, a triangular pluck evolves into a trapezoid, and the resulting step-discontinuities in ψ' result in moving delta-function-forcing terms in Eq. (12), which a discrete chain has difficulty modeling. Nevertheless, rough convergence was obtained for a chain of 400 point particles, though only very short time runs could be made before the memory limits of the computer were exceeded. The results of these simulations suggest, however, that any possible limitations of the uniform tension approximation are likely *not* a problem, as quite good agreement over the first quarter cycle of oscillation was obtained between the transverse wave forms obtained from this chain model and from Eq. (8) with $r_0 = 0$. It would seem, therefore, that the discrepancies between theory and experiment arise from something missing in the model, such as possible influences of the wrapping of the string, measurement inaccuracies, or both.

VI. CONCLUSION

The wave behavior empirically observed in Ref. 10 for large amplitude plucks was compared against the theoretical predictions of the Kirchhoff-Carrier wave equation, which assumes a uniform tension across the string, extended to also include linear damping and flexural rigidity. Despite a number of empirical and theoretical uncertainties, good agreement between theory and experiment was achieved for the length of the first nonlinear period for the model parameters used. However, while a similar depth of bowing was observed in the numerical solutions as was observed in the empirical results at about the same time, the size of the numerically predicted wave at that time appears to be much too small. Whether this discrepancy is due to empirical uncertainties, theoretical deficiencies, or some combination of the two, is unclear. To make a more rigorous assessment of the theory, future research will need to better account for the possible impacts of the wrapping of the string. Specifically, for the wrapped string, a stress-strain curve is needed, as is a determination of the low amplitude frequency and damping coefficient as a function of mode number. It would also be useful to make a comparison between theory and experiment for more than the one instant of time as this might allow a more accurate determination of T_{NL} for the first period.

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[a)]Electronic mail: d.rowland@uq.edu.au

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[14]*Material* coordinates are *fixed* at the equilibrium positions of points in the string and so act as *labels* of these points. Thus, if the point P of the string has the material coordinates [X,0,0], then its longitudinal position at time t is given by the dynamic *spatial* coordinate $x_P(t) \equiv x(X,t) = X + \xi(X,t)$, where ξ is the longitudinal displacement of P from its equilibrium position. Elementary treatments of linear transverse waves do not distinguish between the longitudinal material and spatial coordinates because $\xi(X,t)$ is considered to be negligible. This approximation cannot be made in the derivation of the Kirchhoff-Carrier wave equation, however, and so the fact that the wave equation is a function of the material coordinate X and not the spatial coordinate x is being explicitly made in this Comment.

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