Description of Inclusive Meson Spectra in $e^+e^-$ Annihilation Reactions by a Quark Cascade Model

M. BATISTA,* J. BELLANDI and R. J. M. COVOLAN**

Instituto de Fisica Gleb Wataghin, Universidade Estadual de Campinas
Unicamp 13083-970 Campinas SP
*The Rockefeller University, New York, NY 10021

(Received April 22, 1997)

We present a model to describe inclusive meson production in $e^+e^-$ reactions based on a quark cascade approach whose formulation is put in terms of diffusion equations for three quark flavors ($u,d,s$). These equations are solved by using a formalism previously developed for the problem of the electromagnetic cascade generated in the atmosphere by cosmic-ray interactions. The obtained solutions are given in terms of a combination of power-law functions whose profiles are adequate to describe the characteristics observed in the inclusive spectrum of mesons.

§1. Introduction

Quantum Chromodynamics (QCD), the putative theory of fundamental hadronic interactions, has not yet provided a rigorous scheme to describe how hadrons are produced in electron-positron annihilation processes. The status of art in this matter is represented by QCD-based models which are usually implemented in terms of very complex Monte Carlo algorithms. Such models/algorithms have become able to describe more and more data, but at expenses of increasing complexity. Probably, that is the best that can be done about this subject and trying to simplify the problem would correspond to distort it.

However, simpler analytical models can be helpful in the sense that they (hopefully) may provide a more intuitive view about the role of the elementary interactions underlying such processes (for a general account on Monte Carlo and analytical models see an excellent review in Ref. 1)). The present paper is an attempt to develop one of these models. Its basic idea is not new: it consists fundamentally in expressing the cascade of quarks formed after the collision in terms of diffusion equations. Due to the nature of the problem and to the dynamics of the quark/gluelu interactions, these equations are necessarily coupled which renders this problem very complex in mathematical terms.

Even so, in the present approach we show that it is possible to obtain simple solutions for the case of three flavors by applying a method of calculation previously developed to the problem of the electromagnetic cascade generated in the atmosphere

* Fellow of FAPESP: Instituto de Fisica Teorica, Universidade Estadual Paulista, Unesp, 01405-000 Sao Paulo, SP, Brazil.

**) Permanent address: Instituto de Fisica Gleb Wataghin, Universidade Estadual de Campinas, Unicamp, 13083-970 Campinas, SP, Brazil.
by cosmic-rays. The solutions so obtained are shown to be in agreement with the inclusive spectra of some pseudoscalar mesons.

The paper is organized as follows. In §2 we put the problem of quark cascades in terms of diffusion equations and, in §3, we establish a set of solutions for them. Then, these solutions are used to construct the fragmentation functions and, consequently, the inclusive cross sections (§4). In §5 we apply the model so obtained to describe pions and kaons spectra. Our final remarks are in §6.

§2. Quark cascade model

First of all, let us consider the reaction by which $e^+e^-$ are annihilated, resulting in quark-antiquark pair, that is

$$e^+ + e^- \rightarrow q_i + \bar{q}_i,$$

where the $i$ index refers to the quark flavor (in the present paper, we shall consider only three flavors $(u, d, s)$). After the annihilation process, each quark has a very definite energy and generates a quark cascade by successive emission of mesons. Such a cascade process is exhausted when the remaining quark has not enough energy to give rise to a subsequent emission. The elementary emission processes we are going to consider in the present model are those ones represented in Fig. 1.

In this model we assume that each initial quark fragments into a jet of hadrons independently of the other quark and that the produced hadrons have a limited cylindrical distribution of transversal momentum around the jet axis. For simplicity, we also assume that the hadron momenta distribution can be described in a factorized form as the convolution between the longitudinal and the transversal distributions. This procedure is detailed in §4 where we define the fragmentation functions.

In order to describe the time evolution of the longitudinal component of the quark cascade generated after the annihilation process, we use the following diffusion equation firstly proposed by Fukuda and Iso.

$$\frac{\partial Q(x, t)}{\partial t} = -\lambda \begin{pmatrix} 1 & 1 \\ c & \end{pmatrix} Q(x, t) + \lambda \int_{x_0}^{x} dx' f(x', x') T \ Q(x', t). \quad (1)$$

In the above equation, $Q(x, t)$ is the quark spectral function which in this case is given by

$$Q(x, t) = \begin{pmatrix} U(x, t) \\ D(x, t) \\ S(x, t) \end{pmatrix} = Q_+(x, t) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + Q_-(x, t) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + Q_s(x, t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (2)$$

Above we have already defined spectral functions $U$, $D$ and $S$ in terms of $Q_+$, $Q_-$ and $Q_s$, which is a convenient procedure to solve the differential equations as shown below.

*) In the present approach, we are not going to include processes like $e^+e^- \rightarrow q_i + \bar{q}_i + g$, which can be supposed to constitute 10% - 20% of the reactions at the TRISTAN energies.
In Eq. (1), $\lambda$ is the emission frequency, $x = \frac{2p_H}{\sqrt{s}}$ is the Feynman variable and function $f(x, x')$ represents the probability for a quark with a momentum fraction $x'$ emitting a meson with momentum $x' - x$ and becoming a quark with momentum $x$. The presence of the time variable in these equations is because the emission processes are supposed to be sequential. In this respect, the role of time here is pretty much the same as the role of depth in diffusion equations of the cosmic radiation in the atmosphere (see, for instance, Ref. 4) and references therein).

An interpretation of the terms on the r.h.s. of Eq. (1) is in order. The first one refers to quarks which are originally in the energy interval between $x$ and $x + dx$ and leave it after any kind of emission. The second term corresponds to quarks which initially have energy above such an interval and enter it after a certain emission. In this last case, two aspects must be considered. One is the energy distribution of the emitted particle, which is regulated by function $f(x, x')$, and the second is the possibility for permanence or change of flavor state of the remaining quark.

This last requirement is fulfilled by matrix $T$, whose elements define the probabilities for permanence or change of flavor in emission processes like those sketched in Fig. 1. Such a matrix is given by

$$
T = \begin{pmatrix}
1 - a - b & a & b' \\
1 - a - b & b' & c - 2b' \\
\end{pmatrix},
$$

where parameter $a$ is relative to the processes $u \rightarrow \pi^+ + d$, $\bar{u} \rightarrow \pi^- + \bar{d}$, $d \rightarrow \pi^- + u$, and $\bar{d} \rightarrow \pi^+ + u$. 

Fig. 1. Diagrams illustrating meson emission in (a) quark cascade and (b) antiquark cascade.
and $\bar{d} \rightarrow \pi^+ + \bar{u}$; parameter $b$ corresponds to the processes $u \rightarrow K^+ + s$, $\bar{u} \rightarrow K^- + \bar{s}$, $d \rightarrow K^0 + s$ and $\bar{d} \rightarrow \bar{K}^0 + \bar{s}$; and finally parameter $b'$ corresponds to the processes $s \rightarrow K^- + u$, $\bar{s} \rightarrow K^0 + d$, $\bar{s} \rightarrow K^+ + \bar{u}$, $\bar{s} \rightarrow K^0 + \bar{d}$.

In the present model, we have changed the meaning of the parameters $a$, $b$ and $b'$ with respect to the original definition in Ref. 3). Therein they are defined as emission probabilities and, consequently, are associated with the produced particles. Here, instead, such parameters are associated with flavor change in a given emission. Assuming that $s$ quark is heavier than $u$ and $d$ quarks, whose masses in turn are supposed to be about equal, parameter $a$ represents the transition $\text{light quark} \rightarrow \text{light quark}$, $b$ represents $\text{light quark} \rightarrow \text{heavy quark}$, and $b'$ represents $\text{heavy quark} \rightarrow \text{light quark}$.

This definition reverses the role of $b$ and $b'$ for kaon production in antiquark cascades (Fig. 1(b)) and implies a priori in a hierarchy for the values that these parameters can assume, namely it imposes that $b' > a > b$, incorporating by this way $SU(3)$ violation.

We note that, with the definition of the $T$-matrix given above, the net change in the number of quarks given by the spectral functions is zero, that is

$$\int dx \sum_i \frac{\partial Q_i(x, t)}{\partial t} = 0,$$

provided $f(x, x')$ in Eq. (1) is a normalized function (this is the case as we shall see below). In fact, this result is imposed by construction, so it does not depend on the values assumed by the parameters $a, b, b'$ and $c$.

In the sequence we derive solutions for Eq. (1) and show that, for $c = 1$, these solutions are exact.

§3. Solutions of the diffusion equations

Before starting the solution of the diffusion equations, we introduce a change of variables in Eq. (1). We note that the components of the $Q(x, t)$ spinor define spectral functions by unit of frequency in the time interval between $t$ and $t + dt$. In order to calculate the so-called fragmentation functions (§4), we should take the convolution between $\lambda Q(x, t)$ and the probability function $f(x, x')$ and integrate it over the time variable. However, an equivalent result is obtained by solving the equation in terms of the function $Q(x, z) = \lambda Q(x, t)$, with the new variable $z = \lambda t$, and finally integrating over $z$.*)

With this change of variables, the coupled diffusion equations in (1) can be written in the following way,

$$\frac{\partial Q(x, z)}{\partial z} = -IQ(x, z) + T \dot{D}Q(x, z),$$

*) In fact, this means integrating over $t$ from $t = 0$ to $t = +\infty$. This can be done because after a certain finite time interval $t_e$, the cascade process vanishes due to energy exhaustion and $Q(x, t)$ becomes null. So integrating from $t = 0$ to $t = t_e$ is the same as integrating from $t = 0$ to $t = +\infty$. 

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where $I$ is the diagonal matrix

$$
I = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & c
\end{pmatrix},
$$

$T$ is the matrix given by Eq. (3) and $\hat{D}$ is an operator defined by the following equation,

$$
\hat{D}Q(x, z) = \int_{x}^{x_{0}} dx' f(x, x')Q(x', z).
$$

Equation (5) describes a cascade initiated by a quark ($u$, $d$ or $s$) at $t = 0$ with a fraction of momentum $x = x_{0}$. Thus, the boundary conditions are established by ascribing a delta distribution $\delta(x - x_{0})$ to the flavor initiating the cascade and zero to the remaining flavors.

By introducing Eq. (2) into Eq. (5), we can separate the three coupled equations in a single one for $Q_{-}(x, z),$

$$
\frac{\partial Q_{-}(x, z)}{\partial z} = -Q_{-}(x, z) + (1 - 2a - b)\hat{D}Q_{-}(x, z),
$$

and a coupled system of two equations for $Q_{+}(x, z)$ and $Q_{s}(x, z)$. Defining a two component spinor

$$
\Phi(x, z) = \begin{pmatrix}
Q_{+}(x, z) \\
Q_{s}(x, z)
\end{pmatrix},
$$

this system can be expressed in terms of the following matrix differential equation,

$$
\frac{\partial}{\partial z} \Phi(x, z) = -\Sigma_{1}\Phi(x, z) + \Sigma_{2}\hat{D}\Phi(x, z),
$$

where

$$
\Sigma_{1} = \begin{pmatrix}
1 & 0 \\
0 & c
\end{pmatrix} \quad \text{and} \quad \Sigma_{2} = \begin{pmatrix}
1 - b & b' \\
2b & c - 2b'
\end{pmatrix}.
$$

In order to obtain the solution of Eq. (10), we expand $\Sigma_{1}$ and $\Sigma_{2}$ in terms of Pauli matrices as

$$
\Sigma_{1} = \frac{1}{2}(1 + c)1 + \frac{1}{2}(1 - c)\sigma_{3}
$$

and

$$
\Sigma_{2} = \chi_{0}1 + \vec{\chi} \cdot \vec{\sigma},
$$

where

$$
\chi_{0} = \frac{1 - b + c - 2b'}{2}, \quad \chi_{1} = \frac{2b + b'}{2},
$$

$$
\chi_{2} = \frac{i}{2}(b' - 2b), \quad \chi_{3} = \frac{1 - c + 2b' - b}{2}.
$$
With these definitions, Eq. (10) can be rewritten as
\[
\frac{\partial}{\partial z} \Phi(x, z) = \left[ \left( -\frac{1+c}{2} + \chi_0 \hat{D} \right) \mathbf{1} - \frac{1}{2} (1-c) \sigma_3 + (\vec{\chi} \cdot \vec{\sigma}) \mathbf{D} \right] \Phi(x, z).
\] (14)

Now, reexpressing \( \Phi(x, z) \) as
\[
\Phi(x, z) = e^{-\frac{z + \chi_0}{2}} e^{z \chi_0 \hat{D}} \Psi(x, z),
\] (15)

and introducing it into Eq. (14), the final equation to be solved gets reduced to
\[
\frac{\partial}{\partial z} \Psi(x, z) = \left[ -(1-c) \sigma_3 + (\vec{\chi} \cdot \vec{\sigma}) \mathbf{D} \right] \Psi(x, z).
\] (16)

When \( c = 1 \), the general solution of Eq. (16) is exact and is formally given by
\[
\Psi(x, z) = e^{z(\vec{\chi} \cdot \vec{\sigma}) \hat{D}} \Psi(x, 0).
\] (17)

Following the same procedure of Ref. 4), the two components of \( \Psi \) are found to be
\[
\Psi_+(x, z) = \cosh(z \nu \hat{D}) \Psi_+(x, 0) + \frac{\sinh(z \nu \hat{D})}{\nu} \left[ \mu \Psi_+(x, 0) - b' \Psi_s(x, 0) \right]
\] (18)

and
\[
\Psi_s(x, z) = \cosh(z \nu \hat{D}) \Psi_s(x, 0) + \frac{\sinh(z \nu \hat{D})}{\nu} \left[ 2b \Psi_+(x, 0) - \mu \Psi_s(x, 0) \right],
\] (19)

where \( \nu = b' + \frac{b}{2} \) and \( \mu = b' - \frac{b}{2} \).

Since the solution of Eq. (8) is immediate, our final results for \( Q_- \), \( Q_+ \) and \( Q_s \) are
\[
Q_-(x, z) = e^{-z} e^{z(1-2a-b) \hat{D}} Q_-(x, 0),
\] (20)
\[
Q_+(x, z) = e^{-\frac{1+b}{2} z} e^{z \chi_0 \hat{D}} \Psi_+(x, z),
\] (21)

and
\[
Q_s(x, z) = e^{-\frac{1+b}{2} z} e^{z \chi_0 \hat{D}} \Psi_s(x, z),
\] (22)

with \( \Psi_+(x, z) \) and \( \Psi_s(x, z) \) given by (18) and (19).

In order to get explicit solutions from Eqs. (20)~(22) we need to specify the probability function \( f(x, x') \). Following Fukuda and Iso,\(^3\) we use here for \( f(x, x') \) a distribution inspired in the Bremsstrahlung model,
\[
f(x, x') = \frac{1 + \alpha}{x'} \left( \frac{x}{x'} \right)^\alpha,
\] (23)

which gives the probability that a quark carrying a fraction of momentum \( x' \) becomes a quark with \( x \) after emitting a meson with \( x' - x \).

The motivation for choosing distribution (23) is two fold: it is quite appealing from the physical point-of-view and it is convenient in terms of the analytical calculation that has to be done when one applies the boundary conditions to the solutions.
of the diffusion equations given by Eqs. (20)\text{--}(22). Some additional comments about this function are in order.

First of all, we note that function $f(x, x')$ as defined in (23) is a normalized distribution, that is

$$
\int_0^{x'} f(x, x') dx = 1. \quad (24)
$$

With respect to its role in the emission processes, we observe that the $(x/x')^\alpha$ dependence (with $\alpha > 0$) favors the emission of fast quarks and, consequently, suppresses the production of fast hadrons. This aspect is in qualitative agreement with experimental data since hadron momenta distributions are strongly peaked at low momenta. Another way to see this characteristic feature of function $f(x, x')$ is by looking at the average values that are produced by this function at every emission vertex. After being emitted in a process like those sketched in Fig. 1, the average momentum fraction of the produced quark is

$$
\langle x \rangle = \int_0^{x'} x f(x, x') dx = \frac{1 + \alpha}{2 + \alpha} x'. \quad (25)
$$

and, therefore, the average momentum fraction carried by the meson emitted at the same vertex is

$$
\langle x' - x \rangle = \left[ 1 - \frac{1 + \alpha}{2 + \alpha} \right] x'. \quad (26)
$$

Thus, if $\alpha$ were zero, the produced quark and meson would carry the same amount of momentum on average. Since $\alpha \sim 2$ (as we shall see below), the emission of slow mesons is favored. Of course, Eq. (26) is not valid to estimate the average momentum carried by mesons in the whole cascade process because it assumes emissions from a quark with a definite momentum $x'$ while in the actual cascade development one has a distribution of momenta for the quarks before every emission. Actually, this is one of the reasons why the fragmentation function is defined in terms of convolution between function $f(x, x')$ and the quark spectral function (see details in the next section). Nevertheless, the above reasoning is useful to realize how, in the present scheme, the description of the soft spectrum measured for mesons (and for hadrons in general) is a consequence of the hard profile of the probability distribution $f(x, x')$ given in Eq. (23).

The functional form of $f(x, x')$ has also some advantages from the calculational point-of-view. By using this distribution, it is straightforward to show that the successive action of the operator $\hat{D}$, defined in (7), on a delta function (which are the boundary conditions of this problem) produces the following recurrence relation

$$
\hat{D}^n \delta(x - x_0) = f(x, x_0) \frac{1}{(n - 1)!} \left[ (\alpha + 1) \ln \left( \frac{x_0}{x} \right) \right]^{n-1} \quad (27)
$$

for $n \geq 1$.

Equation (27) is particularly important in the present context since the solutions for the diffusion equations obtained above, (20)\text{--}(22), obviously can only be applied to the boundary conditions in the form of expansions.
In summary, although the formalism developed in Ref. 4) (whose mathematical approach the present model is based on) would allow us to use different functional forms for $f(x, x')$, the probability distribution (23) has several features that makes it an excellent choice for the present purposes.

Now, applying to our solutions (20)~(22) the boundary conditions for the case in which $u\bar{u}$ is the first quark pair produced leads to the following results,

$$Q_-(x, z) = \frac{e^{-z}}{2} \left[ \delta(x - x_0) + \frac{f(x, x_0)}{2g(x, x_0)} \mu_0 I_1(\mu_0) \right],$$

$$Q_+(x, z) = \frac{e^{-z}}{2} \left\{ \delta(x - x_0) + \frac{f(x, x_0)}{4g(x, x_0)} \left[ \left( 1 + \frac{\mu}{\nu} \right) \mu_1 I_1(\mu_1) + \left( 1 - \frac{\mu}{\nu} \right) \mu_2 I_1(\mu_2) \right] \right\},$$

and

$$Q_+(x, z) = \frac{e^{-z} \mu + f(x, x_0)}{2 \nu 2g(x, x_0)} [\mu_1 I_1(\mu_1) + \mu_2 I_1(\mu_2)],$$

where $g(x, x_0) = (a+1) \ln(x_0/x), \mu_0 = 2[(1-2a-b)g(x, x_0)z]^{1/2}, \mu_1 = 2[g(x, x_0)z]^{1/2}, \mu_2 = 2[(1-2\nu)g(x, x_0)z]^{1/2}$, and $I_1(\mu)$ is the modified Bessel function of first order.

By integrating the above expressions over $z$, one obtains the final results for the case in which $u\bar{u}$ is the first quark pair produced. In order to write these results in a more compact way, we first define the following functions,

$$A(x) = \left( \frac{x_0}{x} \right)^{\alpha+1}, \quad B(x) = \beta \left( \frac{x_0}{x} \right)^{\beta(\alpha+1)}, \quad C(x) = 2\gamma \left( \frac{x_0}{x} \right)^{\gamma(\alpha+1)},$$

where $\beta = 1 - 2b' - b$ and $\gamma = 1 - 2a - b$.

Thus, the final solutions for $u\bar{u}$ are

$$U_u(x) = \delta(x - x_0) + \frac{f(x, x_0)}{4} \left[ \left( 1 + \frac{\mu}{\nu} \right) A(x) + \left( 1 - \frac{\mu}{\nu} \right) B(x) + C(x) \right],$$

$$D_u(x) = \frac{f(x, x_0)}{4} \left[ \left( 1 + \frac{\mu}{\nu} \right) A(x) + \left( 1 - \frac{\mu}{\nu} \right) B(x) - C(x) \right],$$

$$S_u(x) = \frac{f(x, x_0)}{2} \frac{b'}{\nu} [A(x) - B(x)].$$

In the case in which the first produced quark pair is $d\bar{d}$, for a matter of symmetry the results are just given by

$$U_d(x) = D_u(x), \quad D_d(x) = U_u(x), \quad S_d(x) = S_u(x).$$

Finally, in the case of producing $s\bar{s}$ firstly, one can follow the same procedure detailed above for $u\bar{u}$ to obtain

$$U_s(x) = D_s(x) = \frac{f(x, x_0) b'}{2} \frac{b'}{\nu} [A(x) - B(x)]$$

and

$$S_s(x) = \delta(x - x_0) + \frac{f(x, x_0)}{2} \left[ \left( 1 - \frac{\mu}{\nu} \right) A(x) + \left( 1 + \frac{\mu}{\nu} \right) B(x) \right].$$
Thus we have shown that, in spite of the intrinsic complexity, the present approach is able to provide a set of very simple solutions to the problem of the quark cascades for the case in which $c = 1$. In fact, this condition ($c = 1$) makes the problem much more treatable. We also present in the Appendix solutions for $c 
eq 1$ which, although more complicated, are also quite manageable. The next step is to use the spectral functions just obtained in this section to construct the fragmentation functions.

§4. Fragmentation functions and inclusive cross sections

The inclusive cross section for $e^+e^-$ annihilation processes can be given by

$$\frac{1}{\sigma_i} \frac{d\sigma}{dx_\mu}(e^-e^+ \rightarrow hX) = \frac{1}{\sigma_i} \sum_q \sigma_{e^-e^+ \rightarrow q\bar{q}} [D^h_q(x_\mu) + D^h_q(x_\mu)],$$  

where the variable $x_\mu = \frac{2\mu}{\sqrt{s}}$ is the fraction of momentum of hadron $h$ and $D^h_q(x_\mu)$ is the fragmentation function of quark $q$ into hadron $h$ (see, for instance, Ref. 5).

In the present model, we start to construct $D^h_q(x_\mu)$ by defining a longitudinal fragmentation function which is given by

$$\tilde{D}^h_q(x) = \beta_q^h \int_{x_\mu}^{x_0} f(y - x, y) Q_q(y, x_\mu) dy,$$  

where $\beta_q^h$ is the probability of flavor change in a given emission, i.e., $\beta_q^h = a, b, b'$, and function $f(y - x, y)$ represents the probability of a quark with momentum fraction $y$ emitting a meson with momentum fraction $x$. According to Eq. (23) this function is given by

$$f(y - x, y) = \frac{1 + \alpha}{y} \left( \frac{y - x}{y} \right) ^\alpha.$$  

In order to respect energy conservation, a constraint has to be imposed on the upper limit of integration in (38). We note that $x_0$ is the momentum fraction carried by each quark right after the annihilation and is the maximum value that can be assumed by the longitudinal momentum fraction of the first produced meson in the case its transversal momentum is zero. This can be seen from the delta distributions appearing in the solutions of the diffusion equations. However, the produced meson can carry also some transversal momentum and that can be put in terms of the fraction $x_t^2 = 4p_t^2/s$. Therefore, the actual maximum value that can be carried by the longitudinal cascade is given by the condition $x_0^2 + x_t^2 = 1$. In order to preserve this condition during the whole cascade development, we introduce a delta function $\delta(x_0 - \sqrt{1 - x_t^2})$ into Eq. (38) by writing

$$\tilde{D}^h_q(x, x_t) = \beta_q^h \int_{x_\mu}^{1} f(y - x, y) \int_{x_0}^{1} \delta(x_0 - \sqrt{1 - x_t^2}) Q(y, x_\mu) dx_0 dy,$$  

which corresponds to calculate (38) with $x_0 = \sqrt{1 - x_t^2}$. The sense of this procedure is completely established when $\tilde{D}^h_q(x)$ so defined is convoluted with the $p_t$-distribution in the definition of the fragmentation function $D^h_q(x_\mu)$ (see Eq. (53) below).
Since $Q_q(y, x_0)$ represents in Eq. (38) the spectral function of the quark that emits a meson, we can write the expression to calculate the longitudinal fragmentation function of some specific pseudoscalar mesons as follows,

\[
\tilde{D}^+_{q}(x, x_t) = a \int_{x}^{1} f(y - x, y)U_q(y, x_0)dy, \tag{41}
\]

\[
\tilde{D}^+_{\bar{q}}(x, x_t) = a \int_{x}^{1} f(y - x, y)\bar{D}_q(y, x_0)dy, \tag{42}
\]

\[
\tilde{D}^-_{q}(x, x_t) = a \int_{x}^{1} f(y - x, y)D_q(y, x_0)dy, \tag{43}
\]

\[
\tilde{D}^-_{\bar{q}}(x, x_t) = a \int_{x}^{1} f(y - x, y)\bar{U}_q(y, x_0)dy, \tag{44}
\]

\[
\tilde{D}^{K^+}_{q}(x, x_t) = \frac{1}{2} a \int_{x}^{1} f(y - x, y)[U_q(y, x_0) + D_q(y, x_0)]dy, \tag{45}
\]

\[
\tilde{D}^{K^+}_{\bar{q}}(x, x_t) = \frac{1}{2} a \int_{x}^{1} f(y - x, y)[\bar{U}_q(y, x_0) + \bar{D}_q(y, x_0)]dy, \tag{46}
\]

\[
\tilde{D}^{K^-}_{q}(x, x_t) = b \int_{x}^{1} f(y - x, y)U_q(y, x_0)dy, \tag{47}
\]

\[
\tilde{D}^{K^-}_{\bar{q}}(x, x_t) = b' \int_{x}^{1} f(y - x, y)\bar{S}_q(y, x_0)dy, \tag{48}
\]

\[
\tilde{D}^{K^-}_{\bar{q}}(x, x_t) = b' \int_{x}^{1} f(y - x, y)S_q(y, x_0)dy, \tag{49}
\]

\[
\tilde{D}^{K^-}_{\bar{q}}(x, x_t) = b \int_{x}^{1} f(y - x, y)\bar{U}_q(y, x_0)dy, \tag{50}
\]

\[
\tilde{D}^{K_0}_{q}(x, x_t) = \frac{1}{2} \int_{x}^{1} f(y - x, y)[bD_q(y, x_0) + b'S_q(y, x_0)]dy, \tag{51}
\]

\[
\tilde{D}^{K_0}_{\bar{q}}(x, x_t) = \frac{1}{2} \int_{x}^{1} f(y - x, y)[b'\bar{S}_q(y, x_0) + b\bar{D}_q(y, x_0)]dy, \tag{52}
\]

where index $q(\bar{q})$ refers to quark flavors $u, d, s(u, d, s)$. All of the integrals in Eqs. (41) ~ (52) must be understood in the sense given by Eq. (40), that is $x_0 = \sqrt{1 - x^2}$.

With the $\tilde{D}^h_q(x, x_t)$ functions defined above, our complete definition of the fragmentation function $D^h_q(x_p)$ (based on Ref. 6)) is established as

\[
D^h_q(x_p) = \int_{\frac{2m_h}{\sqrt{s}}}^{x_p} \int_{0}^{\infty} \tilde{D}^h_q(x, x_t)\rho(p_t^2)\delta \left( x_p - \sqrt{x^2 + \frac{4p_t^2}{s}} \right) dp_t^2 dx, \tag{53}
\]

where $x_p = \sqrt{x^2 + \frac{4p_t^2}{s}}$, $m_h$ is the mass of produced particles and $\rho(p_t^2)$ is the $p_t^2$-distribution of these particles. A convenient and reliable way of introducing the $p_t^2$-distribution is by using the following polar form,

\[
\rho(p_t^2) = a_1 \left[ \frac{1}{1 + a_2p_t} \right]^n, \tag{54}
\]
where $a_1$, $a_2$ and $n$ are parameters to be determined from data.

In the following, we present the description of the momentum spectra of several pseudoscalar mesons obtained with the model established above.

§5. Description of pion and kaon production

In order to simplify our notation, we define the normalization in terms of the weights

$$ p_q = \frac{\sigma_q}{\sigma_h}, $$

where $\sigma_q = \sigma(e^+e^- \rightarrow q\bar{q})$ and $\sigma_h = \sum_q \sigma_q$. Using this definition, the inclusive cross section is given by

$$ \frac{1}{\sigma_t} \frac{d\sigma}{dx_p}(e^-e^+ \rightarrow hX) = \sum_q p_q[D_q^h(x_p) + D_q^h(x_p)]. $$

For the specific case of charged pions, we write

$$ \frac{1}{\sigma_t} \frac{d\sigma}{dx_p}(e^-e^+ \rightarrow \pi^+ X) = p_u[D_u^{\pi^+} + D_u^{\pi^+}] + p_d[D_d^{\pi^+} + D_d^{\pi^+}] + p_s[D_s^{\pi^+} + D_s^{\pi^+}] $$

and

$$ \frac{1}{\sigma_t} \frac{d\sigma}{dx_p}(e^-e^+ \rightarrow \pi^- X) = p_u[D_u^{\pi^-} + D_u^{\pi^-}] + p_d[D_d^{\pi^-} + D_d^{\pi^-}] + p_s[D_s^{\pi^-} + D_s^{\pi^-}]. $$

Now, applying charge conjugation and isospin invariance to relate the fragmentation functions, we obtain for charged pions the following inclusive cross section,

$$ \frac{1}{\sigma_t} \frac{d\sigma}{dx_p}(e^-e^+ \rightarrow \pi^\pm X) = 2(p_u + p_d)D_\pi^{\pi^+}(x_p) + 2(p_u + p_d)D_\pi^{\pi^+}(x_p) + 4p_sD_\pi^{\pi^+}(x_p). $$

Thus, the inclusive cross section of charged pions is reduced to the calculation of just three fragmentation functions (a general result for three flavors which does not depend on the present approach). Following an analogous procedure, we can describe the neutral pion data. In this case, the inclusive cross section is

$$ \frac{1}{\sigma_t} \frac{d\sigma}{dx_p}(e^-e^+ \rightarrow \pi^0 X) = p_u[D_u^{\pi^0} + D_u^{\pi^0}] + p_d[D_d^{\pi^0} + D_d^{\pi^0}] + p_s[D_s^{\pi^0} + D_s^{\pi^0}] $$

which can be reexpressed in terms of the $\pi^+$ fragmentation functions,

$$ \frac{1}{\sigma_t} \frac{d\sigma}{dx_p}(e^-e^+ \rightarrow \pi^0 X) = (p_u + p_d)D_\pi^{\pi^+}(x_p) + (p_u + p_d)D_\pi^{\pi^+}(x_p) + 2p_sD_\pi^{\pi^+}(x_p), $$

that is,

$$ \frac{1}{\sigma_t} \frac{d\sigma}{dx_p}(e^-e^+ \rightarrow \pi^0 X) = \frac{1}{2} \frac{1}{\sigma_t} \frac{d\sigma}{dx_p}(e^-e^+ \rightarrow \pi^\pm X). $$
Kaon spectra can also be described by the same model. In the case of charged kaons, the differential cross sections are

\[
\frac{1}{\sigma_t} \frac{d\sigma}{dx_p} (e^- e^+ \rightarrow K^+ X) = p_u [D_u^{K^+} + D_u^{K^0}] + p_d [D_d^{K^+} + D_d^{K^0}] + p_s [D_s^{K^+} + D_s^{K^0}],
\]

\[
\frac{1}{\sigma_t} \frac{d\sigma}{dx_p} (e^- e^+ \rightarrow K^- X) = p_u [D_u^{K^-} + D_u^{K^0}] + p_d [D_d^{K^-} + D_d^{K^0}] + p_s [D_s^{K^-} + D_s^{K^0}],
\]

and, therefore,

\[
\frac{1}{\sigma_t} \frac{d\sigma}{dx_p} (e^- e^+ \rightarrow K^\pm X) = 2p_u [D_u^{K^+} + D_u^{K^0}] + 2p_d [D_d^{K^+} + D_d^{K^0}] + 2p_s [D_s^{K^+} + D_s^{K^0}].
\]

(60)

Table I. Parameters obtained by fitting the \( \pi^+ \) data shown in Fig. 2.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( \alpha )</th>
<th>( b )</th>
<th>( b' )</th>
<th>( c )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.073</td>
<td>2.0</td>
<td>0.013</td>
<td>219.0</td>
<td>0.110</td>
<td>0.79</td>
</tr>
<tr>
<td>1</td>
<td>8.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 2. Differential cross section for inclusive meson production as function of the normalized momentum \( x_p \). Experimental data are from Refs. 7)~11).
For the case of neutral kaon inclusive production, we have
\[ \frac{1}{\sigma_t} \frac{d\sigma}{dx_p} (e^- e^+ \to K^0 X) = p_u [D_u^{K^0} + D_d^{K^0}] + p_d [D_d^{K^0} + D_d^{K^0}] + p_s [D_s^{K^0} + D_s^{K^0}], \] (61)
and, consequently, the cross section adequate to compare with experimental data as they are given in the $K^0$ case is
\[ \frac{1}{\sigma_t} \frac{d\sigma}{dx_p} (e^- e^+ \to \bar{K}^0 X) = p_u [D_u^{\bar{K}^0} + D_u^{\bar{K}^0}] + p_d [D_d^{\bar{K}^0} + D_d^{\bar{K}^0}] + p_s [D_s^{\bar{K}^0} + D_s^{\bar{K}^0}], \] (62)

Using Eq. (57), we fit the experimental data\(^{7,8}\) for $\pi^\pm$ and the calculated curves for other mesons\(^{7,9,10}\) are shown in Fig. 2. In the description presented above we have used a grand total of seven free parameters. Once these parameters are established for the $\pi^\pm$ case, all of them remain the same irrespective of the reaction which is being described. We note in Table I that the relation $b' > a > b$ pointed out in §2 is satisfied. This is a quite interesting result because it was obtained from the fit in a natural way, that is, no restrictions were imposed on the parameters in the fitting procedure. Thus, data have selected this ordering for parameters $a, b$ and $b'$ spontaneously and this is in agreement with the internal logic of the model.

§6. Final remarks

We have shown in this paper that it is possible to obtain a quite reasonable description for pseudoscalar meson production in inclusive reactions $e^+e^-$ within the context of Quark Cascade Model put in terms of diffusion equations. To the best of our knowledge, the approach presented here represents the first attempt to solve such quark cascade diffusion equations by using a spinorial method successfully applied to cosmic-ray cascades.\(^4\)

Once the parameters are established for $\pi^\pm$ inclusive production, other inclusive processes can be described simultaneously without any additional adjustment. This is a remarkable aspect that explains about the self-consistency of the present approach.

Nevertheless, in order to be accepted as a real alternative to describe inclusive processes in $e^+e^-$ annihilation, such a model needs several improvements. One can think about the formalism presented here as a model for the hadronization of quarks that comes right after the evolution of a cascade of partons given in terms of perturbative QCD. A study in this sense is about to be finished. The inclusive production of baryons is also under study since it is a necessary step in order to include resonances decay. With these improvements plus the inclusion of heavier flavors the description of an extensive variety of data is very likely to be possible within the present approach.
Appendix
— Solutions for \( c \neq 1 \) —

Looking at the diagonal elements of matrix \( T \), Eq. (3), which represent the probability of permanence in a given flavor state, we see that, for \( u \) and \( d \) flavors, these elements are defined in a normalized-to-unity way. Of course, the correspondent term for \( s \) flavor, \( c - 2b' \), obeys the same normalization when \( c = 1 \) (this case was detailed in §3).

Assuming the point-of-view of the original model\(^3\) by which \( SU(3) \) violation would imply \( c \neq 1 \) irrespective of the values assumed by \( a, b \) and \( b' \), we can obtain an approximate solution for Eq. (16) as an expansion in terms of \( (1 - c) \) since this quantity can be supposed to be quite small (the numerical results shown in §5 confirm this hypothesis). We must note, however, that the solutions obtained for \( Q_-(x, z) \) in §3 with \( c = 1 \) are exact and remain the same also in this case.

A first-order approximate solution for the other two components can be written as

\[
\Psi(x, z) = e^{z(\bar{\chi} \cdot \vec{\sigma}) \hat{D}} \Psi(x, 0) - \frac{(1-c)}{2} \int_0^z \hat{D} \lambda e^{(z-\lambda)(\bar{\chi} \cdot \vec{\sigma}) \hat{D}} \sigma_3 e^{\lambda(\bar{\chi} \cdot \vec{\sigma}) \hat{D}} \Psi(x, 0),
\]

for which the boundary conditions is the spinor \( \Psi(x, 0) \) written as

\[
\Psi(x, 0) = \begin{pmatrix} Q^u_1(x, 0) \\ Q^d_1(x, 0) \end{pmatrix} = \frac{1}{2} \delta(x - x_0) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{2} \delta(x - x_0) \psi.
\]

When the cascade is initiated by \( uu \) or \( dd \),

\[
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

and when it is initiated by \( ss \), we have

\[
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.
\]

Below we outline the mathematical development followed to obtain our solutions with \( c \neq 1 \). We limit ourselves to show only the demonstration of the zero-order contribution, \( \Phi_0(x, z) \) (see Eq. (72)). The derivation of the first-order correction follows the same procedure.

The zero order-contribution is given by

\[
\Phi_0(x, z) = e^{-\frac{1}{2} z^2} e^{x_0 \hat{D}} e^{(\bar{\chi} \cdot \vec{\sigma}) \hat{D}} \Psi(x, 0),
\]

where

\[
\Psi(x, 0) = \frac{1}{2} \delta(x - x_0) \psi.
\]

The exponential operator \( \exp(z(\bar{\chi} \cdot \vec{\sigma}) \hat{D}) \) can be decomposed in the following way

\[
e^{z(\bar{\chi} \cdot \vec{\sigma}) \hat{D}} = \cosh(z \omega \hat{D}) 1 + \frac{\sinh(z \omega \hat{D})}{w} (\bar{\chi} \cdot \vec{\sigma}),
\]

with

\[
w = z^2 + \frac{(\bar{\chi} \cdot \vec{\sigma})^2}{4}.
\]
where \( \omega = \sqrt{\chi \cdot \bar{\chi}} \).

Let us first consider the \( \cosh(z\omega \tilde{D}) \) contribution. We can write

\[
e^{z\omega \tilde{D}} \cosh(z\omega \tilde{D}) = \frac{1}{2} (e^{iz\tilde{D} \mu_{+}} + e^{-iz\tilde{D} \mu_{-}}),
\]

where \( \mu_{\pm} = \chi_{0} \pm \omega \).

Applying the above expression to the boundary condition and using Eq. (27), we have

\[
[e^{iz\tilde{D} \mu_{+}} + e^{-iz\tilde{D} \mu_{-}}] \delta(x - x_{0}) = 2 \delta(x - x_{0}) + f(x, x_{0})
\]

\[
\times \sum_{n=1}^{\infty} \frac{z^{n}}{n! (n - 1)!} \left[ \mu_{+}^{n} + \mu_{-}^{n} \right] \left( \alpha + 1 \right) \ln \left( \frac{x_{0}}{x} \right) \right]^{n-1} \tag{69}
\]

The second term on the r.h.s. of this equation can be written in terms of a modified Bessel function of first order and so we have

\[
[e^{iz\tilde{D} \mu_{+}} + e^{-iz\tilde{D} \mu_{-}}] \delta(x - x_{0}) = 2 \delta(x - x_{0})
\]

\[
+ f(x, x_{0}) \left[ z_{\mu_{+}} \frac{2}{u_{1}} I_{1}(u_{1}) + z_{\mu_{-}} \frac{2}{u_{2}} I_{1}(u_{2}) \right], \tag{70}
\]

where \( u_{1,2} = 2\sqrt{z_{\mu_{\pm}} (1 + \alpha) \ln \left( \frac{x_{0}}{x} \right)} \).

The contribution of \( \sinh(z\omega \tilde{D}) \) can be calculated in the same way and we have

\[
\Phi_{0}(x, z) = e^{-\frac{1+z}{2} z} \left\{ \frac{1}{2} \delta(x, x_{0}) + \frac{f(x, x_{0})}{4} \left[ z_{\mu_{+}} \frac{2}{u_{1}} I_{1}(u_{1}) + z_{\mu_{-}} \frac{2}{u_{2}} I_{1}(u_{2}) \right] \right\}
\]

\[
\times \left[ z_{\mu_{+}} \frac{2}{u_{1}} I_{1}(u_{1}) - z_{\mu_{-}} \frac{2}{u_{2}} I_{1}(u_{2}) \right] \left( \vec{\chi} \cdot \vec{\sigma} \right) \psi. \tag{71}
\]

Now we present the final solutions for this case, already integrated over \( z \), as well as for the first-order contribution. The two-component spinor \( \Phi(x) \) can be written as a sum of two spinors

\[
\Phi(x) = \Phi_{0}(x) + \Phi_{1}(x), \tag{72}
\]

where \( \Phi_{0}(x) \) and \( \Phi_{1}(x) \) are, respectively, the zero and the first order contributions which are given by

\[
\Phi_{0}(x) = \frac{\delta(x - x_{0})}{(1 + c)} \psi + \frac{f(x, x_{0})}{2(1 + c)} \left[ U_{1} \mathbf{1} + \frac{\vec{\chi} \cdot \vec{\sigma}}{\omega} U_{2} \right] \psi, \tag{73}
\]

where

\[
U_{1,2} = \frac{\Delta_{1}}{1 + \alpha} \left( \frac{x_{0}}{x} \right)^{\frac{\Delta_{1}}{1 + \alpha}} \pm \frac{\Delta_{2}}{1 + \alpha} \left( \frac{x_{0}}{x} \right)^{\frac{\Delta_{2}}{1 + \alpha}} \tag{74}
\]

with

\[
\Delta_{1,2} = \frac{2(\alpha + 1)}{1 + c} [\chi_{0} \pm \omega]; \tag{75}
\]

and

\[
\Phi_{1}(x, x_{0}) = -\frac{1 - c}{2} \left[ \frac{2\chi_{3} I_{3}}{\omega} \mathbf{1} + 2I_{2} \sigma_{3} + (I_{1} - I_{2}) \frac{2\chi_{3}}{\omega_{2}} (\vec{\chi} \cdot \vec{\sigma}) \right] \psi, \tag{76}
\]

where \( \omega = \sqrt{\chi \cdot \bar{\chi}} \).
where

\[ I_1 = \left( \frac{1}{1+c} \right)^2 \delta(x-x_0) + \frac{2}{1+c} \frac{f(x,x_0)}{8} K_1 \]  \quad \text{(77)}

\[ I_2 = \left( \frac{1}{1+c} \right)^2 \delta(x-x_0) + \frac{2}{1+c} \frac{f(x,x_0)}{8\omega} K_2 \]  \quad \text{(78)}

\[ I_3 = \frac{2}{1+c} \frac{f(x,x_0)}{8} K_3 \]  \quad \text{(79)}

with

\[ K_{1,3} = \frac{8\mu_+}{(1+c)^2} \left[ 1 + \frac{1}{2} \Delta_1 \ln \left( \frac{x_0}{x} \right) \right] \left( \frac{x_0}{x} \right)^{\Delta_1} \pm \frac{8\mu_-}{(1+c)^2} \left[ 1 + \frac{1}{2} \Delta_2 \ln \left( \frac{x_0}{x} \right) \right] \left( \frac{x_0}{x} \right)^{\Delta_2} \]  \quad \text{(80)}

and

\[ K_2 = \left( \frac{2\mu_+}{1+c} \right)^2 \left( \frac{x_0}{x} \right)^{\Delta_1} - \left( \frac{2\mu_-}{1+c} \right)^2 \left( \frac{x_0}{x} \right)^{\Delta_2} \]  \quad \text{(81)}

with \( \mu_{\pm} = \chi_0 \pm \omega \).

**Acknowledgements**

We would like to thank the Brazilian governmental agencies CNPq and FAPESP for financial support. One of us (R.J.M.C.) is grateful to K. Goulianos for the warm hospitality at the Rockefeller University.

**References**