and proposed a "zero-volume change" vibration mode for dynamic stability analysis.

The expression of equation (37d) contains an error as pointed out. However, all the numerical calculation was carried out by using the correctly expressed equation.

Next, let us reexamine Ishii's result for $M \neq 0$ in his Appendix. When $M \neq 0$ and $k = 0$, we obtain from equation (17) of our paper

$$\Phi^*(M \neq 0) = iU\xi/(\beta|\xi|)W^* \exp(-\zeta y)$$

because of $\zeta = \beta|\xi|$, while

$$\Phi^*(M = 0) = iU\xi/|\xi|W^* \exp(-\zeta y)$$

because of $\zeta = |\xi|$ for $M = k = 0$. Comparison between the equations yields $\Phi^*(M \neq 0) = (1 - M^2)^{-1/2}\Phi^*(M = 0)$. Therefore, it does not seem that Ishii's result is incorrect as far as the point just raised is concerned. However, we should repeat that his incompressible aeroforces are invalid for a symmetric natural mode when $k \neq 0$.

As for the proposed paper on the stability analysis, it has already been accepted for publication in this JOURNAL, and will soon be published [9]. In this area, several important problems remain to be resolved. It reexamines again an old and well-known problem, that is, the possibility of postdivergence flutter oscillation which is probably the most controversial point. Since a brief description of this topic is given in [9], we would refrain from repeating it here.

Finally, we'd like to indicate another typographical error: although the explanation in the text is correct, $k = 0$ should read $k = 1$ in the caption of Table 2 of our paper.

Reference

9 Matsuzaki, Y., "Reexamination of Stability of a Two-Dimensional Finite Panel Exposed to an Incompressible Flow," to be published in the ASME JOURNAL OF APPLIED MECHANICS.

A Correct Definition of Elastic and Plastic Deformation and Its Computational Significance[1]

J. Casey[2] and P. M. Naghdi.[3] Our purpose here is to discuss two points raised in a paper of Lubarda and Lee[1] and again in a more recent paper of Lee [1]. These points, which are both related to the use of full invariance requirements in plasticity, in the manner proposed by Green and Naghdi [2] are as follows:

(a) Whether it is necessary to demand independent invariance requirements for intermediate stress-free configurations associated with the decomposition of the deformation gradient in the form $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$; and

(b) whether it is possible to specify $\mathbf{F}_e$ to be a symmetric positive definite tensor in the construction of a general theory.[4]

With regard to (a), Lubarda and Lee[1] and again Lee [1] demand that only partial invariance requirements be satisfied. Similarly, with reference to (b), they assume that without loss in generality the elastic part $\mathbf{F}_e$ of the deformation gradient $\mathbf{F}$ can be chosen to be symmetric. We disagree with the contents of the authors' paper[1] and reference [1] insofar as (a) and (b) are concerned.

[1] By V. A. Lubarda and E. H. Lee, and published in the March, 1981, issue of the ASME JOURNAL OF APPLIED MECHANICS, Vol. 48, pp. 35–40.

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[4] By a "general theory," we mean one that is at least broad enough to include anisotropic materials and to be valid for all possible motions of the elastic-plastic body.

In our recent Brief Note [3], we emphasized the physical grounds for invoking full invariance requirements of the type proposed by Green and Naghdi [2], whenever use is made of intermediate stress-free configurations. Lubarda and Lee (footnote 1, p. 37 after equation (33)) appeal to only a restricted form of invariance requirements, and more recently Lee [1, p. 867] has objected to the use of full invariance requirements because they seem to him "to demand two objectivity checks for the single [current] configuration" and therefore to involve a redundancy. We discuss the two points (a) and (b) above using the notation of [3].

The invariance requirements in [2, 3] are based on the idea that all configurations, global or local, which differ from any physically possible configuration of a body by a rigid displacement are equivalent. All such configurations are physically indistinguishable and should therefore play an equal role in any theory of material behavior. The invariance requirements insure that these configurations enter the theory in a physically meaningful way. The intermediate stress-free configuration, if it exists, *is* a possible configuration of the body and can, in the words of Lee [1, p. 863], "be achieved physically by . . . *destressing*." It was *precisely because the intermediate stress-free* configuration (whenever it exists) is physically realizable, that in [2, 3] it was assumed that this configuration is subject to *exactly the same* invariance requirements as any other possible configuration of the body. As a consequence, two different orthogonal tensors must appear in the transformation rules for certain fields. For example, when the present configuration and the intermediate stress-free configuration are subjected at time t to arbitrary independent rigid displacements, involving rotations $\mathbf{Q}(t)$ and $\bar{\mathbf{Q}}(t)$, respectively, the tensor $\mathbf{F}_e$ is transformed into

$$\mathbf{F}_e^+ = \mathbf{Q}(t)\mathbf{F}_e\bar{\mathbf{Q}}^T(t). \tag{1}$$

The reason for the appearance of two different rigid body rotation tensors in (1) is simple and may be explained even without appealing to either the stress-free nature of the intermediate configuration, or the elastic-plastic character of the body. To see this, consider any deformable body which is first deformed from a reference configuration κ_0 into a configuration κ_1, and then from κ_1 into the present configuration κ. The deformation gradient $\mathbf{F}$ in the configuration κ can then be expressed as[5]

$$\mathbf{F} = \mathbf{F}_2\mathbf{F}_1 \tag{2}$$

where $\mathbf{F}_1$ is the deformation gradient in the configuration κ_1 and $\mathbf{F}_2$ is the gradient in the configuration κ relative to the configuration κ_1. Since both κ_1 and κ are physically realizable configurations of the body, they are both subject to the usual invariance requirements. If κ_1 is replaced at time t_1 by a configuration κ_1^+ that differs from it by an arbitrary rigid displacement whose rotation tensor is $\mathbf{Q}_1(t_1)$, then $\mathbf{F}_1$ is transformed into

$$\mathbf{F}_1^+ = \mathbf{Q}_1(t_1)\mathbf{F}_1. \tag{3}$$

Similarly, if κ is replaced at the present time t by a configuration κ^+ that differs from κ by an arbitrary rigid displacement whose rotation tensor is $\mathbf{Q}(t)$, $\mathbf{F}$ is transformed into

$$\mathbf{F}^+ = \mathbf{Q}(t)\mathbf{F}. \tag{4}$$

The gradient in the configuration κ^+ relative to κ_1^+ is again given by the chain rule in the form

$$\mathbf{F}_2^+ = \mathbf{F}^+(\mathbf{F}_1^+)^{-1}. \tag{5}$$

From (2)–(5), it follows that

$$\mathbf{F}_2^+ = \mathbf{Q}(t)\mathbf{F}_2\mathbf{Q}_1^T(t_1). \tag{6}$$

Since the rigid displacements by which κ_1 is taken into κ_1^+ and κ is

[5] Since the two configurations κ_1 and κ are global, (2) is obtained by means of the chain rule. A similar form holds if κ_1 is only a local configuration.

taken into κ^+ are arbitrary, $\mathbf{Q}(t)$ will not equal $\mathbf{Q}_1(t_1)$ in general. Thus, even though only the usual invariance requirements have been invoked, we still must include two unequal orthogonal tensors in the transformation rule (6) for $\mathbf{F}_2$.

In plasticity, the intermediate configuration κ_1 is a local stress-free configuration, but since it is physically realizable whenever it exists, it is subject to the transformation rule (3). Consequently, in plasticity the tensor $\mathbf{F}_e$ (being the analogue of $\mathbf{F}_2$ above) satisfies the transformation rule (1), which has the same form as (6) with $\mathbf{Q}_1(t_1) = \bar{\mathbf{Q}}(t)$ and with t_1 differing from t by a constant.[6] Rather than involving a redundancy, the invariance requirements of [2, 3] treat all configurations of the body in the same manner. Lubarda and Lee[1] and Lee [1] are not consistent in their treatment of the intermediate stress-free configuration and the present configuration, since they assume that the intermediate stress-free configurations can be subjected only to *special* superposed rigid displacements, whereas the present configuration may be subjected to *arbitrary* rigid displacements. Indeed, in the invariance requirements used in Lubarda and Lee[1] and Lee [1], $\bar{\mathbf{Q}}(t)$ in (1) is taken equal to $\mathbf{Q}(t)$. This obviously involves a loss of generality and is in fact basically no different than what other authors (see [3] for references) have done in choosing $\bar{\mathbf{Q}}(t) = \mathbf{I}$, $\mathbf{I}$ being the identity tensor. As will be seen at the end of our discussion of the second point (b), the restriction $\mathbf{Q}(t) = \bar{\mathbf{Q}}(t)$ which Lubarda and Lee[1] place on their invariance requirements is closely related to their assumption that $\mathbf{F}_e$ be chosen symmetric.

It is appropriate to remark here on a further puzzling aspect of Lee's discussion of invariance requirements in [1]. While Lee admits that the intermediate stress-free configuration is a possible configuration,[7] he later regards it [1, p. 867] as involving only a "thought experiment" and claims [1, p. 866]: "The unstressed state . . . was envisaged and analyzed in order to select convenient variables with which to express the theory, but was not actually achieved during the stressing so that it is not necessary to impose a test of objectivity on that configuration."

Is Lee's distinction in regard to invariance requirements, between those assumed for the actual configurations and for possible configurations, valid? To examine this question, let us consider again the configurations κ and κ_1 that were introduced above in connection with the motion of a general deformable body. Thus, in another motion, let the body once more occupy the configuration κ but suppose that in this motion it did not pass through the configuration κ_1. (This thought experiment is a perfectly valid one.) For whatever purposes desired, one may still use the configuration κ_1 to express the deformation gradient $\mathbf{F}$ as a product in the form (2). Should we now alter the transformation rule (3)? Certainly not, since the configuration κ_1 is still a possible configuration, and must therefore enter the theory in a physically meaningful way. What matters is not whether a configuration is actually occupied in a given motion, but only that it can possibly be occupied in some motion. The transformation rule (6), and its analogue (1) for an elastic-plastic body, should therefore be retained.

Turning now to the second point (b), we recall that a polar decomposition of $\mathbf{F}_e$ yields

$$\mathbf{F}_e = \mathbf{R}_e \mathbf{M}_e, \tag{7}$$

where the tensors $\mathbf{R}_e$ and $\mathbf{M}_e$ are proper orthogonal and symmetric positive definite, respectively. Lubarda and Lee[1] (page 36) state: "For analytical convenience, and with no basic loss of generality, we take the elastic deformation gradient $\mathbf{F}_e$, associated with destressing, to be rotation free and hence given by $\mathbf{M}_e$, a symmetric matrix." Similar statements are repeated by Lubarda and Lee (see footnote 1, p. 36, before equations (14)) and reference [1, p. 864]. Corresponding to their assumption, Lubarda and Lee[1] and Lee [1] choose $\mathbf{R}_e = \mathbf{I}$, a choice which violates the invariance requirements of [2, 3]. Actually, this issue was discussed briefly in [3, p. 674]. To elaborate, even if for a particular choice of intermediate stress-free configuration $\mathbf{F}_e$ is symmetric, then by the transformation rule (1), it is not true that $\mathbf{F}_e$ will be symmetric in all intermediate stress-free configurations that can be obtained from the first by arbitrary rigid displacements; an inspection of (1) shows that even if $\mathbf{F}_e$ is symmetric, $\mathbf{F}_e^+$ will not be symmetric, since in general $\mathbf{Q}(t)$ will not be equal to $\bar{\mathbf{Q}}(t)$. In this connection, it should be noted that the statements "choose $\mathbf{F}_e$ symmetric positive definite" and "choose $\mathbf{F}_p$ symmetric positive definite" are not physically meaningful in that they do not contain invariant ideas. As regards point (b), we conclude that the invariance requirements in [2, 3] imply that $\mathbf{F}_e$ cannot be chosen symmetric in the context of a general theory of plasticity.

Lubarda and Lee[1] justify the restriction $\mathbf{Q}(t) = \bar{\mathbf{Q}}(t)$ on their invariance requirements on the grounds of the admissibility of the choice $\mathbf{F}_e = \mathbf{M}_e$, stating (footnote 1, p. 37): "Since elastic destressing is considered to occur without rotation, each element of the unstressed configuration must be subjected to the same rotation, $\mathbf{Q}(t)$, as the current configuration and this constraint must be introduced into the objectivity requirements." Lubarda and Lee[1] recognize the interconnection between points (a) and (b) and their special assumption that $\mathbf{F}_e$ can be chosen symmetric compels them to adopt restricted invariance requirements. In order to satisfy full invariance requirements their choice $\mathbf{F}_e = \mathbf{M}_e$ must be abandoned.

[6] The invariance requirements always permit a translation in the time variable.

[7] See the quoted phrase in the fourth paragraph of this Discussion.

References

1 Lee, E. H., "Some Comments on Elastic-Plastic Analysis," *International Journal of Solids and Structures*, Vol. 17, 1981, pp. 859–872.

2. Green, A. E. and Naghdi, P. M., "Some Remarks on Elastic-Plastic Deformation at Finite Strain," *International Journal of Engineering Sciences*, Vol. 9, 1971, pp. 1219–1229.

3 Casey, J., and Naghdi, P. M., "A Remark on the Use of the Decomposition $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ in Plasticity," ASME JOURNAL OF APPLIED MECHANICS, Vol. 47, 1980, pp. 672–675.

Authors' Closure

We thank Professors Casey and Naghdi for providing an opportunity to discuss our differences in approach to this question. We agree with the discussers that, in considering a constitutive relation, objectivity or invariance requirements must apply to all physically achievable states falling within its purview. In the paper under discussion we attained this objective by establishing objectivity of an arbitrarily deformed state. This avoids becoming involved in the unnecessarily elaborate mathematical structure used by the discussers in establishing objectivity simultaneously for many states, i.e., the state associated with general deformation and those associated states obtained from it by destressing. Objectivity of the latter states falls within the scope of the proof covering general deformation and there is thus no need to consider it separately.

We should perhaps clarify our use of the concept "physically achievable" for an elastic-plastic state. Since, apart from homogeneous deformation, virtually all elastic-plastic loading of a body will leave it in a state of residual stress when the loads are removed, the corresponding unstressed state can only be achieved by considering the body to be divided into infinitesimal elements. This corresponds to the standard method of measuring residual stresses by machining away parts of the body and measuring the consequent strain change in the remainder. In practice the elements cannot be reconstructed to reform the body because of the material lost in the form of machining chips, but in principle this could be done and is a useful concept in constructing a constitutive relation. Complete investigation of the constitutive relation can be obtained by considering only homogeneous deformation states and then the residual aspect and the appearance of nondifferentiable maps does not arise.

In the subject paper the condition at an arbitrary point on a general elastic-plastic deformation path, for a material element of the body, is analyzed in order to determine the constitutive relation connecting the stress and deformation histories to which the element has been subjected. Because of the incremental or flow-type nature of the plasticity law, strain *rate* appears in the constitutive relation. For application in stress and deformation analysis, total strain rate, and stress and stress rate are needed in the relation. Such a constitutive law is deduced from the kinematic relation expressing the total strain rate in terms of the elastic and plastic components, by substituting from the plasticity law and the time derivative of the usual elasticity law for the plastic and elastic strain rates, respectively. This yields a constitutive relation for the total strain rate in terms of the stress, stress rate, and variables expressing the history of deformation.

For the case when purely elastic deformation could occur in destressing to zero stress, the choice of destressing by pure deformation without rotation, expressed by the symmetric deformation gradient matrix $\mathbf{V}^e$, constitutes a correct definition of elastic deformation, although not the most general one since arbitrary rigid-body rotation of the unstressed state provides an alernative unstressed state. However, utilizing this particular choice of elastic deformation does permit convenient elimination of the elastic strain-rate term in the kinematic relation for total strain. Any rotation which might have been included in the elastic deformation is automatically embodied in the associated plastic deformation and is determined by the solution process to provide whatever compatability is needed in the configuration of the body on the general deformation path. Thus a correct, but not the most general, definition of elastic and plastic deformation is achieved.

In order to investigate objectivity of the resulting constitutive equation (to be used in stress and deformation analysis) which contains stress-rate and total strain-rate variables, we need only consider superposed arbitrary rigid rotation on the current state of the body on the general deformation path. This is done in the paper under discussion and the constitutive relation is found to be objective.

Since this applies for any general deformation history, we can choose the deformation path to follow pure elastic destressing with or without rotation from the considered state on the general deformation path, and this establishes objectivity of the unstressed state for rotations independent of that utilized in the objectivity check carried for the body on the general deformation path prior to destressing. Thus we established the full invariance requirement stressed by the discussers on the basis of a simpler analytical development. We thus assert that the statement that we "demand that only partial invariance requirements be satisfied" is incorrect. The fact that it is necessary to employ the constitutive relation containing only the current rate of total strain to substitute into the usual variational principle for determination of the velocity field eliminates the need to simultaneously include the plastic deformation gradient in establishing objectivity for any state. Although the subject paper considers only isotropic elastic and plastic laws, the approach explained in this Closure can be utilized for anisotropic laws by taking into account rotation of the anisotropic characteristics with the deformation of the unstressed state.

Professors Casey and Naghdi maintain that even when the intermediate (unstressed) configuration is not occupied in a given motion, the transformation law involving simultaneous independent rotations of the intermediate and final states must be included in the analysis. In this connection it is interesting to consider the analogous problem of the evaluation of thermal stresses. The theory can be formulated by considering an intermediate state in which the temperature variation occurs, and hence thermal expansion, but the condition of zero stress is maintained. This configuration would comprise an incompatible state of strain and hence a discontinuous map. Superimposing the thermal stresses then produces a compatible final configuration which can be expressed by a differentiable mapping from the initial undisturbed state at uniform temperature. The geometrical characteristics of the intermediate state have much in common with those of the unstressed state in elastic-plastic theory, yet to our knowledge the alleged difficulties often ascribed to the intermediate unstressed state in elastic-plastic theory have never been mentioned in the thermal stress case. Nor have we heard of prescribing an involved objectivity principle including independent rotations of the thermally expanded configuration and the final thermally stressed configuration. We are not surprised by this omission since it seems natural to us that since the purely thermally expanded configuration does not appear in the final thermoelastic constitutive relation, no advantage would accrue to offset the additional mathematical complexity of including independent rotation of the thermally expanded intermediate state.

We take this opportunity to mention that typographical errors appeared in equations (22), (40), and (61) of the subject paper and they have been corrected in an Erratum which appeared in the September, 1981, issue of the ASME JOURNAL OF APPLIED MECHANICS.

Basic Transport Equations in Ascending Equiangular Spiral Polar Coordinates[1]

C.-Y. Wang.[2] In this paper,[1] Dr. Ali stated that the torous seems to be the only curved tube which has been analyzed. This is not true. Sinusoidal, once-curved and twice-curved planar tubes (center line lies in a plane) have been studied by Murata, Miyake, and Inaba [1]. Helical tubes were recently investigated by Wang [2].

As soon as the Christoffel symbols for a curvilinear system are obtained, it is fairly easy to produce the field equations through tensors. Murata, et al. [1], developed the coordinate system for the general planar curved tube which would include Ali's work as a special case. The planar coordinates are always orthogonal. Wang [3] developed the coordinate system for the general nonplanar curved tube (arbitrary spatial curvature of the center line) which includes both the helical coordinates and Murata's coordinates as special cases.

It is uncertain that the ascending equiangular spiral polar coordinates are suitable for analytical and numerical work as claimed. This kind of geometry does not lead to an asymptotic state and conditions at both ends highly affect the field properties. Both analytical or numerical attempts would be very difficult.

References

1 Murata, S., Miyake, Y., and Inaba, T., "Laminar Flow in a Curved Pipe With Varying Curvature," *Journal of Fluid Mechanics*, Vol. 73, 1976, pp. 735–752.

2 Wang, C.-Y., "The Helical Coordinate System and the Temperature Distribution Inside a Helical Coil," ASME JOURNAL OF APPLIED MECHANICS, Vol. 47, 1980, pp. 951–953.

3 Wang, C.-Y., "On the Low Reynolds Number Flow in a Helical Pipe," *Journal of Fluid Mechanics*, Vol. 108, 1981, pp. 185–194.

[1] By S. Ali, and published in the March, 1981, issue of the ASME JOURNAL OF APPLIED MECHANICS, Vol. 48, pp. 190–192.

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On the Uniqueness and Stability of Endochronic Theory[1]

I. S. Sandler.[2] In the paper,[1] the author presents several conclusions with respect to the uniqueness and stability properties of endochronic theories based on considerations first presented in [1]. Some of the conclusions of the author's paper,[1] however, are incorrect and

[1] By B. J. Hsieh, and published in the December, 1980, issue of the ASME JOURNAL OF APPLIED MECHANICS, Vol. 47, pp. 748–754.

[2] Weidlinger Associates, Consulting Engineers, 110 East 59 St., New York, N. Y. 10022.