Dromion and Lump Solutions of the Ishimori-I Equation

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It is known that the Ishimori-I equation, which is the (2+1)-dimensional generalization of the classical continuous Heisenberg ferromagnet equation, has various localized solutions such as the dromion, lump and rationally-exponentially localized solutions. In this paper localized solutions of the Ishimori-I equation are constructed explicitly in terms of grammian determinants by using the binary Darboux transformation, and it is shown that they include not only the multi-soliton (dromion, lump and so on) solutions but also new localized solutions.

§1. Introduction

The (2+1)-dimensional generalization of the classical continuous Heisenberg ferromagnet equation has been given by Ishimori as follows: ¹)

\[
\vec{S}_t = -\frac{a}{2} \vec{S} \times (\vec{S}_{yy} - \varepsilon^2 \vec{S}_{xx}) + \phi_x \vec{S}_y + \phi_y \vec{S}_x,
\]

\[
\phi_{yy} + \varepsilon^2 \phi_{xx} = \varepsilon^2 a \left( \vec{S} \cdot (\vec{S}_x \times \vec{S}_y) \right),
\]

where \( \vec{S} = (S_1(x, y, t), S_2(x, y, t), S_3(x, y, t)) \), \( \phi = \phi(x, y, t) \) and the constant \( a \) are real. \( \left| \vec{S} \right|^2 = S_1^2 + S_2^2 + S_3^2 = 1 \), and the subscripts \( x, y \) and \( t \) denote partial differentiation. Equation (1) is referred to as the Ishimori-I (Ish-I) equation for \( \varepsilon = -i \) and as the Ishimori-II (Ish-II) equation for \( \varepsilon = +1 \). For the Ish-I equation, various solutions decaying in all directions have been obtained by means of the inverse scattering method. There exist an exponentially localized solution, which is similar to the dromion solution of the Davey-Stewartson (DS)-I equation, and polynomially and polynomially-exponentially localized solutions, which are absent in the DS-I equation, in Ref. 2).

In this paper exact solutions of the Ish-I equation are constructed in terms of grammian determinants by using the binary Darboux transformation (§2) and it is shown that they contain multi-soliton solutions of the above localized solutions, which correspond to the multi-dromion solutions of the DS-I equation (§§3 and 4). In addition we study other localized solutions (§5).

The binary Darboux transformation was first introduced for the KP and DS equation ⁴) and applied to construction of the dromion solutions. ⁶) The binary Darboux transformation for the Ish-I equation was introduced in Ref. 7).

In this connection, we will touch on the gauge equivalence of the DS equation and the Ish equation. The gauge transformation from the Ish equation to the DS

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equation has been constructed and the gauge equivalence of the DS-II equation and
the Ish-II equation has been shown. However, the gauge equivalence of the DS-I
equation and the Ish-I equation has not been shown. Therefore showing existence
of solutions of the Ish-I equation corresponding to solutions of the DS-I equation is
not trivial.

§2. The binary Darboux transformation and exact solutions of the
Ishimori-I equation

We introduce an auxiliary linear system for a $2 \times 2$ matrix valued $\Psi$:

$$
\begin{aligned}
\psi_y &= U \psi_x, \\
\psi_t &= V_2 \psi_{xx} + V_1 \psi_x,
\end{aligned}
$$

where

$$
\begin{aligned}
U &= i \varepsilon \begin{pmatrix}
S_3 & S_1 + i S_2 \\
S_1 - i S_2 & -S_3
\end{pmatrix}, \\
V_2 &= \varepsilon a U, \\
V_1 &= a \left( -\frac{1}{4 \varepsilon^2} [U, U_y] + \frac{1}{2} U_x \right) + \phi_x U + \phi_y I,
\end{aligned}
$$

$I$ is the $2 \times 2$ identity matrix, and $[U, U_y] = U U_y - U_y U$. The integrability condition
for Eq. (2) yields the Ishimori equation (1). We consider only the case $\varepsilon = -i$, or
the Ish-I equation in this paper.

Equation (2) has been shown to be covariant with respect to the binary Darboux
transformation (BDT2) $\Psi \rightarrow \Psi^{(1)}$ defined as

$$
\begin{aligned}
\Psi^{(1)} &= \Psi - \Gamma \partial^{-1} (\Psi_1^* \Psi_x), \\
\Gamma &= \Psi_1 A^{-1}, \\
A &= \partial^{-1} (\Psi_1^* \Psi_1^{-1}),
\end{aligned}
$$

where $\Psi$ and $\Psi_1$ are solution matrices of Eq. (2). The symbol $*$ denotes the Hermitian
conjugate. The integral operator $\partial^{-1}$ is defined as the following linear integration:

$$
\begin{aligned}
\partial^{-1} (\Psi_1^* \Psi_x) &= \int_{(x_0, y_0, t_0)}^{(x, y, t)} L(\Psi_1, \Psi) dx + \frac{1}{2} \Psi_1^* \Psi_1 \big|_{x_0, y_0, t_0} + C, \\
L(\Psi_1, \Psi) &= \Psi_1^* \Psi_x dx + \Psi_1^* \Psi_y dy + (\Psi_1^* \Psi_t - \Psi_1^* \Psi_{1, x}) \Psi_2 \Psi_x dt.
\end{aligned}
$$

Since $d(L(\Psi_1, \Psi)) = 0$ as can be easily shown, Eq. (4) is independent of integral
paths and depends on the initial point $(x_0, y_0, t_0)$ and the final point $(x, y, t)$. $C$
is an arbitrary constant matrix which satisfies the constraint

$$
C + C^* = 0.
$$

The transformed coefficients $U^{(1)}$, $V_2^{(1)}$ and $V_1^{(1)}$ are given as

$$
\begin{aligned}
U^{(1)} &= \Upsilon U Y^{-1}, \\
V_2^{(1)} &= \Upsilon V_2 Y^{-1}, \\
V_1^{(1)} &= \Upsilon \left( V_1 - [V_2, Y^{-1} \Psi_1^*] - 2 V_2 Y^{-1} \Upsilon \right) Y^{-1},
\end{aligned}
$$

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where

\[ \gamma = I - \Gamma \Psi_1 = I - \Psi_1 A^{-1} \Psi_1^* . \]  

(7)

Next, the BDT2 is extended to the \( M \)-times iterated binary Darboux transformation (\( M \)-BDT2). The \( M \)-BDT2 \( \Psi \rightarrow \Psi^{(M)} \) is represented as

\[
\Psi^{(M)} = \Psi - \Gamma_M \partial^{-1}(\hat{\Psi}^* \hat{\Psi}), \\
\Gamma_M = \hat{\Psi} A_M^{-1}, \\
A_M = \partial^{-1}(\hat{\Psi}^* \hat{\Psi}), \]

(8)

where \( \hat{\Psi} \) is given as a \( 2 \times 2M \) matrix consisting of \( M \) solution matrices of Eq. (2), \( \Psi_j (j = 1, 2, \cdots, M) \):

\[
\hat{\Psi} = \left( \begin{array}{c} \Psi_1, \Psi_2, \cdots, \Psi_M \end{array} \right). 
\]

(9)

A \( 2M \times 2M \) matrix \( A_M \) is defined as

\[
A_M = \partial^{-1}(\hat{\Psi}^* \hat{\Psi}) = \int_{(X_0, Y_0, T_0)} L(\hat{\Psi}, \hat{\Psi}) + \frac{1}{2} \hat{\Psi}^* \hat{\Psi} |_{(X_0, Y_0, T_0)} + C_M, 
\]

(10)

where \( C_M \) is a \( 2M \times 2M \) arbitrary constant matrix, which satisfies the constraint

\[ C_M + C_M^* = 0. \]

(11)

As can be seen from the similarity between Eqs. (3) and (8), transformed coefficients \( U^{(M)}, V_2^{(M)} \) and \( V_1^{(M)} \) are written in an expression similar to Eqs. (6) and (7).

We start with the trivial solution of the Ish-I equation

\[ \bar{S} = (0, 0, 1), \quad \phi = \frac{1}{2} \left( \phi_1(\xi, t) + \phi_2(\eta, t) \right), \]

(12)

where \( \xi = x + y, \eta = x - y; \phi_1(\xi, t) \) and \( \phi_2(\eta, t) \) are arbitrary functions. Then the transformed variables \( \bar{S}^{(M)} = (S_1^{(M)}, S_2^{(M)}, S_3^{(M)}) \) and \( \phi^{(M)} \) are obtained from Eqs. (6) and (7) in the form of

\[
S_1^{(M)} + i S_2^{(M)} = \frac{-2fg}{|f|^2 + |g|^2}, \quad S_3^{(M)} = \frac{|f|^2 - |g|^2}{|f|^2 + |g|^2}, 
\]

\[
\phi^{(M)} = \phi + ia \ln \left| \frac{|f|^2 + |g|^2}{D^2} \right|, 
\]

where

\[
f = \begin{vmatrix} A_M & F_1^* \\ F_1 & 1 \end{vmatrix}, \quad g = \begin{vmatrix} A_M & F_2^* \\ F_1 & 0 \end{vmatrix}, \quad D = |A_M|. 
\]

(13)

(14)

(15)

The quantities \( F_1 \) and \( F_2 \) are defined as the first row and the second row of the \( 2 \times 2M \) matrix \( \overline{\Psi} \) respectively. Equation (14) can be rewritten as

\[
\phi^{(M)} = \phi + ia \ln \left| \frac{D^2}{D} \right| = \phi + 2a \arctan \left( \frac{\Im(D)}{\Re(D)} \right) 
\]

(16)
by using the following relation among $f$, $g$ and $D$:
\[ |f|^2 + |g|^2 = |D|^2, \]
where $\Re$ and $\Im$ denote the real part and the imaginary part.

Let the elements of the solution matrices $\Psi_l$ ($l = 1, 2, \cdots, M$) be given by
\[ \Psi_l = \begin{pmatrix} \psi_{2l-1} & \psi_{2l} \\ \varphi_{2l-1} & \varphi_{2l} \end{pmatrix}. \]

Then the $\psi_j$ and $\varphi_j$ ($j = 1, 2, 3, 4, \cdots, 2M-1, 2M$) constitute $2M$ solutions of
the following linear partial differential equations:
\begin{align*}
\psi_j, t &= -i\alpha \psi_j, \xi \xi + \{\phi_1(\xi, t)\}_\xi \psi_j, \xi, \quad \psi_j = \psi_j(\xi, t), \quad (17) \\
\varphi_j, t &= +i\alpha \varphi_j, \eta \eta - \{\phi_2(\eta, t)\}_\eta \varphi_j, \eta, \quad \varphi_j = \varphi_j(\eta, t). \quad (18)
\end{align*}
Moreover, $F_1$, $F_2$, $\Lambda_M$ can be written from Eqs. (9)~(11) as
\begin{align*}
F_1 &= (\psi_1, \psi_2, \psi_3, \psi_4, \cdots, \psi_{2M-1}, \psi_{2M}), \\
F_2 &= (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \cdots, \varphi_{2M-1}, \varphi_{2M}), \\
\Lambda_M &= (\lambda_{jk}), \quad (19)
\end{align*}
where
\begin{align*}
\lambda_{jk} &= \lambda_{jk}[\psi] + \lambda_{jk}[\varphi] + c_{jk}, \quad (20) \\
\lambda_{jk}[\psi] &\equiv \int_{(\xi_0, t_0)}^{(\xi, t)} \left\{ \psi_x^* \psi_k, \xi d\xi + \left( \psi_x^* \psi_k, t - i\alpha \psi_x, \xi \psi_k, \xi \right) dt \right\} + \frac{1}{2} \psi_x^* \psi_k \bigg|_{(\xi_0, t_0)} \quad (21) \\
\lambda_{jk}[\varphi] &\equiv \int_{(\eta_0, t_0)}^{(\eta, t)} \left\{ \varphi_x^* \varphi_k, \eta d\eta + \left( \varphi_x^* \varphi_k, t + i\alpha \varphi_x, \eta \varphi_k, \eta \right) dt \right\} + \frac{1}{2} \varphi_x^* \varphi_k \bigg|_{(\eta_0, t_0)} \quad (22)
\end{align*}
\((j, k = 1, 2, 3, 4, \cdots, 2M-1, 2M)\). The constants $c_{jk}$, which are defined as elements of $C_M$, are arbitrary complex constant numbers which satisfy the constraints
\[ c_{jk} + c_{kj}^* = 0. \quad (23) \]
Therefore substituting $\psi_j$, $\varphi_j$, $\lambda_{jk}[\psi]$ and $\lambda_{jk}[\varphi]$, which are obtained from Eqs. (17), (18), (21) and (22), into Eqs. (13), (15), (16), (19) and (20) leads to exact solutions of the Ish-I equation, $\mathcal{S}(M)$ and $\phi(M)$, in terms of grammian determinants.

**§3. Exponentially or rationally localized solutions**

Hereafter we consider $\phi_1(\xi, t) = \phi_2(\eta, t) = 0$. Then the linear equations (17) and (18) are reduced to
\begin{align*}
\psi_j, t &= -i\alpha \psi_j, \xi \xi, \quad \psi_j, t = +i\alpha \varphi_j, \eta. \quad (24)
\end{align*}
Exponential solutions of Eq. (24) are obtained as follows:
\begin{align*}
\psi_j &= A_j \exp(\theta_j(\xi, t)), \quad (25) \\
\varphi_j &= B_j \exp(\theta_j(\eta, t)) \quad (26)
\end{align*}
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\[ (j = 1, 2, \cdots, 2M - 1, 2M), \]

where

\[ \theta_j(\xi, t) = p_j \xi - i a p_j^2 t, \quad \vartheta_j(\eta, t) = q_j \eta + i a q_j^2 t, \]

and \( A_j, B_j, p_j, q_j \) are arbitrary complex constants. Substituting Eqs. (25) and (26) into Eqs. (21) and (22), respectively, and integrating these equations leads to the following explicit form for \( \lambda_{jk}[\psi] \) and \( \lambda_{jk}[\varphi] \): if \( p_j^* + p_k \neq 0, q_j^* + q_k \neq 0 \), then

\[ \lambda_{jk}[\psi] = \frac{A_j^* A_k p_k}{p_j^* + p_k} \exp(\theta_j(\xi, t)^* + \theta_k(\xi, t)), \quad (27) \]

\[ \lambda_{jk}[\varphi] = \frac{B_j^* B_k q_k}{q_j^* + q_k} \exp(\vartheta_j(\eta, t)^* + \vartheta_k(\eta, t)), \quad (28) \]

and if \( p_j^* + p_k = 0, p_j^* + p_k = 0 \), then

\[ \lambda_{jk}[\psi] = A_j^* A_k \left( p_k \xi - 2 i a p_k^2 + \frac{1}{2} \right), \quad (29) \]

\[ \lambda_{jk}[\varphi] = B_j^* B_k \left( q_k \eta + 2 i a q_k^2 + \frac{1}{2} \right). \quad (30) \]

Hence exponential and rational solutions of the Ish-I equation are constructed from Eqs. (25)~(30). In particular, we can obtain solutions of the Ish-I equation which decay exponentially or rationally in all directions in the \( \xi \eta \) plane (or \( \mathcal{S} \rightarrow (0, 0, 1) \) as \( \sqrt{\xi^2 + \eta^2} \rightarrow \infty \)), under the following choice:

\[ \varphi_j = 0, \quad (1 \leq j \leq J) \]

\[ \psi_j = 0, \quad (J + 1 \leq j \leq J + K = 2M) \quad (31) \]

where \( J \) and \( K \) are positive integers. The choice (31) was first introduced by Nimmo in his study of the DS-I equation. 6)

§4. Dromion, lump and exponentially-rationally localized solutions

In this section it is shown that the exponential and rational solutions constructed in §3 include three types of localized solutions obtained by the inverse scattering method 2) as the simplest case. This implies that the solutions in §3 are the multi-soliton solutions for these three localized solutions. We consider the case that \( M = 1, J = K = 1 \), that is,

\[ \psi_1 = A \exp \theta, \quad \psi_2 = 0, \quad \varphi_1 = 0, \quad \varphi_2 = B \exp \vartheta, \]

where \( \theta = p \xi - i a \xi^2 t, \vartheta = q \eta + i a \eta^2 t \), and \( A, B, p \) and \( q \) are arbitrary complex constants.

4.1. Dromion solution

If \( \Re(p) \) and \( \Re(q) \) are non-zero, \( \lambda_{jk}[\psi] \) and \( \lambda_{jk}[\varphi] \) are obtained as Eqs. (27) and (28). Therefore we obtain the solution decaying exponentially in all directions 2) by...
substituting the following functions $f$, $g$ and $D$ into Eqs. (13) and (16):

$$f = \exp R(\theta + \vartheta) + i\alpha p \exp R(-\theta + \vartheta) - i\beta q^* \exp R(\theta - \vartheta) + \gamma pq^* \exp R(-\theta - \vartheta),$$

$$g = 2\sqrt{R(p)R(q)(\alpha \beta - \gamma)} \exp\{i\Im(\theta - \vartheta) - i\delta\},$$

$$D = \exp R(\theta + \vartheta) - i\alpha p^* \exp R(-\theta + \vartheta) - i\beta q^* \exp R(\theta - \vartheta) - \gamma pq^* \exp R(-\theta - \vartheta),$$

where

$$\alpha = 2iR(p)c_{11}/|pA|^2, \quad \beta = 2iR(q)c_{22}/|qB|^2, \quad \gamma = \alpha\beta - 4R(p)R(q)|c_{12}|^2/|pqAB|^2, \quad \delta = \arg(p^*qAB^*c_{12}).$$

This implies that $\alpha$, $\beta$, $\gamma$, $\delta$ are arbitrary real constants which satisfy the relation

$$R(p)R(q)(\alpha\beta - \gamma) \geq 0.$$  \hspace{1cm} (32)

In this paper the solution (32) is called the ‘dromion solution’ for the localized solution of the DS-I equation.

This solution describes a dromion moving with constant velocity $(v_x, v_y) = -2a(\Im(p), \Im(q))$. When the dromion is stable (or $\Im(p) = \Im(q) = 0$), $S_3^{(1)}$ has a unique minimum, as shown in Fig. 1. This is similar to the dromion solution in the DS-I equation. For the Ish-I equation, there exist the following relations between this minimum value and the form of the auxiliary field $\phi$:

<table>
<thead>
<tr>
<th>$\frac{\alpha\beta pq &gt; 0, \gamma pq &lt; 0}{\text{or} \frac{\alpha\beta pq &lt; 0, \gamma pq &lt; 0}{\text{or} \frac{\alpha\beta pq &gt; 0, \gamma pq &gt; 0}{} } }$</th>
<th>$\min{S_3^{(1)}}$</th>
<th>$\phi$</th>
<th>$\min{S_3^{(1)}}$</th>
<th>$\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>Fig. 2</td>
<td>$\frac{</td>
<td>\alpha\beta pq</td>
<td>^2}{\gamma pq^2 + 1}$</td>
</tr>
</tbody>
</table>

The relation between phase shifts in the plane waves of $\phi$ and the location of the dromion is also shown to be similar to that in the DS-I equation. 5) However, when

![Fig. 1. (a) 3D-plot of $S_3^{(1)}$ when a dromion is stable, (b) contour lines of $S_3^{(1)}$; $p = 0.8$, $q = -1$, $\alpha = 0.8$, $\beta = 0.6$, $\gamma = 3.68$, $a = -2$ and $t = 0$.](attachment:image.png)
the dromion moves (or $\exists(p), \exists(q) \neq 0$), it has features which are not observed in the DS-I equation: Figure 4 shows that the dromion has an arched shape and Fig. 5 shows that it has two minimal values.

In the case $M > 1$ and Eq. (31), if $p_j^* + p_k \neq 0, q_j^* + q_k \neq 0$ for all $j$ and $k$, then we can obtain what is called the $(J, K)$ dromion solution, which describes the interactions of $J \times K$ dromions.

4.2. Lump solution

If $\Re(p) = \Re(q) = 0$, that is, $p = i\mu$ and $q = i\nu$, where $\mu$ and $\nu$ are real, then $\lambda_{jk}[\psi]$ and $\lambda_{jk}[\varphi]$ are obtained as Eqs. (29) and (30). Therefore we obtain the rationally localized solution by substituting the functions

$$f = \left(\dot{\psi} + \frac{i}{2\mu}\right)\left(\dot{\varphi} - \frac{i}{2\nu}\right) - \frac{\gamma^2}{\mu\nu},$$
\[ g = -\frac{\gamma}{\mu \nu} \exp(i\Theta), \]
\[ D = \left( \hat{\xi} - \frac{i}{2\mu} \right) \left( \hat{\eta} - \frac{i}{2\nu} \right) - \frac{\gamma^2}{\mu \nu} \quad (33) \]

into Eqs. (13) and (16), where \( \hat{\xi} = \xi + 2\alpha t + \alpha, \hat{\eta} = \eta - 2\alpha t + \beta, \Theta = \mu \xi - \nu \eta + a(\mu^2 - \nu^2)t + \delta, \) and \( \alpha, \beta, \gamma \) and \( \delta \) are real. The solution (33) is called the 'lump solution' in this paper.

In the case \( M > 1 \) and Eq. (31), if \( \Re(p_j) = 0, \Re(q_j) = 0 \) for all \( j \), then we can obtain what is called the \((J, K)\) lump solution, which describes the interactions of \( J \times K \) lumps.

### 4.3. Exponentially-rationally localized solution

If \( \Re(p) = 0 \) and \( \Re(q) \neq 0 \), that is, \( p = i\mu + i\nu, q = \sigma + i\nu, \) where \( \mu, \sigma \) and \( \nu \) are real, then we obtain the mixed polynomial-exponential decreasing solution. This solution is obtained by substituting the functions

\[
\begin{align*}
\psi_j &= \sum_{m=1}^{N_j} A_{jm} \exp(\theta_{jm}(\xi, t)), \\
\varphi_j &= \sum_{m=1}^{N_j} B_{jm} \exp(\vartheta_{jm}(\eta, t)),
\end{align*}
\quad (35, 36)
\]

\( (j = 1, 2, \ldots, 2M - 1, 2M), \) where

\[
\begin{align*}
\theta_{jm}(\xi, t) &= p_{jm}(\xi - i\alpha p_{jm}^2 t), \\
\vartheta_{jm}(\eta, t) &= q_{jm}(\eta + i\alpha q_{jm}^2 t),
\end{align*}
\]

### §5. Other localized solutions

In this section, new localized solutions of the Ish-I equation are constructed by using other solutions of the linear equation (24).

#### 5.1. Localized solutions constructed from the superposition of the exponential functions

Because Eq. (24) is linear, other solutions of Eq. (24) can be obtained from the superposition of the exponential solutions (25) and (26):

\[
\begin{align*}
\psi_j &= \sum_{m=1}^{N_j} A_{jm} \exp(\theta_{jm}(\xi, t)), \\
\varphi_j &= \sum_{m=1}^{N_j} B_{jm} \exp(\vartheta_{jm}(\eta, t)),\quad (j = 1, 2, \ldots, 2M - 1, 2M),
\end{align*}
\]

where

\[
\begin{align*}
\theta_{jm}(\xi, t) &= p_{jm}(\xi - i\alpha p_{jm}^2 t), \\
\vartheta_{jm}(\eta, t) &= q_{jm}(\eta + i\alpha q_{jm}^2 t),
\end{align*}
\]
\(N_j\) and \(N'_j\) are non-negative integers, and \(A_{jm}, B_{jm}, p_{jm}\) and \(q_{jm}\) are arbitrary complex constants. \(\lambda_{jk}[\psi]\) and \(\lambda_{jk}[\varphi]\) can be easily obtained in the same way as in §3. For example, if \(p_{jm}^* + p_{kl} \neq 0\) and \(q_{jm}^* + q_{kl} \neq 0\) for all \(m\) and \(l\), then

\[
\lambda_{jk}[\psi] = \sum_{m=1}^{N_j} \sum_{l=1}^{N_k} \frac{A_{jm} A_{kl} p_{jm}^* p_{kl}}{p_{jm}^* + p_{kl}} \exp(\theta_{jm}(\xi, t)^* + \theta_{kl}(\xi, t)),
\]

\[
\lambda_{jk}[\varphi] = \sum_{m=1}^{N'_j} \sum_{l=1}^{N'_k} \frac{B_{jm} B_{kl} q_{jm}^* q_{kl}}{q_{jm}^* + q_{kl}} \exp(\vartheta_{jm}(\eta, t)^* + \vartheta_{kl}(\eta, t)).
\]

The localized solutions of the Ishimori-I equation can be constructed under the choice (31).

5.2. **Lump solution changing the minimal values**

For simplicity, we consider the case \(M = 1\). The solutions

\[
\psi_1 = A \xi \exp \theta, \quad \psi_2 = 0,
\]

![Fig. 6](https://academic.oup.com/ptp/article-abstract/98/5/1013/1857776/fig6)

Fig. 6. Lump changing the minimal values; (a) 3D-plot of \(S_3^{(1)}\) at \(t = -2\), (b) Contour lines of \(S_3^{(1)}\) at \(t = -2\), (c) Contour lines of \(S_3^{(1)}\) at \(t = 0\), (d) Contour lines of \(S_3^{(1)}\) at \(t = 2\); \(\mu = 1, \nu = 1.2, A = 1 + i, B = 1.2 + 0.3i, c_{11} = 1.5i, c_{12} = 0.5 + 2.2i, c_{22} = 0.5i, \alpha = \beta = 0, a = -2\).
of Eq. (24) are chosen, where \( \theta = i(\mu \xi + a\mu^2 t) \), \( \varphi = i(\nu \eta - a\nu^2 t) \), \( \xi = \xi + 2a\mu t + \alpha \) and \( \eta = \eta - 2a\nu t + \beta \), \( A \) and \( B \) are arbitrary complex constants, and \( \mu, \nu, \alpha \) and \( \beta \) are real. We obtain a rationally localized solution by substituting the functions

\[
f = \left( \mu \frac{\dot{\xi}^3}{3} - \frac{\dot{\xi}^2}{2i} + a(t - t_1) \right) \left( \nu \frac{\dot{\eta}^3}{3} + \frac{\dot{\eta}^2}{2i} - a(t - t_2) \right) - \gamma^2,
\]

\[
g = -\gamma \dot{\xi} \dot{\eta} \exp(i\Theta),
\]

\[
D = \left( \mu \frac{\dot{\xi}^3}{3} + \frac{\dot{\xi}^2}{2i} + a(t - t_1) \right) \left( \nu \frac{\dot{\eta}^3}{3} + \frac{\dot{\eta}^2}{2i} - a(t - t_2) \right) - \gamma^2
\]

into Eqs. (13) and (16), where \( \Theta = \mu \xi - \nu \eta + a(\mu^2 - \nu^2)t + \delta \), and \( t_1, t_2, \gamma \) and \( \delta \) are real. This solution describes the lump solution which is moving at a constant speed and for which the minimal values of \( S_3^{(1)} \) are changing, as seen from Fig. 6.

**§6. Conclusion**

In §2 exact solutions of the Ish-I equation were constructed in the grammian representation by means of the BDT, and in §§3 and 4 it was shown that these solutions include the multi-soliton solutions for localized solutions such as the dromion, the lump and the polynomially-exponentially localized solution, which were determined using the inverse scattering method. The construction of exact solutions for the Ish-I equation by using either the inverse scattering method (ISM) or the BDT is closely connected with the problem of explicitly solving the linear equation

\[
\psi_t = -ia\psi_{zz} + u(z, t)\psi_z,
\]

where \( z = \xi \) (or \( \eta \)). The coefficient \( u(z, t) \) corresponds to the boundary condition of the auxiliary field \( \phi \) at \( \xi \) (or \( \eta \)) \( \to -\infty \) in the case of the ISM, and the known solution before the transformation is performed in the case of the BDT. Therefore to obtain the three types of localized solutions mentioned above by using the ISM, it is necessary to solve the linear equation with the variable coefficient which corresponds to the respective non-trivial boundary conditions. While in the case that the BDT is used, all of the above localized solutions are obtained from exponential solutions of the linear equation with \( u(z, t) = 0 \). Furthermore, naturally extending the BDT to the \( M \)-times BDT such as Eq. (8) enables one to construct the multi-soliton (dromion, lump and so on) solution explicitly in terms of grammian determinants (13)~(16), (19) and (25)~(31).

The transformation (13) was first given by R. Hirota to construct the bilinear form for the (1+1)-dimensional Heisenberg ferromagnet equation\(^9\) and was applied to the Ish-II equation in Ref. 1. In this paper Eqs. (13) and (14) (or (16)) were derived automatically by the BDT as the generalization to the Ish-I equation of the transformation in Ref. 9.

In §5 the new localized solutions were constructed. These include solutions obtained from the superposition of the exponential functions and the lump solution for which minimal values change.
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References

3) G. Darboux, Compt. Rend. 94 (1882), 1456.