Approximate Sum Rules of CKM Matrix Elements from Quasi-Democratic Mass Matrices

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To extract sum rules of CKM matrix elements, eigenvalue problems for quasi-democratic mass matrices are solved in the first order perturbation approximation with respect to small deviations from the democratic limit. Mass spectra of up and down quark sectors and the CKM matrix are shown to have clear and distinctive hierarchical structures. Numerical analysis shows that the absolute values of calculated CKM matrix elements fit the experimental data quite well. The order of the magnitude of the Jarlskog parameter is estimated by the relation $|J| \approx \sqrt{2}(m_c/m_t + m_s/m_b)|V_{us}|^2|V_{cb}|/4$.

§1. Introduction

Fundamental fermions, quarks and leptons, exist with a broad mass spectrum, ranging from zero or almost zero masses of neutrinos to 180 GeV of the top quark. Magnitude of the weak Cabbibo-Kobayashi-Maskawa (CKM) matrix elements tends to decrease rapidly from the diagonal to the off-diagonal direction. It is impossible to explain such hierarchical structures of fundamental fermions within the framework of the standard model. To do so it is necessary to postulate some working hypothesis on the mass matrices from outside of the model.

Among various forms of mass matrices, democratic mass matrices with small correction terms explain such hierarchical structures in a simple and systematic way. The calculability of the CKM matrix elements in terms of quark mass ratios is examined under the hypothesis of the universal strength of Yukawa couplings in which all Yukawa coupling constants are assumed to have equal moduli. It was shown that it is possible to find the CKM matrix elements from quasi-democratic mass matrices which are consistent with present experimental data to high accuracy. In this article, sum rules of the CKM matrix elements are extracted from Hermitian quasi-democratic mass matrices which have small deviation terms and phases in off-diagonal elements. For this purpose we solve mass eigenvalue problems in a first order perturbation approximation with respect to small deviations around the democratic limit.

We postulate here that mass matrices for up and down quark sectors are quasi-democratic and represented generically in the forms

$$\mathcal{M}_q = M_q \tilde{\Omega}_q, \quad (q = u, d)$$

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where $M_q$ is a mass scale for the $q$-sector and $\tilde{\Omega}_q$ is the Hermitian matrix

$$
\tilde{\Omega}_q = \frac{1}{3} \begin{pmatrix}
1 & a_2^q e^{i\delta_{12}} & a_3^q e^{-i\delta_{12}} \\
a_3^q e^{-i\delta_{12}} & 1 & a_1^q e^{i\delta_{23}} \\
a_2^q e^{i\delta_{23}} & a_1^q e^{-i\delta_{23}} & 1
\end{pmatrix}
$$

(1.2)

with phases satisfying the restriction

$$
\delta_{12}^q + \delta_{23}^q + \delta_{31}^q = 0.
$$

(1.3)

The real parameters $a_j^q$ and $\delta_{jk}^q$ are presumed to take values which deviate slightly from the democratic limit $a_j^q = 1$ and $\delta_{jk}^q = 0$.

Note that, without loss of generality, rephrasing of chiral quark fields enables us to reduce the phases $\delta_{jk}^q$ as

$$
\delta_{12}^q = 0, \quad \delta_{23}^q = \phi_q, \quad \delta_{31}^q = -\phi_q
$$

(1.4)

by a single real number $\phi_q$ for each $q$-sector. This type of mass matrices was investigated first by Branco, Silva and Rebelo $^{15}$ for the quark masses squared in their study of the universal strength for Yukawa coupling constants. Then Teshima and Sakai $^{19}$ applied it to quark masses and carried out numerical analysis.

In the first order perturbation approximation, we obtain CKM matrix elements parameterized by quantities which naturally describe small deviations around the democratic limit. We obtain mass spectra of up and down quark sectors and the CKM matrix. These spectra have distinctive hierarchical structures. It is possible to express the absolute values of the CKM matrix elements in terms of three parameters. Consequently we are able to derive six independent sum rules for the absolute values of nine matrix elements. All of these agree well with available experimental data. In §2 we formulate and solve the mass eigenvalue problems for quasi-democratic quark mass matrices. In §3, we obtain the CKM matrix and extract sum rules from it. Comparison of our results with the experimental data is made in §4, and discussion is given in §5.

§2. Mass eigenvalue problems

It is possible to obtain exact solutions of the eigenvalue problem for the matrix $\tilde{\Omega}_q$. $^{18}$ However, the exact eigenvectors take forms which are too complicated to be used to extract simple and physically-meaningful analytical relations among the CKM matrix elements. Hence we attempt here to solve the eigenvalue problems in a first order perturbation approximation with respect to small parameters $\lambda_q \propto a_1^q - a_2^q$. For this purpose it turns out to be convenient to parameterize $a_j^q$ as

$$
a_1^q = \frac{3}{2\sqrt{2}} \rho_q \sin \theta_q + 3\lambda_q, \quad a_2^q = \frac{3}{2\sqrt{2}} \rho_q \sin \theta_q - 3\lambda_q, \quad a_3^q = 3\rho_q \cos \theta_q,
$$

(2.1)

and then to decompose the matrix $\tilde{\Omega}_q$ into unperturbative and perturbative parts as

$$
\tilde{\Omega}_q = \Omega_q + \Lambda_q,
$$

(2.2)
where

\[
\Omega_q = \frac{1}{3} \begin{pmatrix}
1 & 3\rho_q \cos \theta_q & 3 \rho_q \sin \theta_q e^{i\phi_q} \\
3\rho_q \cos \theta_q & 1 & 3 \rho_q \sin \theta_q e^{i\phi_q} \\
\frac{3}{2\sqrt{2}}\rho_q \sin \theta_q e^{-i\phi_q} & \frac{3}{2\sqrt{2}}\rho_q \sin \theta_q e^{-i\phi_q} & 1
\end{pmatrix}
\] (2.3)

and

\[
\Lambda_q = \lambda_q \begin{pmatrix}
0 & 0 & -e^{i\phi_q} \\
0 & 0 & e^{i\phi_q} \\
-e^{-i\phi_q} & e^{-i\phi_q} & 0
\end{pmatrix}.
\] (2.4)

In this parametrization the democratic limit is realized when

\[
\rho_q = 1, \quad \cos \theta_q = \frac{1}{3}, \quad \lambda_q = 0, \quad \phi_q = 0.
\] (2.5)

Under the assumption \( \lambda_q \ll 1 \), which is confirmed later to be valid for both up and down quark sectors, we solve the eigenvalue problems

\[
\bar{\Omega}_q \tilde{v}_j^q = \tilde{\omega}_j^q \tilde{v}_j^q \quad (q = u, d)
\] (2.6)

in a perturbative approximation. The unperturbed eigenvalue problems

\[
\Omega_q v_j^q = \omega_j^q v_j^q \quad (q = u, d)
\] (2.7)

are readily solved with the eigenvalues \( \omega_j^q \) and eigenvectors \( v_j^q \) in the forms

\[
\omega_1^q = \frac{1}{3} - \rho_q \cos \theta_q : \quad v_1^q = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 \\
-1 \\
0
\end{pmatrix},
\]

\[
\omega_2^q = \frac{1}{3} + \frac{1}{2} \rho_q (-1 + \cos \theta_q) : \quad v_2^q = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sin \frac{\theta_q}{2} \\
\sin \frac{\theta_q}{2} \\
-\sqrt{2} e^{-i\phi_q} \cos \frac{\theta_q}{2}
\end{pmatrix},
\]

\[
\omega_3^q = \frac{1}{3} + \frac{1}{2} \rho_q (1 + \cos \theta_q) : \quad v_3^q = \frac{1}{\sqrt{2}} \begin{pmatrix}
\cos \frac{\theta_q}{2} \\
\cos \frac{\theta_q}{2} \\
\sqrt{2} e^{-i\phi_q} \sin \frac{\theta_q}{2}
\end{pmatrix}.
\] (2.8)

In the first order perturbation with respect to the parameter \( \lambda_q \), the eigenvalues and eigenvectors of \( \bar{\Omega}_q \) are given by

\[
\tilde{\omega}_j^q = \omega_j^q + v_j^q \Lambda_q v_j^q \] (2.9)
and

$$\tilde{v}_j^q = N_{qj} \left( v_j^q - \sum_{k \neq j} \frac{v_k^q \Lambda_q v_j^q}{\omega_k^q - \omega_j^q} v_j^q \right),$$  \hspace{1cm} (2.10)$$
where $N_{qj}$ is the normalization constant fixed below. For these formulae to be acceptable, the parameters $\lambda_q$ must satisfy the restriction

$$\lambda_q^2 \ll (\omega_j^q - \omega_k^q)^2$$  \hspace{1cm} (2.11)$$
for any pairs $(j, k)$ of generations.

Since $v_j^q \Lambda_q v_j^q = 0$ for all $j$, the perturbation $\Lambda_q$ does not affect the eigenvalues at all, i.e.,

$$\tilde{\omega}_j = \omega_j.$$  \hspace{1cm} (2.12)$$
Therefore, the ratios of the quark masses $m_j^q = M_q \tilde{\omega}_j^q$, where $M_q = \sum_j m_j^q$, are related to those of the eigenvalues $\omega_j^q$ in Eq. (2.8) by

$$\frac{m_j^q}{m_k^q} = \frac{\omega_j^q}{\omega_k^q}, \quad (j, k = 1, 2, 3)$$  \hspace{1cm} (2.13)$$

Solving Eqs. (2.8) and (2.13) in terms of the parameters $\rho_q$ and $\theta_q$, we obtain

$$\rho_q = \frac{m_3^q - m_2^q}{m_3^q + m_2^q + m_1^q}$$  \hspace{1cm} (2.14)$$
and

$$\cos \theta_q = \frac{1}{3} \left( 1 + 2 \frac{m_2^q - m_1^q}{m_3^q - m_2^q} \right).$$  \hspace{1cm} (2.15)$$

To examine the behaviour of the eigenvalues and eigenvectors in response to small deviations from the democratic limit, it is useful to introduce parameters $\delta_q$ defined as

$$\delta_q^2 = \frac{m_2^q - m_1^q}{m_3^q - m_2^q}.$$  \hspace{1cm} (2.16)$$
With these parameters, we find the expressions

$$\cos \frac{\theta_q}{2} = \sqrt{\frac{2}{3}} \left( 1 + \frac{1}{2} \delta_q^2 \right)^{1/2}, \quad \sin \frac{\theta_q}{2} = \sqrt{\frac{1}{3}} \left( 1 - \delta_q^2 \right)^{1/2}. \hspace{1cm} (2.17)$$

Therefore the eigenvectors in Eq. (2.8) are determined by the parameters $\delta_q$ and the unknown phases $\phi_q$.

Using the experimental fact $m_1^q \ll m_3^q$, we estimate

$$\rho_q = \left( 1 + 2 \delta_q^2 + \frac{3m_1^q}{m_3^q - m_2^q} \right)^{-1} \simeq 1 - 2 \delta_q^2,$$  \hspace{1cm} (2.18)$$
which results approximately in the hierarchical mass spectrum

$$m_1^q : m_2^q : m_3^q = \omega_1^q : \omega_2^q : \omega_3^q = \delta_q^4 : \delta_q^2 : 1$$  \hspace{1cm} (2.19)$$
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for each $q$-sector. In short, the hierarchical order estimations

$$m_j^q \propto \delta_q^{6-2j} \quad (q = u, d) \quad (2.20)$$

approximately hold. Note here that the best fittings of the quark masses and the CKM matrix elements in §4 give the estimates $\delta_u^2 \approx 10^{-3}$ and $\delta_d^2 \approx 10^{-2}$ (see Eq. (4.3)). Taking these facts into account beforehand, we have introduced the parameter $\delta_q^2$ rather than $\delta_q$ to describe the deviation of mass spectrum from the democratic limit in Eq. (2.16). This definition turns out to be crucial to obtain a simple representation of the CKM matrix in terms of parameters describing small deviations from the democratic limit and to extract sum rules from it.

§3. Sum rules for absolute values of CKM matrix elements

For the restriction in Eq. (2.11) to be fulfilled, the parameter $\lambda_q$ must be subject to the condition $\lambda_q^2 \ll (\omega_q^2 - \omega_1^q)^2 \approx \omega_1^q \delta_q^2 \approx \delta_q^4$. Off-diagonal matrix elements of the perturbation $\Lambda_q$ are calculated to be

$$v_1^q \Lambda_q v_2^q = \sqrt{2} \lambda_q \cos \frac{\theta_q}{2}, \quad v_2^q \Lambda_q v_3^q = 0, \quad v_3^q \Lambda_q v_1^q = -\sqrt{2} \lambda_q \sin \frac{\theta_q}{2}. \quad (3.1)$$

Using these results, we obtain the eigenvectors of the matrix $\tilde{\Omega}_q$ in the forms

$$\left\{ \begin{aligned}
\tilde{v}_1^q &= N_q \left( v_1^q - c_q v_2^q + s_q v_3^q \right), \\
\tilde{v}_2^q &= N_q \left( v_2^q + c_q v_1^q \right), \\
\tilde{v}_3^q &= N_q \left( v_3^q - s_q v_1^q \right),
\end{aligned} \right. \quad (3.2)$$

where

$$c_q = \frac{v_1^q \Lambda_q v_2^q}{\omega_2^q - \omega_1^q} \simeq \sqrt{2} \lambda_q \cos \frac{\theta_q}{2},$$

$$s_q = \frac{v_1^q \Lambda_q v_3^q}{\omega_3^q - \omega_1^q} \simeq \sqrt{2} \lambda_q \sin \frac{\theta_q}{2} \simeq \frac{1}{\sqrt{2} m_q^q} c_q,$$ \quad (3.3)

and

$$N_{q1,2} \simeq 1 - \frac{1}{2} c_q^2 \equiv N_q, \quad N_{q3} \simeq 1. \quad (3.4)$$

Here the terms $s_q^2$ are neglected in comparison with $c_q^2$.

In this scheme, with Hermitian mass matrices, the transformation matrix connecting the chiral quark fields in interaction and mass eigenmodes are constructed to be $U^q = (\tilde{v}_1^q, \tilde{v}_2^q, \tilde{v}_3^q)$. Therefore the CKM matrix $V \equiv U^u U^d = (v_1^u v_1^d)$ is calculated as follows:

$$V = \begin{pmatrix}
N_u N_d (1 + c_u c_d v_2^u v_2^d) & N_u N_d (c_u v_2^u v_2^d) & N_u (s_u - s_d) \\
-c_u s_d v_2^u v_3^d - s_u c_d v_3^u v_2^d & + s_u v_3^u v_2^d & -c_u v_2^u v_3^d \\
N_u N_d (c_u - c_d v_2^u v_2^d) & N_u N_d (v_2^u v_2^d) & N_u (v_2^u v_3^d - c_u s_d) \\
N_d (s_d - s_u - c_d v_3^u v_2^d) & N_d (v_3^u v_2^d - s_u c_d) & v_3^u v_3^d + s_u s_d
\end{pmatrix} \quad (3.6)$$
which is expressed in terms of the parameters $c_q$ and $s_q$ and the unperturbed matrix elements $(v_j^u v_k^d)$ for $j, k = 2, 3$. Note that $s_q$ is given by $c_q$ and that the unperturbed matrix elements calculated from the eigenvectors in Eq. (2.8) are determined as functions of two angles $\theta_q(q = u, d)$ and one phase $\phi = \phi_u - \phi_d$. Therefore the CKM matrix obtained in Eq. (3.6) depends on five parameters $c_q$, $\theta_q$ and $\phi$.

At this stage it is necessary to clarify how the unperturbed matrix elements $(v_j^u v_k^d)$ depend on the small parameters representing the deviations from the democratic limit. Retaining the linear and quadratic terms of $\delta_q$ and $\phi$, we find

\[
\begin{align*}
{v_2^u v_2^d} & = \sin \frac{\theta_u}{2} \sin \frac{\theta_d}{2} + e^{i\phi} \cos \theta_u \cos \frac{\theta_d}{2} \approx 1 - \frac{1}{3} \phi^2 + \frac{2}{3} i\phi, \\
{v_2^u v_3^d} & = \sin \frac{\theta_u}{2} \cos \frac{\theta_d}{2} - e^{i\phi} \cos \theta_u \sin \frac{\theta_d}{2} \approx \sqrt{2} \left[ \frac{1}{2} \phi^2 - \frac{3}{4} (\delta_u^2 - \delta_d^2) - i\phi \right], \\
{v_3^u v_2^d} & = \cos \frac{\theta_u}{2} \sin \frac{\theta_d}{2} - e^{i\phi} \sin \theta_u \cos \frac{\theta_d}{2} \approx \sqrt{2} \left[ \frac{1}{2} \phi^2 + \frac{3}{4} (\delta_u^2 - \delta_d^2) - i\phi \right], \\
{v_3^u v_3^d} & = \cos \frac{\theta_u}{2} \cos \frac{\theta_d}{2} + e^{i\phi} \sin \theta_u \sin \frac{\theta_d}{2} \approx 1 - \frac{1}{6} \phi^2 + \frac{1}{3} i\phi. 
\end{align*}
\]

Substituting these estimations and the approximation for the normalization constant in Eq. (3.5) into Eq. (3.6), we finally obtain

\[
V \simeq \begin{pmatrix}
1 - \frac{1}{2} (c_u - c_d)^2 & c_d - c_u - i \frac{2}{3} c_u \phi & s_u - s_d + i \frac{\sqrt{2}}{3} c_u \phi \\
\frac{c_u - c_d - i \frac{2}{3} c_d \phi}{1 - \frac{1}{2} (c_u - c_d)^2} & 1 - \frac{1}{2} (c_u - c_d)^2 - \frac{1}{3} \phi^2 + \frac{2}{3} i\phi & \frac{\sqrt{2}}{3} \left[ \frac{1}{2} \phi^2 - \frac{3}{4} (\delta_u^2 - \delta_d^2) - i\phi \right] \\
\frac{s_u + i \frac{\sqrt{2}}{3} c_d \phi}{\frac{\sqrt{2}}{3} \left[ \frac{1}{2} \phi^2 + \frac{3}{4} (\delta_u^2 - \delta_d^2) - i\phi \right]} & 1 - \frac{1}{6} \phi^2 + \frac{1}{3} i\phi & \frac{s_d - s_u}{1 - \frac{1}{6} \phi^2 + \frac{1}{3} i\phi}
\end{pmatrix}
\]

for the CKM matrix. Evidently this CKM matrix possesses the hierarchical structure. From the diagonal elements to the most off-diagonal elements, the magnitudes of the dominant terms decrease in the ratio $1 : \vert \phi \vert : \vert c_q \phi \vert$, where $\vert \phi \vert$ and $\vert c_q \vert$ are estimated to be of order of $10^{-1}$ in §4. It is straightforward to verify that this matrix satisfies the condition for unitarity $V^\dagger V \simeq I$ in the present approximation.

For comparison with available experimental results, it is necessary to calculate the absolute values of the matrix elements. Note that, although $V_{cb}$ and $V_{ts}$ in Eq. (3.8) depend on $\delta_q^2$, the dependence becomes higher order and can be ignored in $\vert V_{cb} \vert$ and $\vert V_{ts} \vert$. Consequently, the absolute values of all the CKM matrix elements are expressed solely in terms of the three small parameters $c_u$, $c_d$ and $\phi$. This means that there are six independent sum rules among the absolute values of the nine CKM matrix elements.
Neglecting terms higher than second order in \(c_u\), \(c_d\) and \(\phi\), we readily find

\[
2|V_{ud}| + |V_{us}|^2 \simeq 2 \quad (3\cdot9)
\]

and

\[
|V_{us}| \simeq |V_{cd}| \quad (3\cdot10)
\]

for the first and second generations, and

\[
2|V_{tb}| + |V_{cb}|^2 \simeq 2 \quad (3\cdot11)
\]

and

\[
|V_{cb}| \simeq |V_{ts}| \quad (3\cdot12)
\]

for the second and third generations. Although \(|V_{ud}|\), \(|V_{us}|\) and \(|V_{cd}|\) are functions of \(|c_u - c_d|\), and \(|V_{tb}|\), \(|V_{cb}|\) and \(|V_{ts}|\) are functions of \(\phi\), the formulas in Eqs. (3\cdot9) and (3\cdot10) are of exactly the same form as those in Eqs. (3\cdot11) and (3\cdot12). As the fifth sum rule we obtain

\[
2|V_{ud}| - 2|V_{cs}| \simeq |V_{cb}|^2 \quad (3\cdot13)
\]

for the matrix elements of three generations. To obtain the sixth sum rule, it is necessary to fix the sign of \(c_d - c_u\). Assuming that \(c_d - c_u < 0\) (see Eq. (4\cdot4)) and using the expression in Eq. (3\cdot4), we find the hybrid sum rule

\[
\left(\frac{m_c}{m_t} + \frac{m_s}{m_b}\right) |V_{us}|^2 |V_{cb}|^2 - \left(\frac{m_c}{m_t} - \frac{m_s}{m_b}\right) (|V_{td}|^2 - |V_{ub}|^2)
\]

\[
\simeq \left[4(|V_{td}|^2 + |V_{ub}|^2)|V_{us}|^2 |V_{cb}|^2 - 2|V_{us}|^4 |V_{cb}|^4 - 2(|V_{td}|^2 - |V_{ub}|^2)^2\right]^{\frac{1}{2}} |V_{cb}|
\]

(3\cdot14)

for the quark masses and the CKM matrix elements of three generations.

In the lowest order calculation, the rephasing invariant Jarlskog parameter is calculated to be

\[
J \simeq \frac{\sqrt{2}}{3} (c_u - c_d)(s_u - s_d)\phi, \quad (3\cdot15)
\]

the magnitude of which is obtained by

\[
|J| \simeq \frac{1}{2\sqrt{2}} \left[ \left(\frac{m_c}{m_t} + \frac{m_s}{m_b}\right) |V_{us}|^2 |V_{cb}| - \left(\frac{m_c}{m_t} - \frac{m_s}{m_b}\right) \frac{|V_{td}|^2 - |V_{ub}|^2}{|V_{cb}|} \right]
\]

\[
\simeq \frac{1}{2\sqrt{2}} \left(\frac{m_c}{m_t} + \frac{m_s}{m_b}\right) |V_{us}|^2 |V_{cb}|. \quad (3\cdot16)
\]

To derive the CKM matrix in the Wolfenstein parameterization,\(^{20}\) which is convenient to analyze the unitarity triangle, it is necessary to apply a phase transformation on \(V\) in Eq. (3\cdot8) as follows:

\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{V_{ts}^*}{|V_{ts}|}
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{V_{cb}^*}{|V_{cb}|}
\end{pmatrix}
\]

(3\cdot17)
Then we obtain the expressions

\[
\rho \simeq \frac{\Re(V_{ub}V_{cb}^*)}{(c_d - c_u) |V_{cb}|^2} \simeq \frac{1}{(c_u - c_d) |V_{cb}|^2} \left\{ \frac{2}{9} c_u \phi^2 + \frac{\sqrt{2}}{3} (s_d - s_u) \left[ \frac{1}{2} \phi^2 - \frac{3}{4} (\delta_u^2 - \delta_d^2) \right] \right\},
\]

\[
\eta \simeq \frac{\Im(V_{ub}V_{cb}^*)}{(c_u - c_d) |V_{cb}|^2} \simeq \frac{\sqrt{2}}{3} \frac{1}{|V_{cb}|^2} \frac{s_d - s_u \phi}{c_d - c_u}
\]

(3.18)

for the Wolfenstein parameters.

§4. Numerical results

We have extracted the six (five plus one) sum rules and the expressions for the Jarlskog and Wolfenstein parameters from the quasi-democratic mass matrices \( M_q = M_q \Omega_q \) using the lowest order approximation with respect to the small deviations from the perfect democratic limit. Now it is possible to set the analytical results to the experimental data on the absolute values of the CKM matrix elements.

At present the world averages of the absolute values of the CKM matrix elements have been estimated by the Particle Data Group as follows: \(^{21}\)

\[
\begin{pmatrix}
0.9745 \sim 0.9757 & 0.219 \sim 0.224 & 0.002 \sim 0.005 \\
0.218 \sim 0.224 & 0.9736 \sim 0.9750 & 0.036 \sim 0.046 \\
0.004 \sim 0.014 & 0.034 \sim 0.046 & 0.9989 \sim 0.9993
\end{pmatrix}
\]

(4.1)

For the mixing between \( c \) and \( b \) quarks the new value, \( |V_{cb}| = 0.039 \pm 0.002 \), has been reported. \(^{22},^{23}\)

Evidently the simple relations in Eqs. (3.10) and (3.12) are consistent with the experimental data. Substitution of the central values of the matrix elements into Eqs. (3.9) and (3.11) leads to \( 2 \times 0.9751 + 0.221^2 = 1.9993 \) for the left-hand side (lhs) of Eq. (3.9) and \( 2 \times 0.9991 + 0.039^2 = 1.9997 \) for the lhs of Eq. (3.11), respectively. As for Eq. (3.13), we obtain \( 2 \times 0.9751 - 2 \times 0.9743 = 0.0016 \) for the lhs, which is close to \( (0.039)^2 = 0.0015 \) for the right-hand side. Note that these five sum rules are manifestation of the hierarchical structure of the CKM matrix.

Let us make a numerical estimate for a set of parameters so as to reproduce the experimental values of both the quark masses and the CKM matrix elements. Using a 2-loop renormalization group calculation, Fusaoka and Koide obtained the values of quark masses at 1 GeV as \(^{24}\)

\[
\begin{align*}
m_u &= 4.90 \pm 0.53 \text{ MeV}, \quad m_c = 1467 \pm 28^{+5}_{-2} \text{ MeV}, \quad m_t = 339 \pm 24^{+12}_{-11} \text{ GeV}, \\
m_d &= 9.76 \pm 0.63 \text{ MeV}, \quad m_s = 187 \pm 16 \text{ MeV}, \quad m_b = 6356 \pm 80^{+214}_{-164} \text{ MeV}.
\end{align*}
\]

(4.2)

The best fittings of the absolute values of the CKM matrix elements in Eq. (3.8) to the world averages in Eq. (4.1) and of the quark masses calculated by Eqs. (2.8) and
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(2.13) to the values in Eq. (4.2) are accomplished by choosing

\[
\begin{align*}
\rho_u &= 0.9914 = 1 - 8.632 \times 10^{-3}, \\
\rho_d &= 0.9414 = 1 - 5.856 \times 10^{-2}, \\
\cos \theta_u &= 0.3362 = \frac{1}{3} + 2.888 \times 10^{-3}, \\
\cos \theta_d &= 0.3525 = \frac{1}{3} + 1.915 \times 10^{-2}, \\
\end{align*}
\]

(4.3)

\[
\begin{align*}
\lambda_u &= 2.37 \times 10^{-4}, & (c_u &= 0.0643, & s_u &= 1.939 \times 10^{-4}) \\
\lambda_d &= -3.64 \times 10^{-3}, & (c_d &= -0.1564, & s_d &= -3.024 \times 10^{-3}) \\
\end{align*}
\]

(4.4)

and

\[
\phi \equiv \phi_u - \phi_d = 7.99 \times 10^{-2}. 
\]

(4.5)

The smallness of $|\lambda_u|$ and $|\lambda_d|$ confirms the validity of the lowest order perturbation approximation. From the set of parameters in Eqs. (4.3) ~ (4.5), we reproduce

\[
\begin{pmatrix}
0.9756 & 0.2207 & 0.0040 \\
0.2209 & 0.9749 & 0.0377 \\
0.0067 & 0.0377 & 0.9993
\end{pmatrix}
\]

(4.6)

for the magnitudes of the CKM matrix elements in Eq. (3.8). This result with the data in Eq. (4.2) enables us to calculate the absolute value of the Jarlskog parameter as

\[
|J| \simeq 2.67 \times 10^{-5}
\]

(4.7)

by Eq. (3.15) and its order to be $|J| \approx 2.2 \times 10^{-5}$ by the second relation of Eq. (3.16). We obtain

\[
\rho \simeq 0.187, \quad \eta \simeq 0.386
\]

(4.8)

for the Wolfenstein parameters in Eq. (3.18). All these results are consistent with experimental results. Finally, the left- and right-hand sides of the hybrid sum rule in Eq. (3.14) are estimated, respectively, to be $3.1 \times 10^{-6}$ and $2.9 \times 10^{-6}$.

§5. Discussion

In this way we have developed a simple formalism for the quark masses and the weak mixing matrix which explains all the characteristic features of quark flavours quite well. The mass matrices are postulated to be self-adjoint and to have a democratic skeleton with small deviations from it. It is essential to parametrize the mass matrices and to decompose them into the unperturbative and perturbative parts so that the mass eigenvectors are obtained in simple mathematical forms. The mass eigenvalues $m_j^q = M_q \omega_j^q$ which are not sensitive to perturbations have hierarchical structures and approximately obey the power laws $m_j^q \propto \delta_{q-2j}^6$ with respect to the small parameters $\delta_q^2$. The CKM matrix elements also exhibit hierarchical structures when expanded with respect to small deviations from the democratic limit.
It is worthwhile to observe that the hierarchical structures appear distinctively in the mass spectra of up and down quark sectors and in the CKM matrix elements. The up and down quark sectors have the mass spectra characterized, respectively, by the parameter values $\delta_u^2 \simeq 4.33 \times 10^{-3}$ and $\delta_d^2 \simeq 2.87 \times 10^{-2}$. As for the strength of the perturbations, $|\lambda_u|$ is smaller than $|\lambda_d|$ by one order. Accordingly, the up quark sector is more democratic and hierarchical than the d-sector. On the other hand, the dominant terms in the CKM matrix decrease, from the diagonal elements to the most off-diagonal elements, in the ratio $1 : 14 > 1 : I_c q^4 > 1$, where $I_c$ and $I_c q$ are of order $10^{-1}$. Therefore the behaviour of the CKM matrix is less democratic and hierarchical than that of the quark mass spectra.

The absolute value of the CKM matrix elements turn out to be expressible in terms of three parameters, $c_u, c_d$ and $\phi$, in our approximation. This remarkable fact enables us to derive six independent sum rules among the nine matrix elements. The sum rules in Eqs. (3·9)~(3·13) are rather general consequences of the hierarchical structure of the CKM matrix. The sixth sum rule in Eq. (3·14) and the representation of the Jarlskog parameter in Eq. (3·16) are unique results of our method and consistent with the present experimental data. The set of parameters in Eqs. (4·3)~(4·5) reproduces both the observed data of the quark mass spectra and the CKM matrix elements to high accuracy. From these results it is not unreasonable to infer that our quasi-democratic mass matrices reflect essential elements of quark flavour physics.

As a next step of our approach, we must clarify whether or not the strong restrictions imposed on the quasi-democratic mass matrices such as their self-adjointness and the restriction on phases in Eq. (1·3) are indispensable. At this stage, when the origin of the quasi-democratic mass matrices is not known, it is merely a matter of convenience to assume the mass matrices to be self-adjoint. It is necessary to investigate various models with left-right symmetric and asymmetric quasi-democratic mass matrices. As for the phase restriction, we must examine many other models with mass matrices containing different entries of CP violating phases. Such attempts at model building and comparisons of the models with detailed experimental data bring us closer to the true nature of quark flavour physics. We will study one possible origin of the quasi-democratic mass matrices in Eq. (1·2) in a future publication.

References