A semiclassical theory is developed for billiards with energy-dependent mixed boundary conditions. Explicit expressions are derived for the smooth and oscillatory parts of the level density. The parametric dependence of the spectrum on the boundary conditions provides us a very useful tool in analyzing the classical and quantum correspondence in billiard systems. This is illustrated in the numerical analysis of the spectrum of the stadium billiard that has mixed boundary conditions on the circular parts.

§1. Introduction

The stadium billiard is one of the most extensively studied chaotic systems. It is also one of the few systems which has been mathematically proven to be a K- and B-system.1) - 3) At the same time the stadium billiard is one of the first quantized systems whose level statistics follow expressions derived from random matrix theory (RMT).4) This observation led to intensive study to find out the signature of quantum chaos in the statistics of energy levels of classical chaotic systems.

In order to investigate the correspondence between quantum and classical systems in detail as a function of parameters in physical systems, one can choose shape parameters. In the stadium billiard, for instance, Takami 5) changed the aspect ratio, the ratio of half the length of straight sections to the radius of semicircles. Tomiya and Yoshinaga 6) approximated the semicircles by polygons. In addition to these kinds of shape changes, there is another parameter which can be chosen, that is, the boundary condition. The boundary conditions work only for wave mechanics and they do not give any information in pure classical mechanics. However, one can take their effect into account in semiclassical analysis.

Semiclassical theory has been developed to show that the level statistics are universal and follow the results of RMT on the basis of Gutzwiller's periodic orbit theory for the level density.7) - 9) One of the crucial assumptions for the derivation of universality is that all the periodic orbits of the system are isolated and unstable. In the stadium billiard, however, there exists a set of neutral periodic orbits which bounce perpendicularly between the parallel straight sections of the billiard boundary ('bouncing-ball orbits'). Such a continuous set of neutral orbits gives an additional contribution to the semiclassical expression for the level density.10), 11)
Berry and Tabor\cite{12}) derived semiclassical contributions from the continuous sets of neutral orbits which generically appear in integrable systems. Shudo and Shimizu\cite{13}) analyzed the universality in level statistics and concluded that a non-universal factor comes from the bouncing-ball modes. Primack and Smilansky\cite{14}) investigated the stability of these bouncing-ball modes quantum mechanically. Recently, Sieber et al.\cite{15}) analyzed the quantum-classical correspondence using a mixed boundary condition, which is a mixture of Dirichlet and Neumann conditions. They applied their method to the Sinai billiard for the purpose of removing the contribution coming from the bouncing-ball orbits.

In this paper we adopt an energy-dependent mixed boundary condition which is superior to the energy-independent one in many respects, as discussed below. We compare the results of the semiclassical theory with the corresponding ones in quantum theory focusing mainly on the level density. According to the semiclassical theory, the level density has two parts, a smooth part and an oscillatory part. The oscillatory part of the level density is composed mainly of pieces, the amplitude contribution and the phase contribution. The phase depends on the boundary conditions and we apply semiclassical theory numerically to the stadium billiard to investigate how the phase affects the level density.

This paper is organized as follows. Our definition of the mixed boundary condition is described in §2, where we introduce the energy-dependent boundary condition. In §3 the smooth part of the level density is evaluated and the mean number of energy levels is numerically compared to the corresponding quantum quantity. In §4 the oscillatory part of the level density is discussed. We analyze the Fourier transformation of the level density in §5. We show that the length spectrum can be approximated well in terms of the contribution of the family of bouncing-ball trajectories and the contributions from isolated periodic orbits. In §6 oscillatory parts of both semiclassical and quantum level densities are compared numerically. In §7 we discuss how energy levels depend on the change of a parameter of the boundary condition. A summary and concluding remarks are given in the last section.

§2. Mixed boundary conditions

A prescription for the boundary $\Gamma$ completely defines a classical billiard system. All the dynamics is determined by the requirement that particles have specular reflections from the boundary. In quantum mechanics it is necessary to solve the Schrödinger equation, which reduces for billiards to the Helmholtz equation

$$ (\Delta + k^2)\psi (\vec{r}) = 0. \tag{2.1} $$

Here $k$ is the wave number, and we use natural units where $\hbar = 2m = 1$ and energy $E = k^2$.

In quantum mechanics an additional piece of information is necessary, that is, a condition which the wavefunction has to satisfy at the boundary. If a wavefunction $\psi$ cannot penetrate outside the region, the boundary condition is of the Dirichlet type

$$ \psi (\vec{r}) = 0, \quad \vec{r} \in \Gamma. \tag{2.2} $$
Another common boundary condition is of the Neumann boundary type

$$\partial_n \Psi (\vec{r}) = 0, \quad \vec{r} \in \Gamma,$$

(2.3)

where $\partial_n$ stands for the normal derivative with the normal pointing outside. This boundary condition is most familiar for the magnetic field of the transverse electric (TE) modes in a cavity. One can generalize the above boundary conditions by requiring the mixed boundary condition

$$\lambda \Psi (\vec{r}) + \partial_n \Psi (\vec{r}) = 0, \quad \vec{r} \in \Gamma,$$

(2.4)

where in general $\lambda$ can be a function of the boundary position $\vec{r}$. It is also possible that one can make the boundary condition dependent on energy $E$ or wave number $k$. Balian and Bloch\textsuperscript{16} considered these mixed boundary conditions in a nuclear physics context. Since the $\partial_n \Psi$ term in Eq. (2.4) yields a contribution roughly $k$ times larger than the $\Psi$ term at high energy, the Neumann term dominates over the Dirichlet term at high energy for constant $\lambda$. Therefore, to keep the contributions from Dirichlet and Neumann terms equal on the average, it is desirable to take $\lambda$ proportional to $k$. This can be easily understood in the evaluation of the smooth part of the level density, as will be discussed soon.

In this paper we consider the $2 \times 4$ stadium of Bunimovich.\textsuperscript{17} Half the length of straight sections $a$ and the radius of semicircles $r$ are both 1 in Fig. 1. We assume Dirichlet boundary conditions on the straight sections and the mixed boundary conditions

$$k \cos \alpha \Psi + \sin \alpha \partial_n \Psi = 0 \quad \text{or} \quad k \cot \alpha \Psi + \partial_n \Psi = 0,$$

(2.5)

on the circular parts. That is, $\lambda = k \cot \alpha$ is energy-dependent on the circular parts of the boundary. Here $\alpha$ is a parameter, and $\alpha = 0$ corresponds to Dirichlet boundary conditions, while $\alpha = \frac{\pi}{2}$ corresponds to Neumann boundary conditions.

![Fig. 1. The $2 \times 4$ stadium with various boundary conditions: mixed boundary conditions on the two circular parts and Dirichlet conditions on the two straight sections. Half the length of straight sections is denoted as $a$ and the radius of semicircles is denoted as $r$. Both of them are taken to be 1.](https://academic.oup.com/ptp/article-abstract/98/4/869/1907083)
§3. The smooth part of the level density

We consider the quantum level density defined by

$$d^Q(E; \lambda) = \sum_{n=1}^{\infty} \delta(E - E_n(\lambda)), \quad (3.1)$$

where the $E_n(\lambda)$ are the energy eigenvalues determined by solving the Helmholtz equation (2.1) under the proper boundary condition (2.4). The semiclassical theory approximates this quantum level density in terms of smooth $\tilde{d}(E; \lambda)$ and oscillatory $\ddot{d}(E; \lambda)$ parts:

$$d^Q(E; \lambda) \approx d^S(E; \lambda) = \tilde{d}(E; \lambda) + \ddot{d}(E; \lambda). \quad (3.2)$$

The smooth level density can be expressed as (see Appendix A for a brief derivation)

$$\tilde{d}(E; \lambda) = \frac{A}{4\pi} - \frac{L_D}{8\pi k} - \frac{L_M}{8\pi k} \left[ 1 - \frac{2}{\sqrt{1 + \left(\frac{\lambda}{k}\right)^2}} - 2 \left( \frac{\lambda}{k} \right) \left( \frac{\lambda}{k} \right) - 1 \right] \frac{d\lambda}{dk} + \ldots. \quad (3.3)$$

Here $A$, $L_D$ and $L_M$ are the area of the billiard, the perimeter with Dirichlet boundary conditions and the perimeter with mixed boundary conditions, respectively. This is a generalized Weyl formula.\(^{18}\) As expected, the leading term involving the area of the billiard $A$ is independent of the boundary condition. The higher order corrections, starting from the term containing the perimeter $L$ ($L_D$ or $L_M$), depend on $k$ in a way which interpolates between the known expressions for the Dirichlet and the Neumann boundary conditions. The rest of the terms are the curvature contribution and corner contributions. Since their contributions are relatively small compared to other terms, we neglect them in the following. For a fixed $\lambda$ (constant with respect to $k$), the contribution coming from the mixed-boundary conditions behaves like the Dirichlet term at low energy ($k \to 0$) and like the Neumann term at high energy ($k \to \infty$). From the expression (3.3) it follows that setting $\lambda$ proportional to $k$ leads to nearly equal contributions from the Dirichlet and Neumann terms on the average throughout the range of the spectrum.

In the present case, the boundary conditions $\lambda = k \cot \alpha$ on the circular parts lead to

$$\tilde{d}(E; \alpha) = \frac{A}{4\pi} - \frac{L_D}{8\pi k} - \frac{L_M}{8\pi k} \left( 1 - 2 \tan \frac{\alpha}{2} \right), \quad (3.4)$$

where $A = 4\alpha + \pi r^2 = 4 + \pi$, $L_D = 4\alpha = 4$ and $L_M = 2\pi r = 2\pi$. In this expression higher order contributions such as corner terms are neglected. Here $\alpha = 0$ corresponds to the stadium in which the Dirichlet boundary conditions apply to the complete boundary. In the case $\alpha = \frac{\pi}{2}$ the Neumann boundary conditions apply to the circular parts only.

The mean spectral staircase $\tilde{N}(E)$ is obtained by integrating (3.3) with respect to the energy $E$. In the following we demonstrate that the derivative term which is proportional to $\frac{d\lambda}{dk}$ in Eq. (3.3) is important to count the correct number of levels.
Using the mixed boundary conditions $\lambda = k \cot \alpha$ for the circular parts, one has

$$\tilde{N}(E; \alpha) = \frac{A}{4\pi} E - \frac{L_D}{4\pi} \sqrt{E} - \frac{L_M}{4\pi} \sqrt{E} \left( 1 - 2 \tan \frac{\alpha}{2} \right). \quad (3.5)$$

On the other hand if we neglect the derivative term proportional to $\frac{d\lambda}{dk}$ in Eq. (3.3), we instead obtain

$$\tilde{N}(E; \alpha)|_{\text{No-derivative}} = \frac{A}{4\pi} E - \frac{L_D}{4\pi} \sqrt{E} - \frac{L_M}{4\pi} \sqrt{E} (1 - 2 \sin \alpha). \quad (3.6)$$

Fig. 2. $N(E; \alpha) - \tilde{N}(E; \alpha)|_{\text{No-derivative term}}$ is shown as a function of energy $E$ with $\alpha = 0.1 \times \frac{\pi}{2}$.

Fig. 3. $N(E; \alpha) - \tilde{N}(E; \alpha)|_{\text{With derivative term}}$ is shown as a function of energy $E$ with $\alpha = 0.1 \times \frac{\pi}{2}$. 
In Appendix B we describe our present numerical method of obtaining the eigenmodes. By investigating eigenenergies \( E_n \) as functions of \( \alpha \), we have successfully determined energy levels up to wave number \( k_{\text{max}} = 150 \) without a single missing level. For \( \alpha = 0 \) we have obtained 3172, 3197, 3221 and 3242 eigenstates with the odd-odd, even-odd, odd-even and even-even symmetries, respectively.

Figure 2 displays the number difference \( N(E; \alpha) - N(E; \alpha) \) where \( N(E; \alpha) \) denotes the number of levels up to energy \( E \). Here we take \( \alpha = 0.1 \times \frac{\pi}{2} \). One sees that the \( N(E; \alpha) \) counts three levels more up to \( E = 2500 \) \( (k = 50) \). The quantity \( N(E; \alpha) - N(E; \alpha) \) with derivative term is shown in Fig. 3. From Figs. 2 and 3 it follows that the derivative term is quite important to count the number of levels correctly.

§4. The oscillatory part of the level density

In this section we consider the oscillatory part of the level density under mixed boundary conditions. The semiclassical treatment distinguishes between contributions of unstable, isolated periodic orbits and contributions of neutral periodic orbits. In the standard theory, the former are given by the Gutzwiller trace formula, \(^8\), \(^9\) and the latter were first derived by Berry and Tabor. \(^{12}\) In our example the bouncing-ball orbit is the only example of a neutral orbit and the others are unstable and isolated orbits. Thus the oscillatory part of the level density is made of two terms; the one coming from the bouncing-ball orbits and the other coming from the other isolated and unstable orbits:

\[
\tilde{d}(E; \lambda) = \tilde{d}_{\text{bb}}(E; \lambda) + \tilde{d}_{\text{iso}}(E; \lambda).
\]

As briefly described in Appendix A, to leading order, the introduction of mixed boundary conditions does not affect the amplitude of the oscillating terms. \(^{15}\) Owing to the use of the mixed boundary condition, the phase of each term is changed relative to the case with Dirichlet boundary conditions. The exceptional periodic orbit consists of bouncing-ball modes, since we have Dirichlet boundary conditions on the straight sections.

The Gutzwiller trace formula for the oscillator part of the level density is given for the isolated orbits as

\[
\tilde{d}_{\text{iso}}(E; \lambda) = \frac{1}{2\pi k} \sum_{\gamma} \sum_{j=1}^{\infty} a^{(j)}_{\gamma} \cos \left( j \left( k \ell_{\gamma} - \frac{\pi}{2} \beta_{\gamma} + \phi_{\gamma}(\lambda) \right) \right),
\]

where the amplitude and the phase are given as

\[
a^{(j)}_{\gamma} = \frac{\ell_{\gamma}}{\sqrt{2 - \text{tr} M_{\gamma}^2}}, \quad \beta_{\gamma} = 2m_{\gamma} + \mu_{\gamma}.
\]

In the expression (4.2) the sum is taken over primitive orbits \( \gamma \) with repetitions \( j \). Each primitive orbit has a length \( \ell_{\gamma} \) and a monodromy matrix \( M_{\gamma} \). Here \( m_{\gamma} \) is the number of reflections along the orbit and \( \mu_{\gamma} \) is the maximal number of conjugate points along the orbit (the Maslov index). \(^{11}\) Isolated orbits are classified
into two categories, self-retracting orbits and non-self-retracting orbits. In Eq. (4.2) the non-self-retracting orbits contribute twice because the motion for the positive and negative directions should be treated independently, whereas the self-retracting orbits contribute once.

With Dirichlet boundary conditions \( \psi_{\gamma}(\lambda) = 0 \), but with mixed boundary conditions we need an additional phase,

\[
\psi_{\gamma}(\lambda) = \sum_{i=1}^{n_{\gamma}} 2 \tan^{-1} \left( \frac{k}{\lambda} \cos \theta_{\gamma}^i \right), \tag{4.4}
\]

where \( i \) represents each collision on the mixed-boundary, \( \theta_{\gamma}^i \) is the reflection angle, and \( n_{\gamma} \) is the number of collisions for each orbit (see Appendix A for a brief derivation). In this expression it is natural that \( \lambda \) is set proportional to \( k \), \( \lambda = k \cot \alpha \), as in the previous section, and the phase is given as

\[
\psi_{\gamma}(\alpha) = \sum_{i=1}^{n_{\gamma}} 2 \tan^{-1} \left( \tan \alpha \cos \theta_{\gamma}^i \right), \tag{4.5}
\]

which is independent of the energy range one considers.

Concerning the contribution to the level density from the bouncing-ball, it is expected that it gives no additional phase in the level density formula since this bouncing-ball orbit does not reflect on the circular parts by definition. Therefore we have (see Appendix A)

\[
\tilde{d}_{bb}(E; \lambda) = \tilde{d}_{bb}(E) = \frac{1}{\pi \sqrt{2\pi k}} \sum_{j=1}^{\infty} \frac{A_{bb}}{\sqrt{j} \ell_{bb}} \cos \left( jk \ell_{\gamma} - \frac{\pi}{4} \right), \tag{4.6}
\]

where \( A_{bb} = 4ar \) is the area of the rectangle, and \( \ell_{bb} = 4r \) is the primitive length of the bouncing-ball orbits. Therefore the difference between oscillatory parts with two arbitrary angles \( \alpha \) and \( \beta \) is given by

\[
\tilde{d}(E, \alpha) - \tilde{d}(E, \beta) = \tilde{d}_{iso}(E, \alpha) - \tilde{d}_{iso}(E, \beta). \tag{4.7}
\]

It is concluded from the semiclassical theory that one can remove the contribution from the bouncing-ball modes with this operation. It is now a question whether one can remove the bouncing-ball modes contribution for the quantum level density using the same operation.

§5. Fourier transformation of the level density

Since the invention of the trace formula by Gutzwiller, it has been known that each peak of the Fourier transformation of the level density corresponds to a classical periodic orbit or a one-parameter family of such orbits both in regular and chaotic systems. In the following we express the level density in terms of the wave number \( k \)

\[
d^Q(k; \alpha) \equiv \sum_{n=1}^{\infty} \delta(k - k_n(\alpha)) = 2k \ d^Q(E; \alpha), \tag{5.1}
\]
where we take \( k_n(\alpha) = \sqrt{E_n(\alpha)} > 0 \).

The Fourier transformation of the oscillatory part of the level density (with respect to momentum) is defined as

\[
f(\ell, \alpha) \equiv \int_{-\infty}^{\infty} dk \ d^2(k, \alpha) \cos k \ell = \int_{-\infty}^{\infty} dk \ \sum_{n=1}^{\infty} \delta(k - k_n(\alpha)) \cos k \ell
\]

\[
= \sum_{n=1}^{\infty} \cos (k_n(\alpha) \ell).
\] (5·2)

Since it is impossible to calculate an infinite number of eigenmodes, we truncate the allowed wave numbers at a cutoff \( k_{\text{max}} \). Here we consider two cases. First that in which the boundary condition is of the Dirichlet type, that is, we set \( \alpha = 0 \):

\[
f(\ell; 0) = \sum_{n=1}^{k_n < k_{\text{max}}} \cos (k_n(0) \ell).
\] (5·3)

Next we take the difference between the mixed boundary conditions and the Dirichlet boundary conditions,

\[
f(\ell; \alpha) - f(\ell; 0) = \sum_{n=1}^{k_n < k_{\text{max}}} [\cos (k_n(\alpha) \ell) - \cos (k_n(0) \ell)].
\] (5·4)

In Fig. 4 the Fourier transformation of the level density with Dirichlet conditions \( |f(\ell; 0)|^2 \) is shown up to length \( \ell = 17 \). At lengths 4, 8, 12 and 16, we see the peaks coming from the bouncing-ball orbits. Here, comparison between two different cutoff values \( k_{\text{max}} \) is also made. Comparing Eqs. (4·2) and (4·6) we see that the contribution from the bouncing-ball modes dominate when \( k \) is large, which was already pointed out by Berry. 7) Solid lines denote values using \( k_{\text{max}} = 75 \) and lines in gray denote

![Fig. 4. Fourier transformation of level density with Dirichlet conditions \( |f(\ell; 0)|^2 \) is shown in the length range \( 1 \leq \ell \leq 17 \). Solid lines denote values with \( k_{\text{max}} = 75 \) and lines in gray denote those with \( k_{\text{max}} = 150 \).](https://academic.oup.com/ptp/article-abstract/98/4/869/1907083)
those with $k_{\text{max}} = 150$. We see that the height of the peaks of the bouncing-ball orbits depends largely on the cutoff $k_{\text{max}}$. In Fig. 5 the mixed-Dirichlet difference of Fourier transformation of the level density $|f(\ell; \alpha) - f(\ell; 0)|^2$ is shown. Here we take $\alpha = 0.1 \times \frac{\pi}{2}$ and $k_{\text{max}} = 150$. It is clearly seen that one can remove the contribution coming from the bouncing-ball modes. In Appendix C we demonstrate why the energy-dependent boundary condition is superior to the energy-independent one in obtaining the periods of classical orbits from the Fourier spectra.

In Fig. 6, the lower part of Fig. 5 is magnified ten times to show details. We can see the remaining small peaks at lengths 4 and 8 (there is no peak at lengths 12 and 16). The peak at length 4 is considered to be the edge contribution $^5, ^{11}, ^{19}$ which bounces on edges of those connecting points between straight sections and semicircles. Since our boundary contributions change discontinuously on these edges, we cannot remove this contribution with the mixed-Dirichlet subtraction. Concerning

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**Fig. 5.** Fourier transformation of the mixed-Dirichlet difference of level density $|f(\ell; \alpha) - f(\ell; 0)|^2$ is shown with $\alpha = 0.1 \times \frac{\pi}{2}$.

**Fig. 6.** The same figure as in Fig. 5, but the lower part is magnified ten times.
the peak at length 8, the unstable horizontal orbit that bounces on the center points of semicircles is considered to contribute in addition to the edge contribution.

§6. Comparison between semiclassical and quantum level densities

In this section we compare the oscillatory part of the quantum level density

\[ \tilde{d}^Q(E; \alpha) = d^Q(E; \alpha) - \tilde{d}(E; \alpha) \]  

(6.1)

with the semiclassical level density \( \tilde{d}(E; \alpha) \) defined in Eq. (4.1). Here \( \tilde{d}(E; \alpha) \) is the smooth part of the level density for which the semiclassical expression is substituted. It is impossible to sum over an infinite number of primitive orbits \( \gamma \) in Eq. (4.2), and we thus restrict most of the lengths of the periodic orbits to be shorter than \( \ell = 16 \). In accordance with this restriction we smear out the quantum level density \( d^Q(E, \alpha) \) with the gaussian smoothing function, \( f(E, \varepsilon) = \frac{1}{\sqrt{2\pi\varepsilon^2}} \exp \left( -\frac{E^2}{2\varepsilon^2} \right) \),

(6.2)

where the parameter \( \varepsilon \) should be determined by a suitable method. Using this envelop function we have the smooth quantum level density

\[ d^Q_{\varepsilon}(E, \alpha) = \int_0^\infty dF \ f(E - F, \varepsilon) d^Q(F) \]

\[ = \frac{1}{\sqrt{2\pi\varepsilon^2}} \int_0^\infty dF \exp \left( -\frac{(E - F)^2}{2\varepsilon^2} \right) \sum_{n=1}^\infty \delta(F - E_n(\alpha)) \]

\[ = \sum_{n=1}^\infty \frac{1}{\sqrt{2\pi\varepsilon}} \exp \left( -\frac{(E - E_n(\alpha))^2}{2\varepsilon^2} \right). \]  

(6.3)

According to this change we need an extra factor in the trace formula (4.2),

\[ g(\tau, \varepsilon) = \exp \left( -\frac{\varepsilon^2\tau^2}{2} \right), \]  

(6.4)

which is the Fourier transformation of Eq. (6.2), and its functional form is shown in Fig. 7. The period \( \tau \) is related to the length \( \ell \) of an orbit as \( \tau = \frac{\ell}{2k} \). For instance, if we take \( k = 50 \) (\( E = 2500 \)), contributions with \( \tau \leq 0.16 \) must be taken into account for the orbits shorter than \( \ell = 16 \). From Fig. 7 it is seen that \( \varepsilon = 18 \) is the suitable value in this case.

We obtain classical periodic orbits by the Newtonian method. In the calculation we consider only those periodic orbits which start at points on one semicircle and bounce several times, and then return to the starting points after bouncing on the other semicircle several times (in due course they bounce on the straight sections several times). We consider only these one-cycle periodic orbits (from one semicircle through the other semicircle back to the original semicircle). The number of times an orbit bounces on each semicircle is limited to a value less than or equal to 30 and the number of times an orbit bounces on each straight section is limited to a value less
than or equal to 10. However, in our analysis the extra contribution from boundary edges, connecting points between straight sections and semicircles, is not included in Eq. (4·2). Some whispering gallery modes are also omitted in the analysis. It was reported\textsuperscript{22} that they are less important because of a large cancellation between the contribution of pairs of periodic orbits. Altogether 1106 periodic orbits are used to calculate the semiclassical level density.

In Figs. 8 and 9 the solid gray line denotes the quantum mechanical level density and the solid black line represents the semiclassical level density. In Fig. 8 the quantum level density $\tilde{d}^Q(E, 0) = d^Q(E, 0) - \tilde{d}(E, 0)$, the semiclassical level density $\tilde{d}(E; 0)$, the level density contributed only from the isolated periodic orbits $\tilde{d}_{\text{iso}}(E; 0)$ (dotted line) and that from the bouncing-ball modes $\tilde{d}_{\text{bball}}(E; 0)$ (dotted line in gray) are shown in energy range between $E = 1500$ and $E = 2500$. Here Dirichlet conditions and $\varepsilon = 18$ are used. Good agreement is seen between the quantum and semiclassical level densities (isolated orbits plus bouncing-ball orbits), but there is a large discrepancy between the quantum and semiclassical level densities with only the isolated orbits. It is seen that the major part of the level density comes from the bouncing-ball orbits, but the subtle structure cannot be explained only by those orbits (see the structure around $E = 2000$). In Fig. 9 we show the mixed-Dirichlet difference $\tilde{d}^Q(E, \alpha) - \tilde{d}^Q(E, 0)$ of the quantum level density with $\alpha = 0.1 \times \frac{\pi}{2}$ and the corresponding quantity $\tilde{d}_{\text{iso}}(E, \alpha) - \tilde{d}_{\text{iso}}(E, 0)$ in the semiclassical theory. Excellent agreement is seen between the semiclassical and quantum results. It is clearly seen that by taking the difference, the bouncing-ball orbit contribution is subtracted.
Fig. 8. Level densities between energies $E = 1500$ and $E = 2500$ are shown with the Dirichlet boundary conditions. The solid line in gray and the solid line in black represent the quantum level density with $\varepsilon = 18$ and the semiclassical level density, respectively whereas the dotted line in black represents the semiclassical level density without the bouncing-ball contribution and the dotted line in gray represents that from the bouncing-ball modes.

Fig. 9. The mixed-Dirichlet difference of the level density is shown between energies $E = 1500$ and $E = 2500$ for both the quantum (gray line) level density with $\varepsilon = 18$ and semiclassical (solid line) level density.
§7. Change of energy levels due to the change of mixed boundary condition

In this section we show how the energy eigenvalues change according to the change of the mixed boundary conditions. In Fig. 10 we show a plot of the eigenenergies with odd-odd symmetry as a function of the parameter $\alpha$; (a) 100 levels from the ground state, (b) 100 levels from the 301th state, (c) 100 levels from the 3001th state. Horizontal lines corresponding to the bouncing-ball orbits are indicated by arrows. The energy levels corresponding to the 383th, 387th and 397th states are marked by # 383, # 387 and # 397.

Fig. 10. Eigenenergies with odd-odd symmetry as a function of the parameter $\alpha$; (a) 100 levels from the ground state, (b) 100 levels from the 301th state, (c) 100 levels from the 3001th state. Horizontal lines corresponding to the bouncing-ball orbits are indicated by arrows. The energy levels corresponding to the 383th, 387th and 397th states are marked by # 383, # 387 and # 397.
energies $E_n$ of a part of the spectrum as a function of the parameter $\alpha$ for the odd-odd case corresponding to the quarter stadium billiard. Here $\alpha$ is varied from zero (Dirichlet case) to $\alpha = 0.1 \times \frac{\pi}{2}$. There is a mean decrease of the levels with increasing $\alpha$ which is due to the $\alpha$-dependence of the mean number of energy levels $\bar{N}(E)$ (see Eq. (3.5)).

A striking feature in Fig. 10 is that one can distinguish apparent horizontal lines even at low energies. These lines consist of almost straight pieces which are interrupted by avoiding level crossings. These lines can be attributed to the existence of the families of bouncing-ball orbits since it is expected that bouncing-ball orbits are not affected by the change of the parameter in the boundary conditions.

![Fig. 11. Eigenfunctions $|\psi(\vec{r})|^2$ corresponding to the 383th (a), the 387th (b) and the 397th (c) eigenstates.](https://academic.oup.com/ptp/article-abstract/98/4/869/1907083)
Comparing Figs. 10 (a) ~ (c), one observes that the slopes of most curves become steeper at higher energy. By imposing the condition \( \tilde{N}(E(\alpha); \alpha) = \tilde{N}(E(\alpha + d\alpha); \alpha + d\alpha) \) at high energy, one obtains

\[
\frac{dE}{d\alpha} = -\frac{L_M k}{A - \frac{L_D + L_M}{2k}}
\]

(7.1)

for the mean slope of energy levels as a function of \( \alpha \) around \( \alpha = 0 \). Thus at high energy the slope is proportional to wave number \( k \).

In Fig. 11 eigenfunctions corresponding to the 383rd, 387th and 397th eigenstates (when \( \alpha = 0 \)) are shown to demonstrate the scars of the bouncing-ball modes. The horizontal lines in Fig. 10 clearly correspond to wavefunctions that are mainly scarred along the bouncing-ball orbits. The wavefunctions of two states (383rd and 387th) concentrate on the rectangular region, and there is almost no probability in the circular regions. Therefore it is certain that the wavefunctions are not affected by the change of the boundary conditions on the semicircular parts. For the 397th state, the wavefunction is not completely concentrated in the rectangular region and the eigenenergy does not remain constant, as shown in Fig. 10.

§8. Summary and concluding remarks

In this paper we applied mixed boundary conditions to a billiard problem in order to study the classical and quantum correspondence. Mixed boundary conditions are rarely used in the literature, compared to the Dirichlet and Neumann boundary conditions. Dirichlet boundary conditions are used in quantum mechanics when the surrounding high wall goes to infinity, while Neumann boundary conditions appear for the pressure condition in the field of acoustics.

In some circumstances, however, a perturbative treatment of the Helmholtz equation with respect to changes in boundary conditions can be found in the literature.\(^{23}\) Mixed boundary conditions are useful, for instance, to treat the acoustics of irregularly-shaped rooms in the field of acoustics, scattering from irregular-shaped objects in the field of elasticity and propagation down irregularly-shaped pipes in electromagnetic theory.\(^{24}\)

As far as we know, energy-dependent mixed boundary conditions have never been used in the literature. In some sense the boundary conditions can be viewed as a kind of an approximation scheme which simplifies a problem. For instance, as stated before, Dirichlet boundary conditions are used in quantum mechanics with a high potential wall whose height tends to infinity, but in a realistic case where the wall is not sufficiently high, we must consider the coupling to (or interaction with) the outer region (or to another object). Therefore the boundary conditions provide an approximation scheme with which the problem can be solved within the inner region and on the boundary. Then it is also natural to think that the boundary conditions change as a function of the energy under consideration. At the moment we have no example for this, but in the future we hope our model will be applicable to a realistic example.
Apart from these realistic physical applications, the energy-dependent mixed boundary condition can also be applied to a billiard problem as a mathematical tool to investigate the quantum and classical correspondence. This resulted in:

1. obtaining the generalized Weyl formula for the energy-dependent mixed boundary condition up to the order which includes the term proportional to the circumference of the boundary. Once we decide to introduce the mixed boundary condition, it is natural to make it energy-dependent, since otherwise either the Dirichlet or Neumann terms become dominant, depending on the energy range we consider.

2. The difference of level densities with two different boundary conditions which are different only on the circular parts enabled us to identify the effects of the bouncing-ball modes. As is well known, the oscillatory contribution of the level density is expressed as
   \[ d(E) = \frac{1}{\mu + \frac{1}{2}} \sum_j A_j(E) \exp\{i S_j h\} \]
   where the exponent \( \mu = (N - 1)/2 \) (\( N \): dimension of the system) for integrable systems and zero for chaotic systems. Since the bouncing-ball modes contribution behaves like that of integral systems, in the semiclassical limit the bouncing-ball orbit modes contribution dominates over the isolated periodic orbits. The effect of the bouncing-ball modes is eminent in higher dimensional systems and it is overwhelmingly large compared to that of the isolated orbits, as has been shown numerically in the case of 3D-Sinai billiards.\(^{25}\) As stated in the Introduction, there are several methods to investigate the quantum and classical correspondence in detail. Changing the aspect ratio is one of them. In order to separate out the bouncing-ball modes, it is quite useful to change the aspect ratio. In that case the corresponding eigenenergies change linearly as functions of the aspect ratio, while in our case they remain constant with the change of the mixed boundary conditions. Recently, the distribution of spectral fluctuations, that is, mode fluctuations, has been proposed as an alternative measure to give the hallmark signature of quantum chaos.\(^{26}, 27\) If the system is chaotic, it is shown that mode fluctuations follow the Gaussian distribution. This measure has been demonstrated to work well for the hyperbolic billiards. In this case eliminating the semiclassical contribution of bouncing-ball modes from the spectral staircase function is quite important for the mode fluctuation to show the Gaussian distribution in the stadium billiard.\(^{28}\)

3. Another new aspect is that the energy dependence of the boundary condition leads to the energy-independence of the phase \( \phi_r(\lambda) \) in (C-1). As stated in Appendix C, the Fourier transform of semiclassical level densities have peaks at the corresponding lengths of periodic orbits, without a broadening of the delta functions. We have assumed that the same holds for the quantum level density. Using this advantage we could clearly identify the edge contribution in the Fourier analysis of quantum level density.

4. Finally we can actually identify the wavefunctions corresponding to the bouncing-ball modes. In the analysis by Shudo and Shimizu,\(^{13}\) they concluded that one of the origins of the nonuniversal behavior came from a family of bouncing-ball orbits. In our analysis we can clearly relate the structures of energy levels to those of the bouncing-ball orbit scars in eigenfunctions.

In summary the semiclassical theory has been developed for billiards with energy-
dependent mixed boundary conditions. We have derived explicit expressions for the smooth and oscillatory parts of the level density for the billiard systems. It has been numerically shown that the semiclassical approximation correctly reproduces both the smooth and oscillatory parts of the quantum level density in the stadium billiard. The parametric dependence of the spectrum on the energy-dependent boundary condition provides us a very useful tool in identifying the bouncing-ball modes and thereby analyzing the semiclassical and quantum correspondence in billiard systems.

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Appendix A

Semiclassical Expressions for the Level Density

In this appendix we derive semiclassical expressions of the smooth and oscillatory parts of the level density for a rectangle billiard system with $k$-dependent mixed boundary conditions. The result will be generalized for a general billiard system. The $k$-dependent boundary condition turns out to give an important contribution to the smooth part of the level density which is proportional to the perimeter. For simplicity, we quantize an $L_x \times L_y$ rectangle, with Dirichlet boundary conditions on the edges of length $L_x$ and mixed boundary conditions on the edges of length $L_y$ (see Eq. (2.4)).

The quantization conditions are

$$k_{x,n}L_x + 2 \tan^{-1} \left( \frac{k_{x,n}}{\lambda} \right) = n\pi, \quad n = \pm 1, \pm 2, \cdots \quad (A.1)$$

and

$$k_{y,m}L_y = m\pi, \quad m = \pm 1, \pm 2, \cdots, \quad (A.2)$$

where $k^2 = k_x^2 + k_y^2$ and $(\pm n, \pm m)$ represent the same quantum state.

The level density for the rectangle can be written as

$$d(E) = \sum_{n,m=1}^{\infty} \delta(E - E_{nm})$$

$$= \frac{1}{4} \left[ \sum_{n,m=-\infty}^{\infty} \delta(E - E_{nm}) - \sum_{n=-\infty}^{\infty} \delta(E - E_{n0}) - \sum_{m=-\infty}^{\infty} \delta(E - E_{0m}) + \delta(E - E_{00}) \right], \quad (A.3)$$

where $E = k^2$, $E_{nm} = k_{nm}^2 = k_{x,n}^2 + k_{y,m}^2$, and we take advantage of the antisymmetry relations $k_{x,-n} = -k_{x,n}$, $k_{y,-m} = -k_{y,m}$. We apply the Poisson summation to the
first term in (A·3), using the natural continuations of (A·1) and (A·2) to real \(m, n\):
\[
\sum_{n,m=-\infty}^{\infty} \delta(E - E_{nm}) = \sum_{n,m=-\infty}^{\infty} \delta(E - k_{x,n}^2 - k_{y,m}^2)
\]
\[
= \sum_{p,q=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dn \, dm \, \delta \left[ E - k_{x}^2(n) - k_{y}^2(m) \right] e^{2\pi i (pn+qm)}
\]
\[
= \sum_{p,q=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_{x} \, dk_{y} \left| \frac{\partial(n,m)}{\partial(k_{x},k_{y})} \right| \delta \left[ E - k_{x}^2 - k_{y}^2 \right] e^{2\pi i (pn+qm)}
\]
\[
= \frac{1}{2} \sum_{p,q=-\infty}^{\infty} \int_{0}^{2\pi} d\theta \frac{\partial n}{\partial k_{x}} \frac{\partial m}{\partial k_{y}} e^{2\pi i [pn(k \cos \theta) + qm(k \sin \theta)]}. \quad \text{(A·4)}
\]

The first change of integration variables \((m, n) \rightarrow (k_{x}, k_{y})\) is allowed because of the monotonically increasing functions \(n(k_{x}), m(k_{y})\). The second change of variables is just the transformation to polar coordinates \((k, \theta)\) so that the \(k\) integration can be explicitly performed to eliminate the \(\delta\) term. Similar considerations when Poisson summation is applied to the other terms of (A·3) lead to
\[
d(E) = \frac{1}{8} \sum_{p,q=-\infty}^{\infty} \int_{0}^{2\pi} d\theta \left( \frac{\partial n}{\partial k_{x}} - \delta_{p,0} \delta(k_{x}) \right) \left( \frac{\partial m}{\partial k_{y}} - \delta_{q,0} \delta(k_{y}) \right) e^{2\pi i [pn(k \cos \theta) + qm(k \sin \theta)]}. \quad \text{(A·5)}
\]

For \(p = q = 0\), Eq. (A·5) gives the smooth two dimensional level density
\[
\tilde{d}(E) = \frac{1}{8} \int_{0}^{2\pi} d\theta \left( \frac{\partial n}{\partial k_{x}} - \delta(k_{x}) \right) \left( \frac{\partial m}{\partial k_{y}} - \delta(k_{y}) \right). \quad \text{(A·6)}
\]

In order to calculate the partial derivatives \(\frac{\partial n}{\partial k_{x}}\) and \(\frac{\partial m}{\partial k_{y}}\), we take derivatives with respect to \(k_{x}\) and \(k_{y}\) in Eqs. (A·1) and (A·2):
\[
L_{x} + 2 \left( \frac{1}{\lambda} - \frac{k_{x}}{\lambda^{2}} \frac{\partial \lambda}{\partial k_{x}} \right) \frac{1}{1 + \left( \frac{k_{x}}{\lambda} \right)^{2}} = \frac{\partial n}{\partial k_{x}} \pi \quad \text{(A·7)}
\]
and
\[
L_{y} = \frac{\partial m}{\partial k_{y}} \pi. \quad \text{(A·8)}
\]

Using (A·7) and (A·8), we obtain
\[
\tilde{d}(E) = \frac{1}{8} \int_{0}^{2\pi} d\theta \left[ \frac{L_{x}}{\pi} + 2 \left( \frac{1}{\lambda} - \frac{k_{x}}{\lambda^{2}} \frac{\partial \lambda}{\partial k_{x}} \right) \frac{1}{1 + \left( \frac{k_{x}}{\lambda} \right)^{2}} - \delta(k_{x}) \right] \left[ \frac{L_{y}}{\pi} - \delta(k_{y}) \right]
\]
\[
= \frac{L_{x} L_{y}}{4\pi k} - \frac{L_{x}}{4\pi k} - \frac{L_{y}}{4\pi k} - \frac{1}{2\pi k^{2}} \left( \frac{k}{\lambda} \frac{\partial \lambda}{\partial k_{x}} \right) \left( \frac{\lambda}{k} \right) \frac{1}{1 + \left( \frac{k_{x}}{\lambda} \right)^{2}} + \frac{1}{8} \delta(k)
\]
\[
+ \frac{L_{y}}{4\pi} \int_{0}^{2\pi} d\theta \left( \frac{1}{\lambda} - \frac{k_{x}}{\lambda^{2}} \frac{\partial \lambda}{\partial k_{x}} \right) \frac{1}{1 + \left( \frac{k_{x}}{\lambda} \right)^{2}}. \quad \text{(A·9)}
\]
where the last integration cannot be carried out without knowing the \( k_x \)-dependence of \( \lambda \).

Here we assume that the \( k_x \)-dependence of \( \lambda \) is through \( k \). Then the integration can be carried out, and we obtain

\[
\tilde{d}(E) = \frac{L_x L_y}{4\pi} - \frac{L_x}{4\pi k} - \frac{L_y}{4\pi k} - \frac{1}{2\pi k^2} \left( 1 - \frac{k d\lambda}{\lambda d\lambda} \right) \frac{1}{1 + \left( \frac{\lambda}{k} \right)^2} + \frac{1}{8} \delta(k)
\]

\[
+ \frac{L_y}{2\pi k} \left[ \frac{1}{\sqrt{1 + \left( \frac{\lambda}{k} \right)^2}} + \left( \frac{\lambda}{k} \right)^2 \frac{d\lambda}{dk} \right]. \tag{A.10}
\]

From this result, we can readily find for the general billiard system,

\[
\tilde{d}(E) = \frac{A}{4\pi} - \frac{L_D}{8\pi k} + \frac{L_N}{8\pi k} - \frac{L_M}{8\pi k} \left[ 1 - \frac{2}{\sqrt{1 + \left( \frac{\lambda}{k} \right)^2}} - 2 \left( \frac{\lambda}{k} \right)^2 \frac{d\lambda}{dk} \right] + \cdots, \tag{A.11}
\]

if we assume that in general the smooth part of the level density is expanded in terms of geometrical measures of billiards. Here \( A \) represents the area of the billiard system. In the perimeter terms \( L_D, L_N \) and \( L_M \) represent the perimeters with Dirichlet, Neumann and mixed-boundary conditions, respectively. The neglected higher order terms represent corner contributions.

To derive the oscillatory part of \( d(E) \), as usual the saddle point approximation is made to evaluate the oscillatory integrals \( ((p, q) \neq (0, 0)) \) in (A.5). The phase appearing in these integrals can be divided into two parts as

\[
\varphi_{pq}^{\theta} = 2\pi \left[ pm(k\cos \theta) + qm(k\sin \theta) \right]
\]

\[
= 2 \left[ pkL_x \cos \theta + 2p\tan^{-1} \left( \frac{k\cos \theta}{\lambda} \right) + qkL_y \sin \theta \right]
\]

\[
= \varphi_{pq}^{D} + 4p\tan^{-1} \left( \frac{k\cos \theta}{\lambda} \right) = \varphi_{pq}^{D} + \varphi_{pq}^{N}, \tag{A.12}
\]

where \( \varphi_{pq}^{D} \) is the phase that appears in the pure Dirichlet case and is rapidly oscillating in \( \theta \) for large \( k \). The phase \( \varphi_{pq}^{N} \) is both bounded and slowly oscillating and thus can be absorbed in the slowly varying factor in front. Thus, the saddle points are the same as in the pure Dirichlet case:

\[
\tan \theta_{pq} = \frac{pL_y}{qL_x}. \tag{A.13}
\]

Putting the above results into Eq. (A.5), to leading order we obtain the oscillator part of the semiclassical approximation to \( d(E) \),

\[
\tilde{d}(E) = \frac{A}{4\pi} \sum_{p,q \neq (0,0)} \sqrt{\frac{2}{\pi k L_{pq}}} \cos \left[ kL_{pq} + 4|p|\tan^{-1} \left( \frac{k\cos \theta_{pq}}{\lambda} \right) - \frac{\pi}{4} \right], \tag{A.14}
\]
where \( L_{pq} = 2\sqrt{(pL_x)^2 + (qL_y)^2} \) is the length of the \((p, q)\) periodic orbit and \( \cos \theta_{pq} = 2|p|L_z/L_{pq} \). One can conclude that the effect of the mixed boundary conditions on the oscillatory part is simply to modify the contribution of each periodic orbit by a phase, which is \( 2\tan^{-1}\left[\frac{k}{\sqrt{\cos B_{pq}\cos \theta}}\right] \) for each bounce from a wall with mixed boundary conditions. In particular for bouncing-ball orbits of the stadium billiard, since \( \ell_{bb} = L_{10} = 2L_x \), the contribution in Eq. (A·14) results in

\[
\tilde{d}(E) = \frac{A}{4\pi} \sum_{p \neq 0} \sqrt{\frac{2}{\pi k p \ell_{bb}}} \cos \left[ k |p| \ell_{bb} - \frac{\pi}{4} \right] = \frac{1}{\pi \sqrt{2\pi k}} \sum_{j=1}^{\infty} \frac{A}{\sqrt{j} \ell_{bb}} \cos \left[ kj \ell_{bb} - \frac{\pi}{4} \right].
\]

(A·15)

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**Appendix B**

**Numerical Solutions of Eigenmodes**

In this appendix our numerical method of obtaining the energy spectra is described for the stadium billiard. Here we consider only the quarter part of the billiard system because of reflection symmetries with respect to \( x \) and \( y \) axes. In the following we consider odd-odd type wavefunctions for simplicity.

We take the curvilinear abscissa \( s \) along the billiard perimeter as shown in Fig. 12. The relation \( F(x(s), y(s)) = 0 \) defines the boundary conditions along the circumference. On the straight section we have \( F(x, y) = \Psi(x, y) \) and \( F(x, y) = k \cos \alpha \Psi(x, y) + \sin \alpha \frac{\partial}{\partial r} \Psi(x, y) \) on the quarter of the circle. Assuming that \( F(x(s), y(s)) \) is a function of \( s \) on the boundary, the boundary conditions are that the Fourier coefficients of the periodic function should vanish: \(29\)

\[
\int_0^\ell ds \ F(x(s), y(s)) \sin(k Ms)
\]

![Fig. 12. Quarter stadium billiard system. The curvilinear abscissa \( s \) is taken along the billiard perimeter.](https://academic.oup.com/ptp/article-abstract/98/4/869/1907083)
\[= \int_0^\ell dx \ F(x, r) \sin (k_M x) + \int_0^\ell ds \ F(x(s), y(s)) \sin (k_M s) = 0, \quad (B\cdot1)\]

where \( \ell = a + r \frac{\pi}{2} \) is the perimeter, \( k_M = \frac{\pi M}{\ell} \), and \( M \) is any Fourier mode which we consider.

A wavefunction is written as a linear combination of odd-odd type basis states:
\[
\Psi(x, y) = \sum L a_L \sin k_x^L x \sin k_y^L y. \quad (B\cdot2)
\]

Thus on the straight section one has
\[
\int_0^\ell dx \ \Psi(x, r) \sin (k_M x) = \sum L a_L B_{LM}, \quad (B\cdot3)
\]

where \( B_{LM} \) is given by
\[
B_{LM} = \frac{1}{2} \left[ \frac{\sin ((k_x^L - k_M) a)}{k_x^L - k_M} - \frac{\sin ((k_x^L + k_M) a)}{k_x^L + k_M} \right] \sin k_y^L r. \quad (B\cdot4)
\]

On the quarter of the circular part, putting \( s = a + r \theta \) and \( ds = rd\theta \), we have
\[
\int_0^\ell ds \ \left[ k \cos \alpha \ \Psi(x, y) + \sin \alpha \frac{\partial}{\partial r} \Psi(x, y) \right] \sin (k_M s) = \sum L a_L C_{LM}, \quad (B\cdot5)
\]

where \( C_{LM} \) is given by
\[
C_{LM} = r \int_0^{\frac{\pi}{2}} d\theta \left[ k \cos \alpha \left\{ \sin (k_x^L(a + r \sin \theta)) \sin (k_y^L r \cos \theta) \right\} \\
+ \sin \alpha \left\{ k_x^L \sin \theta \cos (k_x^L(a + r \sin \theta)) \sin (k_y^L r \cos \theta) \right\} \\
+ k_y^L \cos \theta \sin (k_x^L(a + r \sin \theta)) \cos (k_y^L r \cos \theta) \right\} \sin (k_M(a + r \theta)). \quad (B\cdot6)
\]

Putting the contributions together, we have \( \sum_L a_L A_{LM} = 0 \), where \( A_{LM} = B_{LM} + C_{LM} \). Therefore by looking for the roots of determinant \( A_{LM} \) we can obtain the eigenmodes.

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**Appendix C**

**Fourier Transformation of the Level Density**


In this appendix we demonstrate why the energy-dependent boundary condition is superior to the energy-independent boundary condition in obtaining the periods of classical orbits through Fourier transformation of the quantum level density. In the semiclassical theory the oscillatory part of the level density is given as (see Eq. (4\cdot2)),
\[
\tilde{d}_{iso}(E, \lambda) = \frac{1}{2\pi k} \sum_{\gamma} \sum_{j=1}^\infty a_\gamma^{(j)} \cos \left[ j \left( k \ell_\gamma - \frac{\pi}{2} \beta_\gamma + \phi_\gamma(\lambda) \right) \right], \quad (C\cdot1)
\]
which can be rewritten in terms of the level density as a function of wave number $k$:

$$\tilde{d}_{iso}(k, \alpha) = 2k \tilde{d}_{iso}(E, \alpha) = \frac{1}{\pi} \sum_{\gamma} \sum_{j=1}^{\infty} a_{\gamma}^{(j)} \cos \left[ j \left( k \ell_{\gamma} - \frac{\pi}{2} \beta_{\gamma} + \phi_{\gamma}(\lambda) \right) \right]. \quad (C.2)$$

Thus the Fourier transformation of the level density is given as

$$f(\ell, \alpha) \equiv \int_{-\infty}^{\infty} dk \ d(k, \alpha) \ \cos k\ell$$

$$= \frac{1}{\pi} \sum_{\gamma} \sum_{j=1}^{\infty} a_{\gamma}^{(j)} \int_{-\infty}^{\infty} dk \ \cos \left[ j \left( k \ell_{\gamma} - \frac{\pi}{2} \beta_{\gamma} + \phi_{\gamma}(\lambda) \right) \right] \cos k\ell$$

$$= \frac{1}{2\pi} \sum_{\gamma} \sum_{j=1}^{\infty} a_{\gamma}^{(j)} \cos \left[ j \left( \frac{\pi}{2} \beta_{\gamma} - \phi_{\gamma}(\lambda) \right) \right] \delta(\ell - j \ell_{\gamma}), \quad (C.3)$$

which shows that there are peaks at the lengths of the periodic orbits.

It should be noted that in deriving Eq. (C.3) from (C.2) we have assumed that the additional phase $\phi_{\gamma}(\lambda)$ does not depend on the wave number $k$. This is assured only if we use energy-dependent mixed boundary conditions whose Dirichlet part is proportional to the wave number $k$ (see Eq. (4.4)).

References

23) P. M. Morse and H. Feshbach, Methods of Theoretical Physics, Part II (McGraw-Hill, New York, 1953), Chap. 9.2.