Energy and Momentum in the Tetrad Theory of Gravitation

Takeshi SHIRAFUJI and Gamal G. L. NASHED

Physics Department, Saitama University, Urawa 338

(Received June 27, 1997)

We study the energy and momentum of an isolated system in the tetrad theory of gravitation, starting from the most general Lagrangian quadratic in torsion, which involves four unknown parameters. When applied to the static spherically symmetric case, the parallel vector fields take a diagonal form, and the field equation has an exact solution. We analyze the linearized field equation in vacuum at distances far from the isolated system without assuming any symmetry property of the system. The linearized equation is a set of coupled equations for a symmetric and skew-symmetric tensor fields, but it is possible to solve it up to \( O(1/r) \) for the stationary case. It is found that the general solution contains two constants, one being the gravitational mass of the source and the other a constant vector \( E_\alpha \). The total energy is calculated from this solution and is found to be equal to the gravitational mass of the source. We also calculate the spatial momentum and find that its value coincides with the constant vector \( E_\alpha \). The linearized field equation in vacuum, which is valid at distances far from the source, does not give any information about whether the constant vector \( E_\alpha \) is vanishing or not. For a weakly gravitating source for which the field is weak everywhere, we find that the constant vector \( E_\alpha \) vanishes.

§1. Introduction

Einstein concerned himself with the problem of energy and momentum for a system of matter plus gravitational field shortly after he proposed general relativity.\(^1\) He introduced the well-known expression for the energy-momentum complex \( \theta_\mu^\nu \), which is referred to as a canonical expression and satisfies ordinary conservation law as a consequence of the gravitational field equation. For a closed system and for a restricted class of coordinates the quantities \( P_\mu \), obtained from the complex \( \theta_\mu^\nu \) by integrating over spatial coordinates satisfy the following properties: \(^2,3\) (1) They are constant in time, (2) transform as a 4-vector under linear coordinate transformations, and (3) are invariant under arbitrary transformations of spatial coordinates which tend to the identity transformation at infinity. Furthermore (4) the gravitational mass of the system is equal to the total energy. It has recently been shown that a characteristic property of energy thus defined is its positivity, reflecting stability of the physical system.\(^4\)–\(^6\)

Møller revived\(^7\) the issue of energy and momentum in general relativity, and required\(^8\) that any energy-momentum complex \( \tau_\mu^\nu \) must satisfy the following properties: (A) It must be an affine tensor density which satisfies conservation law, (B) for an isolated system the quantities \( P_\mu \) are constant in time and transform as the covariant components of a 4-vector under linear coordinate transformations, and (C) the superpotential \( U_\mu^\lambda = -U_\mu^\lambda \) transforms as a tensor density of rank 3 under the
group of spacetime transformations. The second property requires that $\tau_{\mu\nu}$ around an isolated system should decrease faster than $1/r^3$ for large values of the radial coordinate $r$. The last property is required in order that the four-momentum for any isolated system be a 4-vector under arbitrary spacetime transformations.

The canonical energy-momentum complex $\theta_{\mu\nu}$ does not satisfy (C). An energy-momentum complex that satisfies properties (A) and (C) was constructed,\textsuperscript{7,9} but it was found later\textsuperscript{10} that the property (B) is not satisfied. Lessner\textsuperscript{11} discussed that although this result is inadequate from the viewpoint of special relativity, it may be adequate from the viewpoint of general relativity.

It is not possible to satisfy all the above requirements if the gravitational field is described by the metric tensor alone.\textsuperscript{10} In a series of papers,\textsuperscript{10,12,13} therefore, Møller was led to the tetrad description of gravitation, and constructed a formal form of energy-momentum complex that satisfies all the requirements. The metric tensor is uniquely fixed by the tetrad field, but the reverse is not true, since the tetrad has six extra degrees of freedom. In the tetrad formulation of general relativity, the tetrad field is allowed to undergo local Lorentz transformations with six arbitrary functions. The energy-momentum complex is not a tensor and changes its form under such transformations. Therefore, unless one can find a good physical argument for fixing the tetrad throughout the system, one cannot speak about the energy distribution inside the system. The total energy-momentum obtained by the complex, however, is invariant under local Lorentz transformations with appropriate boundary conditions.\textsuperscript{13}

Møller also suggested another possibility that the tetrad field is uniquely fixed by means of six supplementary conditions.\textsuperscript{12} In this case the underlying spacetime possesses absolute parallelism,\textsuperscript{14} with the tetrad field playing the role of the parallel vector fields, which are allowed to undergo only global Lorentz transformations. The Lagrangian formulation of this tetrad theory of gravitation was first given by Pellegrini and Plebanski.\textsuperscript{15} Hayashi and Nakano\textsuperscript{16} independently formulated the same gravitational theory as a gauge theory of spacetime translation group. The Lagrangian was assumed to be given by a sum of quadratic invariants of the torsion tensor, which is expressed by first-order derivatives of the parallel vector fields. Its most general expression then involves four unknown parameters to be determined by experiment, which we denote here as $a_1$, $a_2$, $a_3$ and $a_4$. The last parameter $a_4$ is associated with a parity-violating term. At first it was required that for the weak field case the gravitational field equation should reproduce the linearized Einstein equation. This restricted the parameters as $a_1 + a_2 = 0$ and $a_4 = 0$.\textsuperscript{16,17} Møller\textsuperscript{17} also suggested a possible generalization of the gravitational Lagrangian by including homogeneous functions of the torsion tensor of degree 4 or higher.

Hayashi and Shirafuji\textsuperscript{18} studied the geometrical and observational basis of the tetrad theory of gravitation\textsuperscript{1} assuming the Lagrangian to be invariant under parity operation, involving three unknown parameters $a_1$, $a_2$ and $a_3$. Two of these parameters, $a_1$ and $a_2$, were determined by comparison with solar-system experiments.

\textsuperscript{1} They coined the name “new general relativity”, because Einstein\textsuperscript{19} was the first to introduce the notion of absolute parallelism into physics after he had constructed general relativity.
while an upper bound was estimated for the $a_3$.\textsuperscript{1)} It was found that the numerical value of $a_1 + a_2$ should be very small, consistent with being zero.

In the tetrad theory of gravitation, as far as we know, the total energy of an isolated system has been calculated only for spherically symmetric case. In the case $a_1 + a_2 = 0 = a_4$, Mikhail et al.\textsuperscript{21}) found a static, spherically symmetric solution $(b^k_{\mu})$ in Cartesian coordinates with $(b^{(0)}_a) \sim 1/\sqrt{r} \sim (b^a_0)$ for $r \to \infty$, and showed that the total energy does not coincide with the gravitational mass. This result was extended to a wider class of solutions with spherical symmetry.\textsuperscript{22}) An explicit expression was given for all the stationary, asymptotically flat solutions with spherical symmetry, which were then classified according to the asymptotic behavior of the components of $(b^a_0)$ and $(b^{(0)}_a)$. It was found that the equality of the gravitational and inertial masses holds only when $(b^a_0)$ and $(b^{(0)}_a)$ tend to zero faster than $1/\sqrt{r}$.

Calculation of the energy of an isolated system was extended to the generic case $(a_1 + a_2) (a_1 - 4a_3/9) \neq 0$ under the assumption of spherical symmetry.\textsuperscript{23}) It was shown that linear approximation can be applied to the field equation at large spatial distances from the source, and that the calculated energy is equal to the gravitational mass of the source. When $(a_1 - 4a_3/9) = 0$, however, the problem of energy of an isolated system is not yet understood well, even for the spherically symmetric case, since it is not always possible to determine the asymptotic behavior of the components $(b^a_0)$ by means of linearized field equation.

It is the purpose of this paper to calculate the total energy and spatial momentum of an isolated system without assuming spherical symmetry. For this purpose we solve the linearized field equation at far distances up to order $O(1/r)$. The general solution has two constants. One of these is related to the gravitational mass of the source, and the other is a constant vector. We then show that the energy is always equal to the gravitational mass of the isolated system. In this sense the equivalence principle is satisfied in the tetrad theory of gravitation. It should be noted that our conclusion is based on the assumption that weak field approximation can be applied at far distances from the source. In the spherically symmetric case mentioned above, however, solutions are known for which the linear approximation does not work even far from the source because the components, $(b^a_0)$ and $(b^{(0)}_a)$, behave like $1/\sqrt{r}$.

In §2 we discuss the most general Lagrangian with a parity-violating term and apply its field equation to the static, diagonal parallel vector fields with spherical symmetry. We find that the exact solution for the parity-conserving case\textsuperscript{18)} satisfies the field equation. In §3 we construct the linearized form of the field equation in vacuum at distances far from an isolated system. Assuming that the system is stationary, we obtain the general solution of the linearized field equation in vacuum. In §4 we derive the superpotential from the general Lagrangian and calculate its components necessary for computing the total energy and momentum of the system, using the solution obtained in §3. The final section is devoted to conclusion and discussion.

\textsuperscript{1)} For macroscopic matter, Nitsch and Hehl\textsuperscript{20)} proposed a tetrad theory of gravitation as the translational gauge limit of Poincaré gauge theory. Their choice of the parameters corresponds to $a_1 = -1/3$, $a_2 = 1/3$, $a_3 = 3/2$ and $a_4 = 0$ in our notation.
\section*{§2. Basic Lagrangian}

In a spacetime with absolute parallelism the parallel vector fields \((b^k_\mu)^*\) define the nonsymmetric connection

\[ \Gamma^\lambda_{\mu\nu} = b^\lambda_b b^k_{\mu\nu} \]  

(1)

with \(b^k_{\mu\nu} = \partial_\nu b^k_\mu\), from which the torsion tensor is given by

\[ T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = b^\lambda_b (b^k_{\mu\nu} - b^k_{\nu\mu}). \]  

(2)

The curvature tensor defined by \(\Gamma^\lambda_{\mu\nu}\) is identically vanishing, however. Here Latin indices are raised or lowered by the Minkowski metric \(\eta_{ij} = \eta^{ij} = \text{diag}(-1, 1, 1, 1)\).

The metric tensor is given by the parallel vector fields as

\[ g_{\mu\nu} = b_{k\mu} b^k_{\nu}. \]  

(3)

Assuming invariance under a) the group of general coordinate transformations, and b) the group of global Lorentz transformations, we write the most general gravitational Lagrangian density quadratic in the torsion tensor as

\[ \mathcal{L}_G = \frac{\sqrt{-g}}{\kappa} \left[ a_1 (e^\mu_{\lambda\lambda} t_{\mu\nu\lambda}) + a_2 (v^\mu_{\nu\mu}) + a_3 (a^\mu a_\mu) + a_4 (v^\mu a_\mu) \right], \]  

(4)

where \(a_1, a_2, a_3\) and \(a_4\) are dimensionless parameters of the theory, and \(t_{\mu\nu\lambda}, v_{\mu}\) and \(a_\mu\) are the three irreducible components of the torsion tensor.\(^{18),\text{**})\)

By applying the variational principle to the Lagrangian (4), we obtain the field equation:

\[ I^{\mu\nu} = \kappa T^{\mu\nu} \]  

(5)

with

\[ I^{\mu\nu} = 2\kappa \left[ D_\lambda F^{\mu\nu\lambda} + v_\lambda F^{\mu\nu\lambda} + H^{\mu\nu} - \frac{1}{2} g^{\mu\nu} L_G \right], \]  

(6)

where

\[ F^{\mu\nu\lambda} = \frac{1}{2} b^{k\mu} \frac{\partial L_G}{\partial b^{k\nu,\lambda}} = -F^{\mu\nu}, \]  

(7)

\[ H^{\mu\nu} = T^{\rho\sigma\mu} F_{\rho\sigma} - \frac{1}{2} T^{\nu\rho\sigma} F_{\rho\sigma} = H^{\nu\mu}, \]  

(8)

\[ T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta L_M}{\delta b_{\mu\nu}}. \]  

(9)

\(^*)\) Latin indices \((i, j, k, \cdots)\) designate the vector number, which runs from \(0\) to \(3\), while Greek indices \((\mu, \nu, \rho, \cdots)\) designate the world-vector components running from \(0\) to \(3\). The spatial part of Latin indices is denoted by \((a, b, c, \cdots)\), while that of Greek indices by \((\alpha, \beta, \gamma, \cdots)\).

\(^{**})\) Throughout this paper we use the relativistic units, \(c = G = 1\). The Einstein constant \(\kappa\) is then equal to \(8\pi\). We will denote the symmetric part by \((\ ),\) for example, \(A^{(\mu\nu)} = (1/2)(A_{\mu\nu} + A_{\nu\mu})\) and the antisymmetric part by the square bracket \([\ ]\), \(A_{[\mu\nu]} = (1/2)(A_{\mu\nu} - A_{\nu\mu}).\)

\(^{***)\) The dimensionless parameters \(\kappa a_i\) of Ref. 18) are here denoted by \(a_i\) for convenience.
Here $L_G = L_G/\sqrt{-g}$, and $\mathcal{L}_M$ denotes the Lagrangian density of material fields, of which the energy-momentum tensor $T^{\mu\nu}$ is nonsymmetric in general.

In static, spherically symmetric spacetime, the parallel vector fields take a diagonal form, and the field equation (5) can be exactly solved. The exact solution so obtained is the same as that obtained by assuming the invariance under parity operation.\(^\text{18}\) This implies that the parameter $a_4$ has no effect when we use diagonal parallel vector fields having spherical symmetry.

In order to reproduce the correct Newtonian limit, the parameters $a_1$ and $a_2$ should satisfy the condition

$$a_1 + 4a_2 + 9a_1a_2 = 0,$$

called the Newtonian approximation condition,\(^\text{18}\) which can be solved to give

$$a_1 = -\frac{1}{3(1-\epsilon)}, \quad a_2 = \frac{1}{3(1-4\epsilon)},$$

in terms of a dimensionless parameter $\epsilon$. Comparison with solar-system experiments indicates that $|\epsilon|$ must be very small.

§3. Solution at far distances

In order to calculate the energy and momentum of an isolated system confined in a finite spatial region, we focus our attention to the solution at far distances from the source without assuming spherical symmetry. In such spatial regions far from the source, the gravitational field is weak and matter fields are not present: Thus, we are allowed to treat the linearized field equation in vacuum. For simplicity we assume that the whole spacetime is covered with a single coordinate system $\{x^\mu; \; +\infty > x^\mu > -\infty\}$ with the origin being located somewhere inside the finite system.

In spatial regions far from the source, the parallel vector fields can be represented as

$$b_k^\mu(x) = \delta_k^\mu + a_k^\mu, \quad |a_k^\mu| << 1,$$

$$b_k^\nu(x) = \delta_k^\nu + c_k^\nu, \quad |c_k^\nu| << 1,$$

and we assume that all the quadratic and higher-order terms of $a_k^\mu$ (or $c_k^\mu$) can be neglected in the field equation. Accordingly, we do not distinguish Latin indices from Greek indices for $a_k^\mu$ and $c_k^\nu$: We use Greek indices which are now raised and lowered by the Minkowski metric $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$. From the relation $b^\mu_k b_k^\nu = \delta^\nu_\nu$, it follows that

$$a_{\nu\mu} + c_{\nu\mu} = 0$$

with $a_{\nu\mu} = \eta_{\nu\lambda}a^\lambda_\mu$ and $c_{\nu\mu} = \eta_{\nu\lambda}c^\lambda_\mu$. We decompose $a_{\nu\mu}$ into symmetric and antisymmetric parts,

$$a_{\nu\mu} = \frac{1}{2} h_{\nu\mu} + A_{\nu\mu},$$

with $h_{\mu\nu} = h_{\nu\mu}$ and $A_{\mu\nu} = -A_{\nu\mu}$. The components of the metric tensor are written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$
The antisymmetric part has no contribution to the spacetime metric, implying that it is associated with the intrinsic spin-1/2 fundamental particles.

Keeping only the linear terms and putting $T_{\mu\nu} = 0$ on the right-hand side, we see that the symmetric part and the skew-symmetric part of the field equation (5) take the form

$$
\frac{3a_1}{2} \Box h_{\mu\nu} - \frac{1}{2} (a_1 + a_2) (\eta_{\mu\nu} \Box h - h_{\mu\nu}) + \left( \frac{a_1}{2} - a_2 \right) \eta_{\mu\nu} h^{\rho\sigma} \partial_{\rho\sigma} \partial_{\mu\nu} = 0
$$

and

$$
- (2a_1 - a_2) h^\rho_{(\mu,\nu)\rho} + 2(a_1 + a_2) A^\rho_{(\mu,\nu)\rho} + \frac{2}{3} a_4 A^\rho_{(\mu,\nu)\rho} = 0
$$

(17)

respectively, in spatial regions far from the source. Here the d'Alembertian operator is given by $\Box = \partial^\mu \partial_\mu$, $h_{\mu\nu}$ denotes

$$
h_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h,
$$

(19)

and $A_{\mu\nu}$ stands for the dual of $A_{\mu\nu}$ defined by

$$
\overline{A}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} A^{\lambda\sigma},
$$

(20)

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor normalized as $\varepsilon^{0123} = +1$. It is clear from (17) and (18) that the symmetric field $h_{\mu\nu}$ and the skew-symmetric field $A_{\mu\nu}$ are coupled to each other unless the parameters satisfy the condition $a_1 + a_2 = 0$ and $a_4 = 0$. As is easily checked, the linearized field equations (17) and (18) are invariant under the gauge transformation

$$
h'_{\mu\nu} = h_{\mu\nu} - 2 \xi_{(\mu,\nu)},
A'_{\mu\nu} = A_{\mu\nu} - \xi_{[\mu,\nu]},
$$

(21)

where the $\xi_\mu$ are small functions which leave the fields weak. By means of this freedom we can require the gauge condition

$$
\partial_\nu \overline{h}^{\mu\nu} = 0,
$$

(22)

which we shall assume henceforth. Then Eqs. (17) and (18) become

$$
\frac{3a_1}{2} \Box h_{\mu\nu} - \frac{1}{2} (a_1 + a_2) (\eta_{\mu\nu} \Box h - h_{\mu\nu}) + 2(a_1 + a_2) A^\rho_{(\mu,\nu)\rho} + \frac{2}{3} a_4 A^\rho_{(\mu,\nu)\rho} = 0,
$$

(23)

(\begin{align*}
(a_1 - \frac{4}{9} a_3) \Box A_{\mu\nu} & \quad - 2 \left( a_2 + \frac{4}{9} a_3 \right) A^\rho_{[\mu,\nu]\rho} - \frac{2}{3} a_4 A^\rho_{[\mu,\nu]\rho} - \frac{1}{3} a_4 \varepsilon_{\mu\nu\lambda\sigma} \partial^\lambda \partial_\sigma A^{\rho\sigma} = 0.
\end{align*})

(24)
In order to calculate the total energy and momentum of the isolated system under consideration, it is enough to obtain asymptotic solutions of (23) and (24) up to order $O(1/r)$. Multiplying $\partial_r$ on (23) [or (24)], we obtain

$$\Box \left[ (a_1 + a_2) A^\nu_{\mu} + \tilde{a}_4 A^\nu_{\mu} \right]_{,\nu} = 0$$

(25)

with $\tilde{a}_4 = (a_4/3)$. Let us define $B_{\mu\nu} = B_{[\mu\nu]}$ by

$$B_{\mu\nu} = (a_1 + a_2) A_{\mu\nu} + \tilde{a}_4 A_{\mu\nu},$$

(26)

which can easily be solved with respect to $A_{\mu\nu}$ to give

$$A_{\mu\nu} = \frac{1}{(a_1 + a_2)^2 + \tilde{a}^2_4} \left[ (a_1 + a_2) B_{\mu\nu} - \tilde{a}_4 B_{\mu\nu} \right],$$

(27)

under the assumption that $(a_1 + a_2)^2 + \tilde{a}^2_4 \neq 0$, or equivalently $(a_1 + a_2) \neq 0$ and/or $a_4 \neq 0$. We note that when this assumption is violated, the theory is reduced to the special case of Hayashi and Nakano \(^{16}\) and Møller. \(^{17}\)

We now assume that the system under consideration is stationary and that the system as a whole is at rest somewhere around the origin. Then $h_{\mu\nu}$ and $A_{\mu\nu}$ are time-independent. Rewriting (23) and (24) in terms of $B_{\mu\nu}$, we have after a slight modification

$$\Delta h = 0,$$

(28)

$$\frac{3a_1}{2} \Delta h_{00} = 0,$$

(29)

$$\frac{3a_1}{2} \Delta h_{0\alpha} - B_{0\beta,\alpha\beta} = 0,$$

(30)

$$f_1 \Delta B_{0\alpha} + f_2 B_{0\beta,\beta\alpha} + f_3 \epsilon_{\alpha\beta\gamma} B_{\delta[\beta,\gamma]\delta} = 0,$$

(31)

$$\frac{3a_1}{2} \Delta h_{\alpha\beta} + \frac{1}{2} (a_1 + a_2) h_{,\alpha\beta} + 2B_{\gamma(\alpha,\beta)\gamma} = 0,$$

(32)

$$f_1 \Delta B_{\alpha\beta} - 2f_2 B_{\delta[\alpha,\beta]\delta} + f_3 \epsilon_{\alpha\beta\gamma} B_{0\delta,\delta\gamma} = 0,$$

(33)

where we have introduced $f_1$, $f_2$ and $f_3$ by

$$f_1 = (a_1 + a_2) \left( a_1 - \frac{4}{9} a_3 \right) + \tilde{a}_4^2,$$

(34)

$$f_2 = (a_1 + a_2) \left( a_2 + \frac{4}{9} a_3 \right),$$

(35)

$$f_3 = \tilde{a}_4 \left( a_2 + \frac{4}{9} a_3 \right).$$

(36)

We note that except for $\Delta B_{0\alpha}$ and $\Delta B_{\alpha\beta}$ Eqs. (30)~(33) involve $B_{0\alpha}$ and $B_{\alpha\beta}$ only in the form of 3-dimensional divergence.
According to (25) with (26) the divergences, $B_{0\alpha,\alpha}$ and $B_{\alpha\beta,\beta}$, satisfy the Laplace equation, and therefore, they are given up to $O(1/r^2)$ by

$$B_{0\alpha,\alpha} = \frac{\dot{B}_\alpha n_\alpha}{r^2}, \quad (37)$$

$$B_{\alpha\beta,\beta} = \frac{\dot{F}_{\alpha\beta} n_\beta}{r^2}, \quad (38)$$

where $\dot{B}_\alpha$ is a constant vector, and $\dot{F}_{\alpha\beta}$ is a nonsymmetric constant tensor. Here the radial unit vector $n^\alpha$ is defined by $n^\alpha = x^\alpha / r$ without making distinction between upper and lower indices. We have omitted from our consideration the possibility that the leading terms of the solutions (37) and (38) begin from $O(1/r)$. This is because if, on the contrary, they begin from $O(1/r)$-terms, Eqs. (30)~(33) will have no solutions of $\bar{h}_{\mu\nu}$ and $B_{\mu\nu}$ which tend to zero for large $r$. Since $B_{\alpha\beta}$ is by definition antisymmetric with respect to $\alpha$ and $\beta$, the constant tensor $\dot{F}_{\alpha\beta}$ must be of the form

$$\dot{F}_{\alpha\beta} = \dot{F}_{[\alpha\beta]} + \frac{1}{3} \dot{F} \delta_{\alpha\beta} \quad (39)$$

with $\dot{F}$ being the trace. With the help of (37)~(39) we can solve the field equations (28)~(33) by following the ordinary procedure to solve the Laplace equation. (See the Appendix for a summary of the procedure.)

From (28) we see that $\bar{h}$ can be expressed by $\bar{h} = p/r$ up to $O(1/r)$, where $p$ is a constant. Using this expression for $\bar{h}$ and Eq. (38) in (32), we obtain

$$\bar{h}_{\alpha\beta} = \frac{\dot{G}_{\alpha\beta}}{r} + \frac{1}{6a_1} \left\{ 4\delta_{\delta(\alpha} \dot{F}_{\beta)\gamma} - \frac{4}{9} \dot{F} \delta_{\alpha\beta} \delta_{\gamma\delta} + p(a_1 + a_2) \left( \delta_{\alpha\gamma} \delta_{\beta\delta} - \frac{1}{3} \delta_{\alpha\beta} \delta_{\gamma\delta} \right) \right\} \frac{n_\gamma n_\delta}{r}, \quad (40)$$

where $\dot{G}_{\alpha\beta}$ is a symmetric, constant tensor. Applying the condition (22) to $\bar{h}_{\alpha\beta}$ of (40), we see that $\dot{G}_{\alpha\beta}$ is given by

$$\dot{G}_{\alpha\beta} = \frac{2}{3a_1} \left[ \frac{4}{9} \dot{F} \delta_{\alpha\beta} + \dot{F}_{[\alpha\beta]} + \frac{1}{3} (a_1 + a_2) p \delta_{\alpha\beta} \right]. \quad (41)$$

Since $\dot{G}_{\alpha\beta}$ is symmetric, Eq. (41) implies

$$\dot{F}_{[\alpha\beta]} = 0. \quad (42)$$

As for the field equation for $B_{\alpha\beta}$ we note that the second term on the left-hand side of (33) vanishes owing to (39) and (42). Applying the same procedure to (33), we obtain

$$B_{\alpha\beta} = \frac{\dot{H}_{\alpha\beta}}{r} + \frac{f_3}{6f_1} \frac{\epsilon_{\alpha\beta\gamma}}{r} (\dot{B}_\gamma - 3n_\gamma n_\delta \dot{B}_\delta), \quad (43)$$

where we have assumed that the constant $f_1$ of (34) is nonvanishing. Here $\dot{H}_{\alpha\beta}$ is an antisymmetric constant tensor to be fixed by the relation (38): We have

$$\dot{H}_{\alpha\beta} = \frac{f_3}{3f_1} \epsilon_{\alpha\beta\gamma} \dot{B}_\gamma, \quad (44)$$
Energy and Momentum in the Tetrad Theory of Gravitation

\( \dot{F} = 0, \) \hspace{1cm} (45)

the latter of which together with (39) and (42) implies that \( B_{\alpha\beta\gamma} \) vanishes up to \( O(1/r^2) \).

We can analyze the field equations for \( h_{0\alpha} \) and \( B_{0\alpha} \) in the same manner. We here summarize the asymptotic solution up to \( O(1/r) \) for \( h_{\mu\nu} \) and \( B_{\mu\nu} \) as follows:

\[
\begin{align*}
\bar{h} &= \frac{p}{r}, \\
\bar{h}_{00} &= \frac{(2a_2 - a_1)p}{3a_1 r}, \\
\bar{h}_{0\alpha} &= \frac{\dot{B}_\alpha + n_\alpha n_\beta \dot{B}_\beta}{3a_1 r}, \\
\bar{h}_{\alpha\beta} &= \frac{(a_1 + a_2)p}{6a_1 r} (\delta_{\alpha\beta} + n_\alpha n_\beta), \\
B_{0\alpha} &= -\frac{1}{r} \left[ \dot{B}_\alpha + \frac{f_2}{2f_1} \left( \dot{B}_\gamma + n_\gamma n_\delta \dot{B}_\delta \right) \right], \\
B_{\alpha\beta} &= \frac{f_3}{2f_1} \epsilon_{\alpha\beta\gamma} \left( \dot{B}_\gamma - n_\gamma n_\delta \dot{B}_\delta \right),
\end{align*}
\] 

where \( p \) is an unknown constant and \( \dot{B}_\alpha \) is an unknown constant vector. From (19) and (27), we obtain the most general expression for \( h_{\mu\nu} \) and \( A_{\mu\nu} \) up to \( O(1/r) \):

\[
\begin{align*}
h &= -\frac{p}{r}, \\
h_{00} &= \frac{-3a_2 p}{2r}, \\
h_{0\alpha} &= \frac{\dot{B}_\alpha + n_\alpha n_\beta \dot{B}_\beta}{3a_1 r}, \\
h_{\alpha\beta} &= -\frac{p}{2r} \delta_{\alpha\beta} + \frac{p(a_1 + a_2) (\delta_{\alpha\beta} + n_\alpha n_\beta)}{6a_1} \\
A_{0\alpha} &= -\frac{1}{2f_1 r} \left[ 2\dot{B}_\alpha \left( a_1 - \frac{4}{9} a_3 \right) + \left( a_2 + \frac{4}{9} a_3 \right) \left( \dot{B}_\alpha + n_\alpha n_\beta \dot{B}_\beta \right) \right], \\
A_{\alpha\beta} &= \frac{\check{a}_4}{f_1 r} \epsilon_{\alpha\beta\gamma} \check{B}_\gamma.
\end{align*}
\]

The covariant components of the parallel vector fields are then represented up to \( O(1/r) \) by

\[
\begin{align*}
b^{(0)}_{0} &= 1 - \frac{m}{r}, \\
b^{(0)}_{\alpha} &= \frac{\left( a_1 - \frac{4}{9} a_3 \right) \dot{B}_\alpha}{f_1} - \left\{ \frac{1}{3a_1} - \frac{\left( a_2 + \frac{4}{9} a_3 \right)}{f_1} \right\} \left( \dot{B}_\alpha + n_\alpha n_\beta \dot{B}_\beta \right),
\end{align*}
\]

\begin{align*}
\dot{F} &= 0, \\
\bar{h} &= \frac{p}{r}, \\
\bar{h}_{00} &= \frac{(2a_2 - a_1)p}{3a_1 r}, \\
\bar{h}_{0\alpha} &= \frac{\dot{B}_\alpha + n_\alpha n_\beta \dot{B}_\beta}{3a_1 r}, \\
\bar{h}_{\alpha\beta} &= \frac{(a_1 + a_2)p}{6a_1 r} (\delta_{\alpha\beta} + n_\alpha n_\beta), \\
B_{0\alpha} &= -\frac{1}{r} \left[ \dot{B}_\alpha + \frac{f_2}{2f_1} \left( \dot{B}_\gamma + n_\gamma n_\delta \dot{B}_\delta \right) \right], \\
B_{\alpha\beta} &= \frac{f_3}{2f_1} \epsilon_{\alpha\beta\gamma} \left( \dot{B}_\gamma - n_\gamma n_\delta \dot{B}_\delta \right),
\end{align*}

\[
\begin{align*}
h &= -\frac{p}{r}, \\
h_{00} &= \frac{-3a_2 p}{2r}, \\
h_{0\alpha} &= \frac{\dot{B}_\alpha + n_\alpha n_\beta \dot{B}_\beta}{3a_1 r}, \\
h_{\alpha\beta} &= -\frac{p}{2r} \delta_{\alpha\beta} + \frac{p(a_1 + a_2) (\delta_{\alpha\beta} + n_\alpha n_\beta)}{6a_1} \\
A_{0\alpha} &= -\frac{1}{2f_1 r} \left[ 2\dot{B}_\alpha \left( a_1 - \frac{4}{9} a_3 \right) + \left( a_2 + \frac{4}{9} a_3 \right) \left( \dot{B}_\alpha + n_\alpha n_\beta \dot{B}_\beta \right) \right], \\
A_{\alpha\beta} &= \frac{\check{a}_4}{f_1 r} \epsilon_{\alpha\beta\gamma} \check{B}_\gamma.
\end{align*}
\]

\[
\begin{align*}
b^{(0)}_{0} &= 1 - \frac{m}{r}, \\
b^{(0)}_{\alpha} &= \frac{\left( a_1 - \frac{4}{9} a_3 \right) \dot{B}_\alpha}{f_1} - \left\{ \frac{1}{3a_1} - \frac{\left( a_2 + \frac{4}{9} a_3 \right)}{f_1} \right\} \left( \dot{B}_\alpha + n_\alpha n_\beta \dot{B}_\beta \right),
\end{align*}
\]
with \( m \) being a constant given by
\[
m = \frac{3a_2 p}{4}.
\]

Thus the parallel vector fields are asymptotically characterized by the constant vector \( \mathbf{E}^{ex} \) besides the constant \( m \) at large spatial distances from an isolated, stationary system. The constant \( m \) can be interpreted as the gravitational mass of the source, as seen from (53). The physical meaning of the constant \( \mathbf{E}^{ex} \) will become clear when we calculate in the next section the total energy and momentum of the isolated system.

The parallel vector fields of (58)~(61) are not spherically symmetric because they involve the constant vector \( \mathbf{E}^{ex} \). This is to be contrasted with the metric in general relativity, which is governed by the Einstein equation.\(^{24} \) More precisely, if one selects a system of coordinates with respect to which the isolated system is as a whole at rest, and if one neglects gravitational radiation by the system, the Einstein equation requires that the metric is spherically symmetric and static up to \( O(1/r) \) for large \( r \). We note, however, that when \( \mathbf{E}^{ex} = 0 \), the parallel vector fields (58)~(61) reduce to the static, spherically symmetric solution expanded up to \( O(1/r) \).

We have been assuming that the parameters satisfy \((a_1 + a_2)^2 + a_4^2 \neq 0\) and \( f_1 \neq 0 \). Now that we have obtained the solution of \((b^\alpha_{\mu})\) up to \( O(1/r) \), we can study the limiting cases \( a_4 = 0 \neq (a_1 + a_2) \) and \((a_1 + a_2) = 0 \neq a_4 \). In both cases we can use \( A_{\mu\nu} \) directly instead of \( B_{\mu\nu} \). In the case \( a_4 = 0 \) and \((a_1 + a_2) \neq 0\), we can further take the limit \((a_1 + a_2) \to 0\): It is easy to check that the resultant \( h_{\mu\nu} \) and \( A_{\mu\nu} \) satisfy Eqs. (23) and (24) with \((a_1 + a_2) = 0\) and \( a_4 = 0 \). In the case \((a_1 + a_2) = 0\) and \( a_4 \neq 0\), on the other hand, it is possible to take the limit \( a_4 \to 0 \) only when \((a_1 - 4a_3/9) \to 0\) is taken simultaneously. The resultant theory is just the tetrad formulation of general relativity.

\section*{§4. Calculation of energy and momentum}

Since the total Lagrangian is a scalar under general coordinate transformations, the total energy-momentum complex \((T^\mu_\nu + t^\mu_\nu)\) is represented as
\[
\sqrt{-g}(T^\mu_\nu + t^\mu_\nu) = U^\mu_\nu \eta_\lambda
\]
with the superpotential \( U^\mu_\nu \eta_\lambda \) being antisymmetric in \( \nu \) and \( \lambda \). Here \( t^\mu_\nu \) is the canonical energy-momentum complex for the gravitational field derived from the gravitational Lagrangian density (4).\(^{22} \) The superpotential \( U^\mu_\nu \eta_\lambda \) is given by
\[
U^\mu_\nu \eta_\lambda = 2\sqrt{-g} F^\mu_\nu \eta_\lambda
\]
\[ \frac{2\sqrt{-g}}{\kappa} \left[ \left( a_1 - \frac{a_3}{3} \right) T_{\mu}^{\nu \lambda} + \left( \frac{a_1}{2} + \frac{a_3}{3} \right) \left( T_{\lambda\nu}^{\mu} - T_{\nu\lambda}^{\mu} \right) \right. \\
\left. - \left( \frac{a_1}{2} - a_2 \right) \left( \delta^\mu_{\nu} \delta^\lambda_{\lambda} - \delta^\mu_{\lambda} \delta^\lambda_{\nu} \right) \right] + \frac{a_4}{12} \left( \delta^\mu_{\nu} \epsilon_{\lambda\rho\sigma\tau} T_{\rho\sigma\tau} - \delta^\mu_{\lambda} \epsilon_{\nu\rho\sigma\tau} T_{\rho\sigma\tau} - 2 \epsilon_{\mu\nu\lambda\rho} \epsilon_{\nu\lambda\rho} \right) \right], \tag{64} \]

which satisfies the Møller condition (C). Since we have solved the field equation approximately at far distances, it is better to rewrite the superpotential (64) in the asymptotic form

\[ \mathcal{U}_{\mu}^{\nu \lambda} = \frac{2}{\kappa} \left[ \left( a_1 - \frac{a_3}{3} \right) b^\mu_{\nu \lambda} - \left( a_1 - 2a_2 \right) \left( \delta^\mu_{\nu} \partial^\lambda_{\lambda} b^\rho_{\rho} - b^\sigma_{\sigma \lambda} \delta^\nu_{\mu} \right) \right. \\
\left. + \left( a_1 + \frac{2a_3}{9} \right) \left( b^\lambda_{\nu \mu \lambda} - \partial^\nu_{\nu} b^\mu_{\nu \lambda} \right) + \frac{a_4}{3} \left\{ \delta^\mu_{\nu} \epsilon_{\lambda\rho\sigma\tau} b_{\rho\sigma\tau} + \epsilon_{\mu \nu \lambda \rho} b_{\lambda \rho} \right\} \right]. \tag{65} \]

keeping only \(O(1/r^2)\)-terms. Here we use only Greek indices and raise (or lower) them with the Minkowski metric: Thus, for example, \( b^\lambda_{\mu} = \delta^\lambda_{\mu} b_k^k \) and \( b^\nu_{\lambda \mu} = \eta^{\nu \rho} \partial^\xi b_{\rho \mu \lambda}. \) The formula for the energy and spatial momentum are given by\(^7\)

\[ E = - \lim_{r \to \infty} \int_{r = \text{const}} \mathcal{U}_0^{0\alpha} dS_{\alpha}, \tag{66} \]
\[ P^\alpha = \lim_{r \to \infty} \int_{r = \text{const}} \mathcal{U}_0^{0\beta} dS_{\beta}. \tag{67} \]

The gravitational Lagrangian is assumed to be a quadratic invariant of the torsion tensor, and accordingly the canonical complex \( t^\mu_{\nu} \) behaves as \(O(1/r^4)\) at far distances since the parallel vector fields are asymptotically given by (58)-(61). Thus, the Møller condition (B) is also satisfied, and the energy and momentum are transformed as a 4-vector.

Now let us calculate the energy by using the asymptotic form of the parallel vector fields. The necessary components of the superpotential are given by

\[ \mathcal{U}_0^{0\alpha} = \frac{2}{\kappa r^2} \left[ \left( a_1 + a_2 \right) + \frac{\left( 2a_2 - a_1 \right)^2}{9a_1 a_2} \right] m n_{\alpha} + \frac{3a_1 a_4}{2f_1} \epsilon_{\alpha\gamma\eta} n_{\beta} \hat{B}_{\gamma} \]
\[ = - \frac{2}{\kappa r^2} \left( m n_{\alpha} - \frac{3a_1 a_4}{2f_1} \epsilon_{\alpha\beta\gamma} n_{\beta} \hat{B}_{\gamma} \right), \tag{68} \]

where we have used the Newtonian approximation condition (10) to rewrite the first term inside the square bracket. It is easily shown that Eq. (68) satisfies \( \mathcal{U}_0^{0\alpha} = 0 \) up to \(O(1/r^3)\), as it should. Using (68) in (66) we find that the last term of (68) does not contribute to the integral and that the energy is given by

\[ E = m. \tag{69} \]
This shows that the most general Lagrangian (4) including a parity-violating term is consistent with the equivalence principle.

Next, we turn to the spatial momentum $P_\alpha$. The necessary components of the superpotential are given by

$$u_\alpha^{0\beta} = \frac{2}{\kappa f_1 r^2} \left[ \left( a_1 - \frac{4a_3}{9} \right) \left\{ \frac{3a_1}{2} n_\beta \dot{B}_\alpha - \frac{1}{2} (a_1 - 2a_2) \delta_{\alpha\beta} n_\gamma \dot{B}_\gamma \right\} ight] + \frac{2}{\kappa f_1} \left( a_1 + \frac{2a_3}{9} \right) (a_1 + a_2) n_{[\alpha} \dot{B}_{\beta]} - \tilde{a}_4 \frac{1}{6a_2} f_1 mn_{\rho} \epsilon_{\alpha\beta\rho} - \tilde{a}_4^2 \left( n_{[\alpha} \dot{B}_{\beta]} - \delta_{\alpha\beta} n_\gamma \dot{B}_\gamma \right) - \frac{1}{4} \left( f_1 + 3a_1 \left( a_2 + \frac{4a_3}{9} \right) \right) \left\{ 2n_{[\alpha} \dot{B}_{\beta]} + (\delta_{\alpha\beta} - 3n_\alpha n_\beta) n_\gamma \dot{B}_\gamma \right\}. \tag{70}$$

Taking the divergence of (70), we find that $u_\alpha^{0\beta} = 0$, which means that the leading term of the $u_\alpha^{0\beta}$ is of $O(1/r^4)$, as it should. Now we are ready to calculate the spatial momentum to determine if it is vanishing or not. Using (70) in (67), we find after a lengthy calculation

$$P_\alpha = \dot{B}_\alpha, \tag{71}$$

which shows that the constant vector $\dot{B}_\alpha$ has the physical meaning of the spatial momentum of the isolated system.

Since the isolated system is assumed to be stationary and as a whole at rest near the origin, its spatial momentum should be vanishing, and hence the constant vector $\dot{B}_\alpha$ must be zero. This means that the divergence $B_{0\alpha,\alpha}$ should be at most of $O(1/r^3)$ for parallel vector fields at distances far from a stationary, isolated system. As we have seen in the previous section, however, such an asymptotic behavior of $B_{0\alpha,\alpha}$ does not follow from the linearized field equation in vacuum. Accordingly the asymptotic condition,

$$\lim_{r \to \infty} B_{0\alpha,\alpha} = O(1/r^3), \tag{72}$$

must be imposed by hand, in addition to the condition (12) (or (13)). By contrast, as we noted below (45), the linearized field equation implies the condition $B_{\alpha\beta,\beta} = O(1/r^3)$.

The asymptotic condition (72) is indeed satisfied when the gravitational field is weak everywhere and the linearized field equation, which is just given by Eqs. (17) and (18) with the right-hand sides being replaced by $\kappa T_{(\mu\nu)}$ and $\kappa T_{[\mu\nu]}$, respectively, is valid throughout whole space. Here $T_{\mu\nu}$ is the energy-momentum tensor in the special relativistic limit and satisfies the conservation law $\partial_\mu T^{\mu\nu} = 0$. Multiplying both sides of the linearized field equation by $\partial^\nu$, and using the conservation law, we obtain the inhomogeneous Laplace equation for $B_{0\alpha,\alpha}$

$$\Delta B_{0\alpha,\alpha} = \kappa T_{[0\alpha],\alpha} = \frac{\kappa}{2} S_{0\alpha\beta,\alpha\beta}, \tag{73}$$

where $S^{\mu\nu\lambda}$ is the intrinsic spin tensor of the source, and we have used the Tetrode.
formula $T^{\mu \nu} = (1/2) \partial_{\lambda} S^{\mu \nu \lambda}$ in the last step. For an isolated system confined to a finite spatial region, the solution $B_{0 \alpha, \alpha}$ of (73) is of $O(1/r^3)$ for large $r$.

In the parity-conserving case with $a_4 = 0$ it is more convenient to use $A_{0 \alpha}$ and $A_{\alpha \beta}$ directly. Writing the asymptotic form of $A_{0 \alpha, \alpha}$, which satisfies the Laplace equation, as $B_{\alpha \alpha}/r^2$, Eq. (71) reads

$$P_{\alpha} = (a_1 + a_2) B_{\alpha}.$$  \hspace{1cm} (74)

As explained at the end of §3, the solution of $h_{\mu \nu}$ and $A_{\mu \nu}$ has a well-defined limit when we put $a_4 = 0$ and then $(a_1 + a_2) = 0$. Although the resultant solution involves the constant vector $B_{\alpha}$, the total spatial momentum is vanishing as is shown in (74). This situation can be understood as follows. In the special case $(a_1 + a_2) = 0 = a_4$, the linearized field equation for $A_{\mu \nu}$ is decoupled from that for $h_{\mu \nu}$, and therefore, the former equation is invariant under the gauge transformation

$$A'_{\mu \nu} = A_{\mu \nu} + \zeta_{[\mu, \nu]}$$  \hspace{1cm} (75)

with $\zeta_\mu$ representing arbitrary small functions. We can use this freedom to choose a gauge in which $A_{0 \alpha}$ and $A_{\alpha \beta}$ are vanishing up to $O(1/r)$. Thus, in this special case, the constant vector $B_{\alpha}$ is unphysical and can be eliminated by a gauge transformation.

§5. Conclusion and discussion

We have studied the energy and momentum of an isolated system in the tetrad theory of gravitation, starting from the most general Lagrangian that is quadratic in torsion and involves four unknown parameters $a_1, a_2, a_3$ and $a_4$, the last of which is associated with a parity-violating term. As a first application we considered the static, spherically symmetric case, where the parallel vector fields take a diagonal form. The solution of the field equation in vacuum is found to be the same as the exact solution of the parity-conserving case with $a_4 = 0$. This is due to the fact that the axial-vector part of the torsion tensor is identically vanishing for diagonal parallel vector fields.

The total energy and momentum of an isolated system are expressed by a surface integral over a large closed surface enclosing the system. It is then sufficient to know the asymptotic form of the parallel vector fields at distances far from the source, where the gravitational field is weak. In view of this, we analyzed the linearized field equation in vacuum which follows from the most general gravitational Lagrangian.

It is well known that in general relativity the Einstein equation ensures that the metric tensor at far distances is the same as the Schwarzschild metric up to $O(1/r)$ for any isolated stationary system. This is usually shown by solving the linearized Einstein equation in vacuum, which is the Laplace equation in stationary case and can easily be solved.

In our case, the linearized field equation in vacuum consists of 16 equations for the symmetric field $h_{\mu \nu}$ and the skew-symmetric field $A_{\mu \nu}$, and is invariant under a gauge transformation, which allows us to impose the harmonic condition on the
It is found, however, that the symmetric part and skew-symmetric part of the linearized field equation are coupled with each other unless the parameters satisfy \( a_1 + a_2 = a_4 \). Nevertheless, we can solve the coupled equation up to \( O(1/r) \) assuming that the parameters satisfy \( (a_1 + a_2)^2 + (a_4/3)^2 \neq 0 \) and \( (a_1 + a_2)(a_1 - 4a_3/9) + (a_4/3)^2 \neq 0 \). It is found that the general solution has two constants, the gravitational mass \( m \) and a constant vector \( \vec{B}_\alpha \). The general solution of the linearized field equation up to \( O(1/r) \) is, therefore, not spherically symmetric, due to the presence of \( \vec{B}_\alpha \).

Using the definition of energy-momentum complex given by Møller,\(^{12}\) we derive the superpotential \( U_{\mu}^{\nu\lambda} \) from the general Lagrangian. It transforms as a tensor density under general coordinate transformations. Using the general solution obtained, we give an explicit expression for the components \( U_0^{\alpha\alpha} \) and \( U_\alpha^{0\beta} \), which are shown to be divergenceless up to \( O(1/r^3) \), implying that the leading term of the divergence of those components is of order \( O(1/r^4) \).

From the components \( U_0^{\alpha\alpha} \) we calculate the total energy, and show that the result is equal to the gravitational mass of the isolated system, i.e. \( E = m \). Using the components \( U_\alpha^{0\beta} \), we then calculate the spatial momentum, and find that \( P_\alpha = \vec{B}_\alpha \). Thus, we arrive at the result that the general solution at far distances is characterized by the total energy and the spatial momentum of the isolated system under consideration.

The asymptotic solution with vanishing \( \vec{B}_\alpha \) is acceptable as a description of the parallel vector fields far from a stationary isolated system which is at rest as a whole. It indicates that the total momentum of the isolated system at rest is vanishing. The asymptotic solution coincides with the exact solution with spherical symmetry up to \( O(1/r^4) \).

It is not yet clear whether or not the asymptotic solution with \( \vec{B}_\alpha \neq 0 \) actually describes the asymptotic behavior of exact solutions. Such exact solutions, if they exist, should describe exotic systems which have nonvanishing total momentum although at rest as a whole.

The vanishing of the constant vector \( \vec{B}_\alpha \) implies that the divergence \( B_{0\alpha,\alpha} \) of \( B_{0\alpha} = (a_1 + a_2)A_{0\alpha} + (a_4/3)\vec{A}_{0\alpha} \) behaves at most as \( O(1/r^3) \) for large \( r \). This asymptotic behavior does not follow from the linearized field equation alone. By contrast, the asymptotic behavior of \( B_{\alpha\beta} = (a_1 + a_2)A_{\alpha\beta} + (a_4/3)\vec{A}_{\alpha\beta} \) is more severely governed by the linearized field equation: In fact, it is shown that \( B_{\alpha\beta,\beta} \) vanishes up to \( O(1/r^2) \).

For a weakly gravitating source, for which the field is weak everywhere and the weak field approximation can be applied, we can show that \( B_{0\alpha,\alpha} \) behaves as \( O(1/r^3) \). For the fully relativistic case we do not know the reason why \( B_{0\alpha,\alpha} \) should behave in this manner. Therefore, further study is required to establish the asymptotic condition of the parallel vector fields, in particular, of the skew-symmetric field \( A_{\mu\nu} \).

The study of the weak field case also needs more investigation. Although in the present paper we solved the linearized field equation for the stationary case, the same procedure may be applicable also for the time-dependent case. This will be studied in future work.

The present analysis is based on the assumption that the linearized theory can
Energy and Momentum in the Tetrad Theory of Gravitation

be applied at distances far from the source. The solution obtained by Mikhail et al.\textsuperscript{21} which violates $E = m$, does not satisfy this assumption. It can also be shown that their solution does not satisfy the Møller condition (B), because the divergence $\mathcal{U}_a^{\alpha_3, \beta}$ is $O(1/r^{5/2})$ for their solution.

Acknowledgements

One of the authors (G. N.) would like to thank the Japanese Government for supporting him with a Monbusho Scholarship and also wishes to express his deep gratitude to all the members of Physics Department at Saitama University, in particular, Professor K. Kobayashi, Professor Y. Tanii, Professor K. Tanabe and Professor N. Yoshinaga.

Appendix A

---

Solution of an Inhomogeneous Laplace Equation

Here we derive the general solution of an inhomogeneous Laplace equation of the form

$$\Delta U(x, y, z) = \frac{1}{r^3} \sum_{N=1}^{\infty} A_{\alpha_1 \alpha_2 \cdots \alpha_N} n^{\alpha_1} n^{\alpha_2} \cdots n^{\alpha_N}, \tag{A.1}$$

where $A_{\alpha_1 \alpha_2 \cdots \alpha_N}$ is totally symmetric and traceless, and $n^\alpha$ represents for the unit vector $x^\alpha / r$. We look for a solution in the form

$$U(x, y, z) = \frac{1}{r} \sum_{N=0}^{\infty} B_{\alpha_1 \alpha_2 \cdots \alpha_N} n^{\alpha_1} n^{\alpha_2} \cdots n^{\alpha_N}, \tag{A.2}$$

where $B_{\alpha_1 \alpha_2 \cdots \alpha_N}$ is totally symmetric and traceless. Applying the Laplacian operator on (A.2), and substituting the result in (A.1) we obtain

$$U(x, y, z) = \frac{B}{r} - \frac{1}{r} \sum_{N=1}^{\infty} \frac{1}{N(N+1)} A_{\alpha_1 \alpha_2 \cdots \alpha_N} n^{\alpha_1} n^{\alpha_2} \cdots n^{\alpha_N}, \tag{A.3}$$

where $B$ is an arbitrary constant. Here we have used the following proposition:

$$\sum_{N=1}^{\infty} A_{\alpha_1 \alpha_2 \cdots \alpha_N} n^{\alpha_1} n^{\alpha_2} \cdots n^{\alpha_N} = 0 \quad \Rightarrow \quad A_{\alpha_1 \alpha_2 \cdots \alpha_N} = 0, \tag{A.4}$$

when $A_{\alpha_1 \alpha_2 \cdots \alpha_N}$ is totally symmetric and traceless.

References

   The earlier version of Møller's requirements can be found in Ref. 10 cited below.
1627.
24) C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco,