Gap Equations from Fermionic Constraints on the Light-Front

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We investigate how the gap equations are obtained in the light-front formalism within the four-Fermi theories. Instead of the zero-mode constraint, we find that the “fermionic constraints” on non-dynamical spinor components are responsible for symmetry breaking. Careful treatment of the infrared divergence is indispensable for obtaining the gap equation.

1. Introduction Development of our understanding of broken phase physics in the light-front (LF) formalism has been somewhat one-sided with scalar models. The idea that the spontaneous symmetry breaking in a scalar model is described by solving a constraint on a longitudinal zero mode (ZM), i.e. the ZM constraint, has been examined by several authors for simple scalar models. Compared with such extensive works, the dynamical symmetry breaking (DSB) in fermionic systems is not sufficiently understood at present. In particular, the four-Fermi theories which are typical examples of DSB should be investigated in more detail before we study the chiral symmetry breaking in QCD. Absence of bosonic degrees of freedom in the four-Fermi theories implies that we cannot follow the same idea above. Then, how can we formulate DSB analogously to the idea of ZM constraints in scalar models? In this paper, we consider this problem by viewing how the gap equations appear in the four-Fermi theories defined by

\[ \mathcal{L} = \bar{\Psi}_a (i\not\partial - m_0) \Psi_a + \frac{g^2}{2} \left\{ (\bar{\Psi}_a \Psi_a)^2 + \lambda (\bar{\Psi}_a \gamma_5 \Psi_a)^2 \right\}, \quad a = 1, \ldots, N. \]  

Here, we consider an \( N \)-component fermion \( \Psi_a \) and include the bare mass. In the 3+1-dimensional case, the Lagrangian (1) represents the Nambu-Jona-Lasinio (NJL) model (\( \lambda = 1 \)), while in 1+1 dimensions, the chiral Gross-Neveu (GN) model (\( \lambda = 1 \)) and the non-chiral GN model (\( \lambda = 0 \)). The standard understanding of the equal-time (ET) formulation is as follows. When the bare mass is zero, the (chiral or discrete) symmetries of the system break down spontaneously due to \( \langle \bar{\Psi} \Psi \rangle = 0 \). This is a result of an analysis of the gap equation, which is essentially the self-consistency condition on the fermion self-energy. This result yields information with regard to the phase transition as well as the nonzero value of \( \langle \bar{\Psi} \Psi \rangle \). Therefore, it is important to see how the gap equations emerge in the LF formalism. Despite several works on the LF four-Fermi theories, however, it is not yet clear how the gap equations are obtained exactly on the light-front frame without auxiliary fields included.

In the next section, we consider the implication of a nonlinear constraint on the non-dynamical component of the spinor (i.e. the “fermionic constraint”) and insist that solving the fermionic constraint is the key to the broken phase description. In §3, it is shown that the fermionic constraint to leading order in the \( 1/N \) expansion
indeed becomes the gap equation. Discussion is given in the final section.

2. Implication of the fermionic constraints

One of the most outstanding features of the LF four-Fermi theories is the complexity of the fermionic constraints. In the LF formalism, half of the spinor field is a dependent variable, and there always exists a constraint on it. While the fermionic constraints are usually solved without any effort, they are nonlinear and thus are difficult to solve in the four-Fermi theories. For example, one of the Euler-Lagrange equations of the GN model is given by

\[ i\partial_- \chi_a = \left\{ \frac{m_0}{\sqrt{2}} - \frac{g^2}{2} (\psi_b \Gamma_b \chi_a + \chi_a \psi_b) \right\} \psi_a, \]

where \( \psi_a = 2^{-1/4} (\psi_a, \chi_a)^T \). This includes only the spatial derivative \( \partial_- \) and thus is a constraint. This is the fermionic constraint. It is this complexity that makes the analysis of the LF four-Fermi theories very difficult.

Now let us consider the implication of the fermionic constraint. We should note that the complexity of the fermionic constraint is not just a difficulty but rather a key to a broken phase physics. We discuss this making the two observations below.

Let us first survey the Yukawa approach for the GN model of Ref. 13). The GN model can be defined as a limit of a Yukawa-like theory. At the limit, a scalar field becomes an auxiliary field \( \sigma = g\bar{\psi}\psi \), and we recover the GN model. In Ref. 13), non-dynamical degrees of freedom (\( \chi \) and the ZM of \( \sigma \)) were eliminated by solving two coupled equations (the ZM and fermionic constraints) step by step. Inclusion of the auxiliary field made it easy to solve the fermionic constraint, and essentially the ZM constraint became the gap equation in this approach. This method leads us to the following conjecture: even without an auxiliary field included, solving only the fermionic constraint may correspond to solving the above two constraints.

The second observation is on symmetry. Since the lower component of the spinor \( \chi \) is a dependent field to be expressed in terms of \( \psi \), any transformations should be imposed only on \( \psi \). On the other hand, transformation of \( \chi \) is not specified until we solve the fermionic constraint. So we cannot check whether the models are symmetric under the transformations. From this fact, it is reasonable to consider that if we solve the constraint appropriately, we can obtain \( \chi \) with some definite transformation property. Since a perturbative solution turns out to be symmetric, we must use some non-perturbative method to solve the fermionic constraint.

From these considerations, we expect that, without auxiliary fields, solving the fermionic constraint non-perturbatively must provide us with a description of dynamical symmetry breaking. As is usually done in the ET formalism, the \( 1/N \) expansion is useful for obtaining non-perturbative and nontrivial solutions.

\(^\ast\) Our notation is as follows. In 3+1 (1+1) dimensions, the light-front coordinates are \( x^\pm = (x^0 \pm x^3)/\sqrt{2} \) and \( x_i^i = x^i \) for \( i = 1, 2 \) (\( x^\pm = (x^0 \pm x^i)/\sqrt{2} \)) and the derivatives \( \partial_\pm = \partial/\partial x^\pm \). We use the following two-component representation for the \( \gamma \) matrices:

\[ \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix} \]

(\( \gamma_0 = \sigma_1, \gamma_1 = i\sigma_2 \) and \( \gamma_5 = \gamma^0\gamma^1 = \sigma_3 \) in 1+1 dimensions).
To this end, we set up the problem as follows. First, we work in infinite space. That is, we do not put the system in a finite box. This is because the important quantity is not the ZM of $\tilde{\Psi}$ but the ZM of the composite operator $\tilde{\Psi}\tilde{\Psi}$. Most of the latter is made of nonzero modes of $\tilde{\Psi}$. Second, we rewrite the Euler-Lagrange equations entirely with the bi-fermion operators since the fermion condensate is given as the vacuum expectation value of the fermion bilinear. We introduce two kinds of $U(N)$ singlet bi-local operators at equal LF time. For the GN models,

$$M(x^-,y^-) = \sum_{a=1}^{N} \psi^+_a(x^-,x^+)\psi_a(y^-,x^+), \quad C(x^-,y^-) = \sum_{a=1}^{N} \psi^+_a(x^-,x^+)\chi_a(y^-,x^+),$$

and similarly for the NJL model ($M_{\alpha\beta}$ and $C_{\alpha\beta}$, where $\alpha, \beta = 1, 2$ are the spinor indices). For example, the fermionic constraint (2) in the GN model is rewritten as

$$i\frac{\partial}{\partial y^-}T(x,y) = \frac{m_0}{\sqrt{2}} \left( M(x,y) - M(y,x) \right) - \frac{\theta^2}{2} \left( M(x,y)T(y,y) - T(y,y)M(y,x) \right),$$

where $T(x,y) = C(x,y) + C^l(x,y)$, and we have omitted suffixes of $x^-$ and $y^-$. We define the theory with these equations and this ordering*) along with the usual quantization condition on the dynamical fermions. This setup is the same as that in Ref. 10, which treated the GN models. However, their description of the broken phase did not resort to the gap equation, which means that it is not clear whether their formulation can predict the existence of the phase transition (though we are always in the broken phase, in fact, in the GN model up to leading order of $1/N$). Moreover, their method cannot be applied to the NJL model. Instead, we search for an alternative description which includes the gap equation and the physics of the phase transition.

3. The gap equations  Let us expand the bi-local operators as

$$M(p,q) = N \sum_{n=0}^{\infty} \left( \frac{1}{\sqrt{N}} \right)^n \mu^{(n)}(p,q),$$

and so on. Here $M(p,q)$ is the Fourier transformation**) of $M(x,y)$. A systematic $1/N$ expansion of $M(p,q)$ is given by the boson expansion method, which is a familiar technique in many-body physics. Among various methods of boson expansions, the Holstein-Primakoff type is the most useful for large $N$ theories. For example, the first two orders are given by $\mu^{(0)}(p,q) = \theta(p)\delta(p+q)$ and $\mu^{(1)}(p,q) = B(q,p)\theta(p)\theta(q) + B^l(-p,-q)\theta(-p)\theta(-q)$, where $B(p,q)$ is a bosonic operator satisfying $[B(p_1,p_2),B^l(q_1,q_2)] = \delta(p_1-q_1)\delta(p_2-q_2)$, and so on. Rewriting

*) It is almost hopeless to discuss the ordering ambiguity at this stage. This is because we cannot define the ordering of $\psi$ and $\chi$ unless we solve the fermionic constraint.

**) Note that we allow $p$ and $q$ to be negative because we define the Fourier transformation by

$$M(x,y) = \int_{-\infty}^{\infty} \frac{dpdq}{2\pi} e^{-ipx} e^{-iqy} M(p,q).$$
the $1/N$-expanded constraint equation in terms of bosonic operators, we can, in principle, solve the equation order by order and find $C(p, q)$ expressed by $B$ and $B^\dagger$.

Incidentally, if the lowest order equation has a nontrivial solution, it gives rise to a physical fermion mass $M = m_0 - g^2 \langle \bar{\Psi} \Psi \rangle$. Using $M$ and $\mu^{(0)}$, the lowest order fermionic constraint is reduced to $(g_0^2 = g^2 N)$

$$\frac{M - m_0}{M} = \frac{g_0^2}{2\pi} \int_0^\infty \frac{dk^+}{k^+}$$

(6)

for the chiral and non-chiral GN models, and

$$\frac{M - m_0}{M} = \frac{2g_0^2}{(2\pi)^3} \int_0^\infty \frac{dk^+}{k^+} \int_{-\infty}^{\infty} dk^1 dk^2$$

(7)

for the NJL model. Here we chose $\langle \bar{\Psi} i\gamma_5 \Psi \rangle = 0$ for the chiral GN model and the NJL model. If we set $m_0 = 0$, these equations seem to become "independent of $M"", and thus they are not the gap equation in this form. However, this observation is not correct because they are not well-defined until the divergent integral is regularized. Indeed, by carefully treating the infrared divergences, these equations become the gap equations which give nonzero values for $M$ even in the $m_0 = 0$ case.

Essentially, the same equation as Eq. (6) was obtained in Ref. 13 as the result of the ZM constraint for the auxiliary field. There, the gap equation was derived by using the damping factor as an infrared regulator. Instead of this, let us here introduce some IR cutoff. As was explicitly pointed out in Ref. 13, we obtain the correct gap equation if we follow the same cutoff schemes as those of the standard methods in the ET formulation, such as the covariant four-momentum cutoff. Indeed, in Ref. 8, a noncovariant (rotationally invariant) three-momentum cutoff was performed to obtain the known result. But such a cutoff seems very artificial as a light-front theory, and we here propose another cutoff scheme, the parity invariant cutoff. Usually, it is natural and desirable to choose a cutoff so as to preserve the symmetry of the system as much as possible. For the LF coordinates $x^\pm$ and $x^2$, it would be natural to consider the parity transformation $(x^+ \leftrightarrow x^-, x^2 \rightarrow -x^2)$ and a two-dimensional rotation in the transverse plane. In fact, parity invariance is not manifest, because the LF time $x^+$ is treated distinctively in the usual canonical formulation. However, we find it useful for obtaining the gap equation. First consider the 1+1-dimensional case. In momentum space, the parity transformation is the exchange of $k^+$ and $k^-$, and therefore the parity invariant cutoff is given by $k^\pm < \Lambda$. Using the dispersion relation**, $2k^+k^- = M^2$, we find that the parity...
invariant regularization inevitably relates the UV and IR cutoffs:

$$\frac{M^2}{2\Lambda} < k^+ < \Lambda.$$  

(8)

In 3+1 dimensions, the condition $k^\pm < \Lambda$ gives $(2k^+k^- - k_\perp^2 = M^2)$

$$\frac{k_\perp^2 + M^2}{2\Lambda} < k^+ < \Lambda.$$  

(9)

Note that this also implies the planar rotational invariance $k_\perp^2 < \Lambda^2 = 2\Lambda^2 - M^2$.

Since both of the IR cutoffs include $M$, the right-hand sides of Eqs. (6) and (7) become dependent of $M$. Indeed they are estimated as

$$\frac{M - m_0}{M} = \frac{g_0^2}{2\pi} \frac{2\Lambda^2}{M^2},$$  

(10)

$$\frac{M - m_0}{M} = \frac{g_0^2\Lambda^2}{4\pi^2} \left\{ 2 - \frac{M^2}{\Lambda^2} \left( 1 + \ln \frac{2\Lambda^2}{M^2} \right) \right\}.$$  

(11)

These are the gap equations. Both equations have nontrivial solutions $M \neq 0$ even in the $m_0 \to 0$ limit. The somewhat unfamiliar equation (11) of the NJL model exhibits the same property as the standard gap equations of the ET quantization. For example, there is a critical coupling $g^2_{cr} = 2\pi^2/\Lambda^2$, above which $M \neq 0$. In the GN models, we must renormalize the divergence. The same renormalized gap equation and thus the same dynamical fermion mass as the ET result are obtained if we renormalize the UV and IR divergences together. Now it is evident that the essential physical role of Eq. (6) is not a “renormalization condition” as was discussed in Ref. 10), but “the gap equation”.

Though it was already pointed out in Ref. 13), the following is very significant and worth mentioning again. The essential step to obtain the gap equation is the inclusion of mass information as the regularization rather than the fact that the UV and IR cutoffs are related to each other. If we regulate the divergent integral without mass information (e.g., introducing the UV and IR cutoffs independently, as in Ref. 10)), we cannot reproduce the gap equation. The loss of mass information is closely related to the fundamental problem of the LF formalism, and the parity invariant regularization can be considered as one of the prescriptions for it. The problem is that the light-front quantization gives a mass-independent two-point function (at equal LF time), which contradicts the result from general arguments concerning the spectral representation. A symptom of this problem can be easily observed as the absence of mass dependence in the usual mode expansion on the LF. For example, a 1+1-dimensional fermion field is expanded as

$$\psi(x) = \int_0^\infty \frac{dk^+}{\sqrt{2\pi}} \left[ b_k e^{-ik^+x^-} + d_k^\dagger e^{ik^+x^-} \right],$$  

(12)

with $\{ b_k^+, b_k^\dagger \} = \delta(k^+ - p^+)$, and so on. In order to restore the necessary mass dependence, a parity invariant IR cutoff is sometimes introduced in a scalar field.
4. Discussion  We found that the gap equation was derived from the fermionic constraints in the LF four-Fermi theories. To obtain the gap equation, it was necessary to include the mass parameters and use the constituent masses as the IR cutoffs. This requirement can be understood as a prescription for the common pathology in the LF field theories. It is suggestive that we are naturally guided by symmetry, such as parity, to remedy this problem.

If we solve the fermionic constraints to the next leading order and insert the solution into the equations of motion, we can derive the fermion-antifermion bound-state equations.\textsuperscript{17,18} For the GN models, these are essentially the same as the results of Ref. 10). There are solutions for the NG bosons in the chiral limit. In particular, in the chiral GN model, we can obtain an analytic solution to the bound-state equation even in the massive case and its mass spectrum near the chiral limit.\textsuperscript{18}

The way of describing the DSB on LF in our models may be summarized as follows. By solving the fermionic constraints, which are characteristic of the LF fermionic theories, we have in general two kinds of solutions, the symmetric and broken solutions. Substituting these into the Hamiltonians or the equations of motion, we reach symmetric or broken theories with a trivial vacuum. Instead of finding a new vacuum, the effect of the broken phase on the LF can be seen as modifications of the Hamiltonians. However, these arguments do not predict which theory should be realized in reality. This problem will be discussed in more detail elsewhere.\textsuperscript{17}

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