Choosing a Conformal Frame in Scalar-Tensor Theories of Gravity with a Cosmological Constant

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Cosmological solutions of the Brans-Dicke theory with an added cosmological constant are investigated with an emphasis on selecting a conformal frame in order to implement the scenario of a decaying cosmological constant, featuring an ever-growing scalar field. We focus particularly on the Jordan frame, the original frame with nonminimal coupling, and the conformally transformed Einstein frame without it. For the asymptotic attractor solutions as well as the "hesitation behavior", we find that none of these conformal frames can be accepted as the basis of analyzing primordial nucleosynthesis. As a remedy, we propose to modify the prototype BD theory by introducing a scale-invariant scalar-matter coupling, thus making the Einstein frame acceptable. The invariance is broken as a quantum anomaly effect due to non-gravitational interactions, naturally entailing a fifth force, characterized by a finite force-range and violation of weak equivalence principle (WEP). A tentative estimate shows that the theoretical prediction is roughly consistent with the observational upper bounds. Further efforts to improve experimental accuracy are strongly encouraged.

§1. Introduction

The cosmological constant is a two-step problem. First, the observational upper bound to $\Lambda$ is more than 100 orders smaller than what is expected naturally from most of the models of unified theories. Second, some recent cosmological findings seem to suggest strongly that there is a lower bound as well, though it might be premature to draw a final conclusion. To understand the first step of this problem, theoretical models of a "decaying cosmological constant" have been proposed. These are based on some versions of scalar-tensor theories of gravity essentially of the Brans-Dicke type. Attempts toward understanding the second step have also been made by extending the same type of theories.

As a generic aspect of the scalar-tensor theories, however, one faces an inherent question on how one can select a physical conformal frame from two obvious alternatives, conveniently called the J frame (for Jordan) and the E frame (for Einstein), respectively. The former is a conformal frame in which there is a "nonminimal coupling" that characterizes the Jordan-Brans-Dicke theory but which can be removed by a conformal transformation, sometimes called a Weyl rescaling, thus moving to the E frame in which the gravitational part is of the standard Einstein-Hilbert form.

None of the realistic theories of gravity are conformally invariant. Consequently, the physics looks different from frame to frame, though physical effects in different

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conformal frames can be related to each other unambiguously. The latter fact is often expressed as "equivalence", though sometimes it results in confusion.

A conformal transformation is a local change of units. In the context of Robertson-Walker cosmology, it is a time-dependent change of the choice of the cosmic time, measured by different clocks. In the prototype BD model with the scalar field decoupled from matter in the Lagrangian, the time unit in the J frame is provided by the masses of matter particles, whereas the time in the E frame is measured in units of the gravitational constant, or the Planck mass. As will be demonstrated explicitly, the way the universe evolves is quite different in the two frames. We attempt to determine how one can use this difference to select a particular frame.

We confine ourselves mainly to the analysis of the primordial nucleosynthesis, which is known to provide the strong support for the standard cosmology. We also focus on the simplest type of the theories, which may apply only to the first step of the problem as stated above. The result obtained here will still serve as a basis of more complicated models to be applied to the second step.

Suppose first that at the onset of the process of nucleosynthesis, the universe had already reached the asymptotic phase during which it evolved according to the "attractor" solution for the BD model with \( \Lambda \) added. We find that, unlike in many analyses based on the BD model without the cosmological constant, the physical result here is acceptable in neither of the two frames for any value of \( \omega \), the well-known fundamental constant of the theory. Conflicts with the standard picture are encountered also in the early epoch, just after inflation and in the dust-dominated era.

We then point out that the cosmological solution may likely show the "hesitation behavior", in which the scalar field remains unchanged for some duration. If nucleosynthesis occurred during this phase, the two frames are equally acceptable, with no distinction existing between them. Outside the era of nucleosynthesis, however, we inherit the same conflicts for both conformal frames; hesitation itself offers no ultimate solution.

Fortunately, most of these conflicts can be avoided in the E frame, as we find, if, contrary to the original model, the scalar field is coupled to matter in a way which is not only simple and attractive from a theoretical point of view, but which is roughly consistent with observations currently available.

In reaching this conclusion in favor of the E frame, we emphasize that the manner in which a time unit in a certain conformal frame changes with time depends crucially on how the scalar field enters the theory. Searching for a correct conformal frame is intricately coupled with the search for a theoretical model which would lead to a reasonable overall consistency with cosmological observations.

In §2, we start by defining the model first in the J frame, and then we apply a conformal transformation moving to the E frame. We discuss in §3 the attractor solution in some detail, including elaborated comparison between the two conformal frames. In §4 we discuss the comparison with phenomenological aspects of standard cosmology, particularly primordial nucleosynthesis, the dust-dominated universe, and the pre-asymptotic era. We enter into discussion of the "hesitation behavior" in §5, still finding disagreement with standard cosmology. To overcome the difficulty we
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face, we propose in §6 a revision of the theoretical model by abandoning one of the premises in the original BD model, but appealing to a rather natural feature of scale invariance. The analysis is made both classically and quantum theoretically. As we find, the effect of a quantum anomaly naturally entails a "fifth force", featuring a finite-force range and violation of the weak equivalence principle (WEP). Importance of further experimental studies is emphasized. Section 7 is devoted to concluding remarks. Three appendices are added to provide some details on (A) another attractor solution, (B) the mechanism of hesitation, and (C) loop integrals resulting in the anomaly.

§2. The model

We start with the Lagrangian in the J frame,

$$\mathcal{L} = \sqrt{-g} \left( \frac{1}{2} \xi \phi^2 R - \frac{1}{2} \epsilon \eta^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + \phi^n \Lambda + \mathcal{L}_{\text{matter}} \right),$$

(1)

where our scalar field $\phi$ is related to BD's original notation $\varphi \equiv \phi_{\text{BD}}$ by

$$\varphi = \frac{1}{2} \xi \phi^2,$$

(2)

and with the constant $\xi$ related to their $\omega$ by $\xi \omega = 1/4$. Note that we use units in which $c = \hbar = 8\pi G(= M_{\text{P}}^{-2}) = 1$.\(^1\)

We prefer using $\phi$ defined above because by so doing we write equations in a form more familiar from conventional relativistic field theory, avoiding the apparent singularity $\varphi^{-1}$ in the kinetic term. Also, $\epsilon$ in (1) can be ±1 or 0. Examples with $\epsilon = -1$ are provided by the dilaton field coming from 10-dimensional string theory,\(^8\) and the scalar field representing the size of the internal space arising from compactifying $N$-dimensional space-time with $N \geq 6$, while the latter with $N = 5$ gives $\epsilon = 0$.\(^2\)

Various models have been proposed for different choices of the function of $\phi$ in the nonminimal coupling.\(^7\) We adhere for the moment, however, to the simplest choice in (1), expecting it to be applied to situations of immediate physical interest. The factor $\phi^n$ is inserted because a cosmological constant in higher dimensions may appear in 4 dimensions multiplied with some power of $\phi$ depending on the model.

We assume the RW metric, specializing to the radiation-dominated universe after the inflationary epoch, because it not only simplifies the calculation considerably but also applies to the era of nucleosynthesis. Also choosing $k = 0$, we obtain the equations

$$6 \dot{\varphi} H^2 = \frac{1}{2} \epsilon \phi^2 + \phi^n \Lambda + \rho_r - 6H \dot{\phi},$$

(3)

\(^1\) In units of the Planck time ($2.71 \times 10^{-43}$ sec), the present age of the universe $\sim 1.4 \times 10^{10}$ y is given by $\sim 1.6 \times 10^{60}$, while "3 minutes" is $\sim 10^{65}$.

\(^2\) A negative value of $\epsilon$ is found to be essential in the calculations in Ref. 2) carried out in the J frame.
\[ \ddot{\phi} + 3H\dot{\phi} = (4 - n)\zeta^2\phi^n\Lambda, \]
\[ \dot{\rho}_r + 4H\rho_r = 0, \]

where
\[ H = \frac{\dot{a}}{a}. \]

Now consider a conformal transformation defined by
\[ g_{*\mu\nu} = \Omega^2(x)g_{\mu\nu}, \quad \text{or} \quad ds^2_\star = \Omega^2(x)ds^2. \]

By choosing
\[ \Omega = \xi^{1/2}\phi, \]
we rewrite (1) as the Lagrangian in the E frame,
\[ \mathcal{L} = \sqrt{-g_\star} \left( \frac{1}{2} R_\star - \frac{1}{2} g^{\mu\nu}_\star \partial_{\mu}\sigma \partial_{\nu}\sigma + V(\sigma) + L_{\text{matter}} \right), \]

where
\[ \phi = \xi^{-1/2}e^{\xi\sigma}, \]
with the coefficient \( \zeta \) defined by
\[ \zeta^{-2} = 6 + \epsilon\xi^{-1} = 2(3 + 2\epsilon\omega), \]
and
\[ V(\sigma) = \Omega^{-4}\phi^n\Lambda = \Lambda\xi^{-n/2}e^{(n-4)\xi}\sigma. \]

Note that the \( \Lambda \) term in (1) now acts as a potential even with \( n = 0 \).

We point out that the canonical field \( \sigma \) is a normal field (not a ghost) if the right-hand side of (11) is positive,
\[ 6 + \epsilon\xi^{-1} > 0, \]
even with \( \epsilon = -1 \). This positivity condition is satisfied trivially for any finite value of \( \xi \) if \( \epsilon = 0 \). The condition (13) should be obeyed even in the analysis in the J frame; in the presence of the nonminimal coupling that causes mixing between \( \phi \) and the spinless part of the tensor field, the sign of the total energy of the scalar-tensor sector is not determined solely by \( \epsilon \). We note that the conformal transformation serves also as the relevant diagonalization procedure. We impose \( \xi > 0 \) to keep the energy of the tensor gravitational field positive.

With the RW metric also in the E frame, the cosmological equations are
\[ 3H_\star^2 = \rho_{*\sigma} + \rho_{*r}, \]
\[ \ddot{\sigma} + 3H_\star\dot{\sigma} + V'(\sigma) = 0, \]
\[ \dot{\rho}_{*r} + 4H_\star\rho_{*r} = 0, \]

where
\[ \rho_{*\sigma} = \frac{1}{2}\dot{\sigma}^2 + V(\sigma). \]
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The overdot here implies a differentiation with respect to \( t_* \), the cosmic time in the E frame, defined by \(^3\)

\[
dt_* = \Omega dt,
\]
\[
a_* = \Omega a, 
\]

where the scale factor \( a_* \) has also been introduced to define \( H_* = \dot{a}_*/a_* \).

§3. Attractor solution

The solutions of the cosmological equations can be obtained more easily in the E frame than in the J frame. We in fact find a special solution

\[
a_*(t_*) = t_*^{1/2}, 
\]
\[
\sigma(t_*) = \bar{\sigma} + \frac{2}{4 - n} \zeta^{-1} \ln t_*, 
\]
\[
\rho_{\sigma\tau}(t_*) = \rho_{\sigma\tau 0} t_*^{-2}, \quad \text{with} \quad \rho_{\sigma\tau 0} = \frac{3}{4} \left[ 1 - \left( \frac{2}{4 - n} \right)^2 \zeta^{-2} \right], 
\]

where the constant \( \bar{\sigma} \) is given by

\[
\Lambda e^{(n-4)\zeta \bar{\sigma}} = \frac{1}{(n-4)^2} \zeta^{n/2} \zeta^{-2}. 
\]

We also find

\[
\rho_{\sigma\sigma}(t_*) = \frac{3}{(n-4)^2} \zeta^{-2} t_*^{-2}, 
\]

which is independent of \( \Lambda \).

It is easy to see that, since \( V(\sigma) \) is an exponentially decreasing function if \( n < 4 \), \( \sigma \) is pushed toward infinity, hence giving \( \rho_{\sigma\sigma} \sim t_*^{-2} \), which implies that \( \Lambda_{\text{eff}} \equiv \rho_{\sigma\sigma} \) does decay with time even with \( n = 0 \), a purely constant cosmological constant in 4 dimensions. \(^3\) According to (21), the scalar field continues to grow. This is a unique feature that distinguishes our model from other models \(^7\) with the scalar field designed to settle eventually to a constant. \(^*\)

To ensure the natural condition \( \rho_{\sigma\tau} > 0 \), we impose

\[
\zeta^2 > \frac{4}{(4 - n)^2}. 
\]

Though there is a subtlety, as will be discussed in Appendix A, this is one of the peculiar outcomes of including \( \Lambda \). According to (11), this yields a prediction for \( \omega \) which is too small to be consistent with the currently accepted lower bound. Below, however, we propose a revised theoretical model in which \( \zeta \) is not related directly to phenomenological parameters. We also note from (11) that \( \zeta > 1/\sqrt{6} \) can be realized only for \( \epsilon = -1. \) \(^**\)

\(^*\) If \( \Lambda \) is added to this type of model, this constant would yield a \( \Lambda_{\text{eff}} = \text{constant} \) which is too large. We attempt to avoid this without fine-tuning the parameters.

\(^**\) This is equivalent to choosing a negative \( \omega \) according to several authors. \(^{14, 15}\)
It is already known that the solution (20) ∼ (24) is an attractor, though many investigations concerning the scalar field have been aimed at the question of whether it leads to sufficient inflation.\(^9\)\(^{−15}\) In contrast, we are interested in how the cosmological constant relaxes in a manner consistent with other known aspects of cosmology.\(^*)\) For this reason we start integrating the equations sometime after the end of inflation, when a sufficient amount of matter energy is created due to reheating, though related details are not yet well understood. Figure 1 shows how a typical solution of (14) ∼ (16) (with \(n = 0\)) tends asymptotically to the attractor behavior for \(\log t* \geq 20\). We emphasize that the behavior is essentially the same for any value of \(n\), as long as \(n < 4\); according to (12) differences in the value of \(n\) are absorbed into different \(\zeta\).

In this example we chose the initial value of \(\sigma\) at the time \(t_{1*} = 10^{10}\) in such a way that the resultant solution varies "smoothly" around \(t_{1*}\). To be more specific, we require \(t*H*\), the local effective exponent in \(a*(t*)\), to remain order unity. This is in accordance with assuming that the "true" initial condition on a more fundamental level is given at a much earlier time.

The same attractor solutions can be obtained also in the J frame. This can be done most easily by substituting (20) ∼ (24) into (18) and (19). After straightforward calculations, we find

\[
t = \begin{cases} 
A_n t_*^{(2-n)/(4-n)}, & \text{if } n \neq 2, \\
A_2 \ln t_*, & \text{if } n = 2,
\end{cases}
\]

and hence

\[
a(t) = \begin{cases} 
t^\alpha, \text{ with } \alpha = \frac{1}{2} \frac{n}{n-2}, & \text{if } n \neq 2, \\
\exp \left( -A^2 \xi^{-1/2} \zeta t \right), & \text{if } n = 2,
\end{cases}
\]

where \(A_n\) is a constant given in terms of \(\Lambda, \xi\) and \(n\).

\(^*)\) See also Ref. 16\) for some of our earlier results.
Of special interest is the choice \( n = 0 \), a purely constant cosmological constant in 4 dimensions, giving

\[
t = A_0 t^{1/2},
\]
\[
a(t) = \text{const}.
\]

The asymptotic behavior (28) is seen in Fig. 2 for the same solution as in Fig. 1, under the condition \( t_1 = t_{*1} \) at the "initial" time. Note that this behavior is preceded by a period of an extremely slow take-off of \( t(t_*) \), which can be traced back to the large value of \( \Omega \) appearing in (18), related to the remark stated above on the initial value \( \sigma(t_{*1}) \). Figure 3 shows how the asymptotic behavior (29) is reached. A closer look at the curve toward the initial time reveals oscillatory behavior too rapid to be drawn here exactly. This might have a disturbing effect, as will be discussed later.

Also for \( n = 2 \), for which some exact solutions have been obtained, \(^{11,14,15}\) we find an exponentially contracting universe, though this can be converted to an exponentially expanding universe by reversing the direction of time.

The exponent \( \alpha \) in the first of (27) is larger than 1 for \( n > 2 \), while it is negative for \( 0 < n < 2 \). The exponent never reaches 1/2 for any finite value of \( n \); to obtain 0.45, for example, requires \( n = -18 \). We also point out that the behavior differs considerably depending on whether the "ordinary" matter (\( \rho_r \) or \( \rho_{*r} \)) is included or not.

Summarizing, we expect substantial differences in the behavior of the scale factor not only between the conformal frames but also among different values of \( n \) in the J frame. This demonstrates how crucial it is to select the right conformal frame to discuss any of the physical effects. In the next section we analyze the
physical implications to be compared with the results of standard cosmology.

§4. Comparison with standard cosmology

4.1. Nucleosynthesis

According to the standard scenario, light elements were created through nuclear reactions in the radiation-dominated universe with the temperature dropping in proportion to the inverse square root of the cosmic time. The entire process is analyzed in terms of nonrelativistic quantum mechanics, in which particle masses are taken obviously as constant. In scalar-tensor theories, on the other hand, particle masses depend generically on the scalar field, and hence on time. The prototype BD theory is unique in that masses are true constants due to the assumption that the scalar field is decoupled from matter in the Lagrangian. For this reason the mass $m$ of a particle is a pure constant in the J frame, specifically denoted by $m_0$.

We should recall that we have so far considered relativistic matter alone. One may nevertheless include massive particles which play no role in determining the overall cosmological evolution, but may nonetheless serve to provide standards of time and length.

To find how masses become time-dependent after a conformal transformation, it is sufficient to consider a toy model of a free massive real scalar field $\Phi$, described by the matter Lagrangian in the J frame,

$$ L_{\text{matter}} = \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} m_0^2 \Phi^2 \right), \quad (30) $$

where no coupling to $\phi$ is introduced. After the conformal transformation (7), we find

$$ L_{\text{matter}} = \sqrt{-g_*} \left( -\frac{1}{2} \Omega^{-2} g_*^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} \Omega^{-4} m_0^2 \Phi^2 \right). \quad (31) $$

The kinetic term can be made canonical in terms of a new field $\Phi_*$ defined by

$$ \Phi = \Omega \Phi_*, \quad (32) $$

thus putting (31) into

$$ L_{\text{matter}} = \sqrt{-g_*} \left( -\frac{1}{2} g_*^{\mu\nu} \partial_\mu \Phi_* \partial_\nu \Phi_* - \frac{1}{2} m_*^2 \Phi_*^2 \right), \quad (33) $$

where

$$ D_\mu \Phi_* = [\partial_\mu + \zeta (\partial_\mu \sigma)] \Phi_*, \quad (34) $$

$$ m_*^2 = \Omega^{-2} m_0^2. \quad (35) $$

We point out that this relation holds true generally beyond the simplified model considered above.

*) This is a major point that distinguishes between the BD model and that due to Jordan who pioneered the nonminimal coupling.
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From (8), (10) and (21) we find

$$\Omega \sim \phi \sim t_*^{2/(4-n)},$$

hence giving a *time-dependent mass* in the E frame. This would result in the reduction of masses by as much as $1 - 1/\sqrt{10} \approx 70\%$ if $n = 0$, for example, in the period 100-1000 sec, during which the major part of the synthesis of light elements is supposed to have taken place with the temperature dropping at the same rate. Obviously this is in complete conflict with the success of the standard scenario.

In this way we come to a dilemma: The J frame is selected uniquely because of the constancy of masses, as taken for granted in conventional quantum mechanics to analyze the physical processes, while the E frame is definitely preferred to have a universe that cooled down sufficiently for light elements to form.

In passing we offer a simple intuitive interpretation on the relation between the two conformal frames. In the present context, the time unit $\tau$ in the E frame is provided by $m_0^{-1} \sim \Omega$. The time $\tilde{t}$ measured in units of $\tau$ may be defined by $d\tilde{t} = dt_\tau / \tau \sim \Omega^{-1} dt_*$. Comparing this with (18), we find $\tilde{t} = t$. On the other hand, the only constant in the E frame which is not dimensionless is $M_0$. In this sense the E frame corresponds to the time unit provided by the gravitational constant. For $n = 0$, for example, the microscopic length scale provided by $m_0^{-1}$ expands at the same rate as the scale factor $a_\tau(t_\tau)$, hence showing no expansion in $a(t)$ in (19).

4.2. *Dust-dominated era*

In the standard theory the radiation-dominated era is followed by the dust-dominated universe. Its description is, however, likely problematic, as we now show.

Due to the equation

$$\square_\tau \sigma = c \zeta T_\tau,$$

with $T_\tau = -\rho_\tau$, the right-hand side of (15) acquires the additional term

$$c \zeta \rho_\tau.$$

Here, for the non-relativistic matter density $\rho_\tau$, $c = 1$ for the prototype BD model, but we allow $c$ to be different in the proposed revision, which will be discussed later. Suppose the total matter density is the sum $\rho_\tau + \rho_\tau$, which would replace $\rho_\tau$ in (14). Corresponding to (16), we find

$$\dot{\rho}_\tau + 3H_\tau \rho_\tau = -c \zeta \rho_\tau \dot{\phi},$$

where the right-hand side is included to meet the condition from the Bianchi identity.

We obtain the attractor solution with

$$a_\tau(t_\tau) = t_*^{\alpha_*} \quad \text{with} \quad \alpha_* = \frac{4-c}{6},$$

and

$$\rho_\tau = \frac{4-c}{3} - \frac{1}{4} \zeta^{-2}.$$

Equation (21) still holds. Note, however, that (40) gives $\alpha_* = 1/2$ for $c = 1$, — the same behavior as in the radiation-dominated era. This seems "uncomfortable" if we
wish to stay close to the realm of the standard scenario, though we may not entirely rule out a highly contrived way for a reconciliation. On the other hand, we would obtain the conventional result $\alpha_* = 2/3$ for $c = 0$.

We also find that the relation $t \sim t_*^{1/2}$ remains unchanged (for $n = 0$), and then it follows that $a = \text{const}$ in the J frame again for $c = 1$. This is certainly disfavored, as in the analysis of nucleosynthesis.

4.3. Pre-asymptotic era

As we noted in Fig. 3, $H$ in the J frame seems to be oscillatory around zero in early epochs, arousing suspicion of a contracting universe. More details toward $t_1 = t_{*1}$ are shown in Fig. 4, in which $tH$ is plotted against $\log t_*$ instead of $\log t$; the dip in $tH$ would be too sharp to be shown if plotted against $\log t$. We also plotted $\log a$, which does show a decrease of the scale factor in the J frame. The example here shows that the scale factor $a$, which tends to a constant in the asymptotic era is even smaller than at $t_{*1}$. The exact amount of contraction depends on the choice of the parameters, still making it considerably difficult to reach a compromise with the idea that the early universe had cooled down sufficiently to trigger the process of nucleosynthesis.

§5. Hesitation behavior

We have so far concentrated on the attractor solution, to which some solutions — depending on the initial conditions — do tend smoothly, as demonstrated in Fig. 1, for example. We point out, however, there is an important pattern of deviation from this smooth behavior. As illustrated in Fig. 5, the scalar field may remain almost at rest temporarily before entering the asymptotic phase in which it resumes increasing toward the attractor solution. This "hesitation" behavior may occur if $\rho_{\sigma r} \ll \rho_{\sigma \sigma}$ at the initial time, as will be shown in detail in Appendix B. 18)

Note that $t_* H_*$, which equals the effective exponent $\alpha_*$ if the scale factor is approximated locally by $a_*(t_*) \sim t_*^{\alpha_*}$, tends to $1/2$ after some wiggle-like behavior that separates the plateau of the same value $1/2$ in the hesitation period (also see Appendix B for discussion of the mechanism behind another plateau of $t_* H_* = 1/3$).

With the scalar field nearly constant during this hesitation phase, particle masses are also nearly constant, and all the other cosmological effects are virtually the same.
as those in the standard theory as long as $\rho_{s\sigma} \ll \rho_{s\tau}$, as is the case for $\log t_* \gtrsim 35$ in the example of Fig. 5, in which we chose the parameters in such a way that the hesitation period $\log t_* = 35-48$ covers the era of nucleosynthesis. Moreover, the constant scalar field makes the conformal transformation (18) and (19) trivial, implying that the two conformal frames are essentially equivalent.

When considered in more detail, however, we find some differences between them. In our example, we carried out the transformation (18) and (19), showing first in Fig. 6 how the two time variables $t_*$ and $t$ are related to each other: Corresponding to the hesitation behavior, we find the period of $t \sim t_*$ in addition to the behavior as shown in Fig. 2 without the hesitation period.

As in Fig. 4, we find in Fig. 7 that the universe in the J frame had experienced a considerable contraction prior
to the epoch of nucleosynthesis, making the scenario of the evolution in early epochs quite different from the standard theory (see Appendix B for discussion of its origin).

No such problems will occur if we are still within the hesitation period at the present time. If this happens, however, there is no distinction between the two frames. Thus in this case also, there is no reason not to choose the E frame. From this point of view, we do not consider this possibility any further.

One might suspect that all of these "conflicts" with the standard picture come directly from the theoretical models to start with. In fact it seems obvious that we would be in a much better position if particle masses were constant in the E frame rather than in the J frame. This can be achieved, as we will show, by modifying one of the assumptions in the prototype model in a natural manner.

§6. A proposed revision

6.1. Classical theory

Unlike in the original BD model, let us start with the matter Lagrangian in the J frame

$$\mathcal{L}_{\text{matter}} = \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} f^2 \phi^2 \Phi^2 - \frac{1}{4!} \lambda \Phi^4 \right), \quad (42)$$

in place of (30). We have introduced the coupling of $\phi$ to the matter field $\Phi$, hence abandoning one of the premises of the original BD theory. Also there is no mass term of $\Phi$; the "mass" of the field $\Phi$ is $f\phi$, which is no longer constant. With the choice (42), therefore, the J frame loses its privilege as the basis of the theoretical analysis of nucleosynthesis. We also introduced the self-coupling of $\Phi$ to illustrate the effect of the quantum anomaly.

After the conformal transformation, we obtain the same Lagrangian (33) (plus the self-coupling term) but with (35) replaced by

$$m_{\sigma 0}^2 = f^2 \phi^2 \Omega^{-2} = \xi^{-1} f^2, \quad (43)$$

which is obviously constant. This makes the E frame now a relevant frame for realistic cosmology. We point out that no matter coupling of $\sigma$ is present in the E frame. Absence of the coupling in the Lagrangian having no nonminimal coupling implies a complete decoupling, unlike the corresponding situation in the J frame in the prototype model. This corresponds to the choice $c = 0$ in (37) $\sim$ (41), thus leaving the results in the E frame the same as those of standard cosmology also in dust-dominated universe. On the other hand, the scalar field cannot be detected in any way by measuring its contribution to the force between objects, or the conventional tests of general relativity, hence removing the constraints on $\omega$ (or $\xi$) obtained so far. The effect of the scalar field may still be manifested through cosmological phenomena.

The time scale in the E frame is provided commonly by particle masses and the gravitational coupling constant. This implies that no time variability of the gravitational constant should be observed if measured by atomic clocks with their
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units given basically by particle masses.*)

The above scheme is attractive because the coupling constant $f$ is dimensionless, implying scale invariance in the gravity-matter system, except for the $A$ term. By applying Noether's procedure, we obtain the dilatation current as given by

$$J^\mu = \frac{1}{2}\sqrt{-g} g^{\mu\nu} \left[ 2\zeta^{-1} \partial_\nu \sigma + (\partial_\nu + 2\zeta \partial_\nu \sigma) \Phi_*^2 \right],$$

(44)

which is shown to be conserved by using the field equations. This conservation law remains true even after the conformal transformation, with a nonzero mass as given by (43). This implies that the scale invariance is broken *spontaneously* due to the trick by which a dimensional constant $M_p(=1)$ has been "smuggled" in (8). In this context $\sigma$ is a Nambu-Goldstone boson, a dilaton. This invariance together with the constancy of particle masses will be lost, however, if one includes quantum effects due to non-gravitational couplings *among* matter fields, as will be briefly discussed.

6.2. Quantum anomaly

Consider one-loop diagrams for the coupling between $\Phi_*^2$ and $\sigma$ assumed to carry no momentum, as illustrated in Fig. 8, arising from the non-gravitational coupling $(\lambda/4!)\Phi^4$. They will result, according to quantum field theory, generally in divergent integrals, which may be regularized by means of continuous spacetime dimensions. Corresponding to this, we rewrite previous results extended to $N$ dimensions. Equation (32) is then modified to

$$\Phi = \Omega^{N-1} \Phi_*,$$

(45)

where $\nu = N/2$. Also, the relation (8) is changed to

$$\Omega = \zeta^{1/(N-2)} \phi^{1/(\nu-1)}.$$

(46)

The Lagrangian (42) is now put into

$$L_{\text{matter}} = \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} D_\mu \Phi_* D_\nu \Phi_* - S \right),$$

(47)

where $D_\mu$ is given by (34) while $S$ is defined by

$$S = e^{2(\nu-2)\zeta^\sigma} S_0,$$

(48)

with

$$S_0 = \xi^{-1} \frac{f^2}{2} \Phi_*^2 + \frac{1}{4!} \lambda \Phi_*^4.$$

(49)

*) Strictly speaking, the time unit of atomic clocks depends also on the fine-structure constant, which is assumed constant, however, at this moment.
As a result, the right-hand side of (43) is multiplied by $e^{(N-4)\zeta \sigma}$, which might be expanded with respect to $(N-4)\zeta \sigma$ according to (10); decoupling occurs only in $4$ dimensions. Following the rule of dimensional regularization, we keep $N$ different from 4 until the end of the calculation including the evaluation of loop integrals, which are finite for $N \neq 4$.

The divergences coming from the $\Phi_*$ loops are represented by poles $(N-4)^{-1}$ which cancel the factor $N-4$ that multiplies the $\sigma$ coupling, as stated above, hence yielding an effective interaction of the form

$$-L_{\sigma \Phi} = g_\sigma \frac{m_*^2 \phi^2}{M_p} \sigma,$$

for $N \to 4$, where the coupling constant $g_\sigma$ is given by

$$g_\sigma = \frac{\zeta \lambda}{8\pi^2}. \quad (51)$$

(The details of this computation are presented in Appendix C.) In (50) we reinstalled $M_p^{-1} = \sqrt{8\pi G}$ to remind us that this coupling is basically as weak as the gravitational interaction.

Suppose the $\Phi_*$ field at rest is a representative of dust matter. We may then take $\rho_\star \approx m_*^2 \phi_*^2$. Comparing (50) with (37), we find

$$g_\sigma = -c\zeta. \quad (52)$$

In this way we obtain $g_\sigma$, and hence $c$, which are nonzero finite due to a non-gravitational interaction.

It should be noted that our determination of $c$ in (37) never implies that $\sigma$ couples to the trace of the energy-momentum tensor. From (51) we find that the coefficient $g_\sigma$ depends on $\lambda$, which is not related to the mass directly, and hence it may differ from particle to particle. If we include many matter particles, the right-hand side of (37) should be the sum of corresponding components with different coefficients. For this reason, the force mediated by $\sigma$ fails generically to respect WEP. This was shown more explicitly in our QED version. 19)

We note that the above calculation is essentially the same as those from which various "anomalies" are derived, particularly, the trace anomaly. 20) The relevance to the latter can be shown explicitly if we consider the source of $\sigma$ in the limit of weak gravity.

We first derive

$$\partial_\mu J^\mu = 2\zeta(\nu - 2)\sqrt{g_*} S,$$

indicating that scale invariance is broken explicitly for $N \neq 4$. Now, deriving (50) is essentially equivalent to calculating the quantum theoretical expectation value of $S_0$.
between two 1-particle states of $\Phi_*$,
\[ <2\zeta(\nu - 2)S_0 >_{\Phi_*} = g_\nu m^2_{*0}, \]  
(54)
ignoring terms higher order in $\zeta(= \zeta M^{-1}_P)$. Combining this with (37), we may write
\[ \Box \sigma = <\partial_\mu J^\mu >_{\Phi_*}. \]  
(55)
This equation shows that $\sigma$ couples to the breaking of scale invariance effected by the quantum anomaly,\(^1\) though this simple relation is justified only up to lowest order in $M^{-1}_P$.

One might be tempted to extend the analysis by identifying $\Phi$ with the Higgs boson in the standard electro-weak theory or grand unified theories, hence predicting observable consequences. As we find, however, realistic analysis is likely to be more complicated; we must take contributions from other couplings — including the Yukawa and QCD interactions — into account. Even more serious is the fact that we are still short of a complete understanding of the "content" of nucleons, the dominant constituent of the real world. We nevertheless attempt an analysis, as will be sketched briefly, leaving further details to the future publications.

From a practical point of view, we need the coupling strength of $\sigma$ to nucleons, through the couplings to quarks and gluons. By a calculation parallel to that leading to (51), we obtain\(^\text{**}^\text{)\)}
\[ c_q = -\frac{5\alpha_s}{\pi} \sim 0.3, \]  
(56)
for a $\sigma$-quark coupling, where $\alpha_s$ is the QCD analog of the fine-structure constant, most likely of the order of $\sim 0.2$. We then evaluate
\[ c_N m_N = c_q <\sum_i m_i q_i q_{*i} >_N, \]  
(57)
where the nucleon matrix element has been estimated to be $\sim 60$ MeV,\(^2\) which together with (56) would give
\[ c_N \sim 1.8 \times 10^{-2}. \]  
(58)
It is interesting to note that this is rather close to the constraints obtained from observations, as will be shown.

6.3. Fifth force and cosmology

By replacing $\zeta$ in (11) by $c\zeta$ and also choosing $\epsilon = +1$ in accordance with the conventional analysis, we obtain
\[ c^{-2} \zeta^{-2} = 8(3 + 2\omega), \]  
(59)
\(^\text{*1}\) In this respect we rediscover the proposal by Peccei, Sola and Wetterich,\(^2\) though their approach to the cosmological constant problem is different from ours.
\(^\text{**}^\text{)\)} The coefficient 6 in Eq. (52) of Ref. 19 should be replaced by $15/4$ (also $\beta$ is our present $\zeta$). The QCD result, our (56), is obtained by multiplying further by the factor $(N^2 - 1)/2N$ which is $4/3$ for $N = 3$.\)
or
\[
\alpha \zeta \sim \frac{1}{4\sqrt{\omega}} \lesssim 0.8 \times 10^{-2},
\]
if we accept \( \omega \gtrsim 10^3 \) obtained from solar-system experiments, assuming the force-range of \( \sigma \) to be longer than 1 solar unit. Note that \( \sigma \) is now a pseudo Nambu-Goldstone boson which likely acquires a nonzero mass. Combining this with the constraint (25), we find
\[
|c| \lesssim 1.6 \times 10^{-2},
\]
which we find is nearly the same as (58).

Similar constraints may come from the “fifth force” phenomena which are characterized by a finite force-range and WEP violation, both of which are generic in the present model. The parameter \( \alpha_5 \), the relative strength of the fifth force, is given by
\[
\alpha_5 \sim 2g_s^2 \sim c_N^2,
\]
expecting that the coupling comes mainly from that to nucleons.

Since \( g_\sigma \) may depend on the object to which \( \sigma \) couples, as was pointed out, \( \alpha_5 \) may also depend on the species of nuclei, for example, between which \( \sigma \) is exchanged, to be denoted by \( \alpha_{5ij} \).

From the observations carried out to this point, we have the upper bound given roughly by \( 23) \)
\[
|\alpha_{5ij}| \lesssim 10^{-5}.
\]
In view of the fact that the result depends crucially on the assumed value of the force-range as well as the model of WEP violation, we may consider that the estimate (62) with (58) is approximately consistent with (63), hence providing renewed motivation for further studies of the subject both from theoretical and experimental points of view.

The analysis is still tentative, particularly because (56) is justified only to lowest order with respect to \( \alpha_5 \), though the renormalization-group technique can be used to include the leading-order terms. Potentially more important would be an estimation of the contribution from gluons. Corresponding to the second term of (C·1), we should include the direct coupling of \( \sigma \) to the QCD coupling constant \( g_s \) as given by
\[
\lim_{\nu \to -2} \zeta (\nu - 2) g_s Z_g \sigma = 2\pi \zeta \beta_0 g_s \alpha_5 \sigma, \tag{64}
\]
where \( \beta_0 = (4\pi)^{-2}(11 - 2n_f/3) \) with \( n_f \) the number of flavors. It is yet to be studied how this coupling would affect the simple result (58) through the gluon content of a nucleon.

It should also be emphasized that the right-hand side of (57) (even with the modification stated above) is quite different from the matrix element of the conventional energy-momentum tensor or its trace related directly to observations; only the anomalous part participates.\(^a\)

\(^a\) An analysis due to Ji\(^24\) showing that the trace anomaly contributes approximately 20 % of the nucleon mass might also be suggestive.
One might argue that a value of $c_N$ which would be too large to be allowed by the phenomenological constraints could emerge if we apply the same type of calculation to a nucleon considered to be an elementary particle, as was attempted in a simpler QED version.\textsuperscript{19} We point out, however, the finite size of a composite nucleon would serve to suppress ultra-violet divergences, thus failing to produce an anomaly, which is a manifestation of the fact that the underlying theory is divergent.

It is rather likely that $\sigma$ couples also to the nuclear binding energy, which is supposedly generated by the exchanged mesons. This would make the analysis of composition dependence even more complicated.\footnote{Most of the past analyses on the composition dependent experiments\textsuperscript{25} have been made based on the assumption that the fifth force is decoupled from the nuclear binding energy. This might be too simplified from our point of view.}

We now turn to cosmological aspects. Adding $(50)$ to the “classical” mass term, we typically obtain

$$-I_{m\Phi}' = \frac{1}{2}m^2_\phi \Phi^2 \quad \text{with} \quad m^2_\phi = m^2_\phi (1 - c \zeta \sigma + \cdots).$$

(65)

We focus on the cosmological background $\sigma(t_*)$ rather than the space-time fluctuating part which would mediate a force between objects, as considered above. In this sense $m_\phi$ depends on time. We must then apply another conformal transformation to cancel this effect. If, however, $c$ is sufficiently small, as indicated in $(61)$, one may expect that the expansion in $(65)$ is exponentiated giving

$$m_\phi (\sigma) \approx m_{\phi 0} e^{-c \zeta \sigma} \sim t_*^{-c/2},$$

(66)

where we have used the asymptotic behavior (21) with $n = 0$. We expect (66) to hold approximately for realistic nucleons and nuclei.

Note that the final expression of the time variation is independent of $\zeta$. The resulting conformal frame is expected to be close to the E frame. A small $|c|$ is also favored from (40) for the attractor solution in the dust-dominated universe. It would further follow that $\bar{G}/G$ is somewhat below the level of $10^{-10} \text{ y}^{-1}$, in accordance with the observations.\textsuperscript{26}

If, on the contrary, $|c|$ is “large”, we may not even be able to compute the required conformal transformation unless we determine higher-order terms in the parenthesis in (65). All in all, we would be certainly “comfortable” if $|c|$ is sufficiently small. On the other hand, it seems unlikely that $|c|$ would be smaller than unity by many orders of magnitude, because we know no basic reason why it should be so. The present constraint (61) might already be close to the limit which one can tolerate in any reasonable theoretical calculation. In this sense, probing $\alpha_5$ with accuracy improved by a few orders of magnitude would be crucially important to test the proposed model of broken scale invariance. If we come to discover any effect of this kind, it would provide us with valuable clues of how nucleons and nuclei are composed of quarks and gluons.
§7. Concluding remarks

Having introduced a scalar field in order to relax the cosmological constant within the realm of the standard scenario, we come to a conclusion: At the classical level, the E frame is the only choice, provided that the J frame version has a scale invariant coupling between the scalar field and the matter fields without the intrinsic mass terms. The most crucial points are the constancy of particle masses and the expansion law of the universe during the epoch of primordial nucleosynthesis. A quantum anomaly serves naturally to break scale invariance explicitly, offering yet more support for the existence of a fifth force featuring WEP violation. A tentative calculation based on QCD yields a coupling strength roughly consistent with the observational upper bounds. The physical conformal frame should then remain close to the E frame. Improved efforts to probe for the fifth force are encouraged, though detailed theoretical predictions are yet to be attempted.

We point out, however, there is a possible way to leave $\sigma$ completely decoupled even with quantum effects included. We may demand that the $\phi$-matter coupling in (42) has a coupling constant $f$ which is dimensionless in any dimensions. This can be met if we replace the second term in the parenthesis in (42) by

$$ -\frac{1}{2} f^2 \phi^2/\sqrt{-\lambda} \Delta^2, $$

resulting in $m_*$, which is shown to be completely $\sigma$-independent in the E frame, even with the quantum effect included.

We find, on the other hand, that the term of the nonminimal coupling of the form $\phi^2 R$, as in the first term of (1), is multiplied always by a dimensionless constant for any dimensions. This is a fact that underlies the whole discussion of scale invariance, constituting another difference from the models which allow more general functions of the scalar field. Scale invariance is respected also in a new approach to the scalar-tensor theory based on $M_4 \times Z_2$. 27)

The present study is limited because a model with a single scalar field might be too simple to account for a possible nonzero cosmological constant. 4) Looking further into the hesitation behavior would be useful to acquire more insight into the time-(non)variability of various coupling constants, probably a related issue which seems to deserve further scrutiny. 18)

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4) See, for example, Ref. 5) for a model with another scalar field implementing the idea of a "sporadic" occurrence of a small but nonzero cosmological constant.
Appendix A

Another Attractor Solution

It seems interesting to consider the fate of the matter energy $\rho_{*r}$ if we assume $\zeta$ which violates (25). Rather unexpectedly, $\rho_{*r} < 0$ is evaded automatically. The solutions for $\zeta$ smaller than the critical value given by (25) tend to the "vacuum" solution given by ($n = 0$ for simplicity)

$$a(t_*) = t^{\alpha_*} \quad \text{with} \quad \alpha_* = \frac{1}{8}\zeta^{-2}, \quad (A\cdot1)$$

$$\sigma(t_*) = \frac{1}{2\zeta} \ln t_* + \bar{\sigma}, \quad (A\cdot2)$$

$$\rho_{*r}(t_*) = 0, \quad (A\cdot3)$$

where $\bar{\sigma}$ is given by

$$\Lambda e^{-4\zeta \bar{\sigma}} = \frac{3 - 8\zeta^2}{64\zeta^4}. \quad (A\cdot4)$$

Note that these agree with (20) ~ (23) with $n = 0$ for $\zeta = \zeta_0 = 1/2$.

The two solutions may be depicted schematically as in Fig. 9. For $\zeta > \zeta_0$, the solution with $\rho_{*r} > 0$, represented by Track 1, is an attractor. Suppose we descend along Track 1. At $\zeta = \zeta_0$, one switches to Track 2 for $\rho_{*r} = 0$ instead of yielding negative matter energy. We in fact have examples to show that the lower half of Track 1 is a repeller.

We may respect (25) as long as we are interested only in a universe which accommodates nontrivial matter content asymptotically.
Appendix B
— Hesitation Behavior —

Suppose there was a period in which \( \rho_{*\sigma} \gg \rho_{*r} \) in the very early universe. This is reasonably expected if reheating after inflation was not too large to allow overwhelming recovery of \( \rho_{*r} \), which had been extremely red-shifted during inflation. Note that \( \rho_{*\sigma} \) should have stayed basically of the order of \( \Lambda \) when \( \sigma \) was rolling down the slope of the exponential potential, as given by (12).

Also, the exponential slope is so steep that \( V(\sigma) \) rapidly becomes small, leading to a \( K_{\sigma} \)-dominated universe, where

\[
K_{\sigma} = \frac{1}{2} \dot{\sigma}^2. \tag{B·1}
\]

As is well known, the matter energy density of only the kinetic energy of a scalar field is equivalent to the equation of state \( p = \rho \), resulting in the expansion law

\[
H_* = \frac{1}{3} t_*^{-1}. \tag{B·2}
\]

Using this in (15) also with \( V' \) ignored, we find the solution

\[
\dot{\sigma} \sim \beta t_*^{-1}, \tag{B·3}
\]

and hence

\[
\sigma \sim \beta \ln t_*. \tag{B·4}
\]

It also follows that

\[
\rho_{*\sigma} \approx K_\sigma \sim \frac{1}{2} \beta^2 t_*^{-2}. \tag{B·5}
\]

The coefficient \( \beta \) can be determined if we substitute (B·2) and (B·5) into (14), also with \( V(\sigma) \) dropped:

\[
\beta = \sqrt{\frac{2}{3}}. \tag{B·6}
\]

In fact, the solution given by (B·2) and (B·4) with (B·6) is found to be an attractor.

On the other hand, substituting (B·2) in (16) yields

\[
\rho_{*r} \approx t_*^{-4/3}, \tag{B·7}
\]

which falls off more slowly than \( \rho_{*\sigma} \) as given by (B·5). For this reason \( \rho_{*r} \) soon catches up and eventually surpasses \( \rho_{*\sigma} \). After this time, the universe enters the radiation-dominated epoch with

\[
H_* = (1/2) t_*^{-1}. \tag{B·8}
\]

Using this in (15) we obtain

\[
\dot{\sigma} \sim t_*^{-3/2}. \tag{B·9}
\]
showing that $\sigma$ comes to rest quickly. This is the beginning of the hesitation phase. We also find

$$\rho_{*\sigma} \approx K_\sigma \sim t_*^{-3},$$

which decreases much faster than $\rho_{*\tau} \sim t_*^{-2}$.

The potential $V(\sigma) \sim e^{-4k_\sigma}$ had already been very small at the onset of the hesitation period, remaining there since. Eventually, however, $K_\sigma$ reaches this small value, so that (15) must be solved with $V'$ included again, and hence the hesitation ends.

Figure 5 shows a period during which (B·2) is obeyed. We can also derive the behavior of $a$ in the $J$ frame during the $K_\sigma$-dominated epoch. From (B·4) we first find

$$\Omega \sim \phi \sim t_*^{\xi \beta}.$$  

Using this together with (B·2) in (19) we obtain

$$a(t_*) \sim t^\tilde{\alpha}, \quad \text{with} \quad \tilde{\alpha} = \frac{1}{3} - \zeta \beta. \tag{B·12}$$

From (25) with $n = 0$ and (B·6) we find

$$\tilde{\alpha} \leq \frac{1}{3} - \frac{1}{\sqrt{6}} = -0.108 < 0, \tag{B·13}$$

implying contraction of the universe in the $J$ frame during the $K_\sigma$-dominated universe. In Fig. 7 we recognize the slope of log $a$ against log $t_*$ for $\zeta = 1$,

$$\tilde{\alpha} = \frac{1}{3} - \sqrt{\frac{2}{3}} = -0.483. \tag{B·14}$$

**Appendix C**

--- Loop Integrals ---

We show some of the details of calculating loop diagrams of Fig. 8. The basic coupling to the linear $\sigma$ is obtained by expanding the exponential in (48):

$$-\tilde{L}_{\sigma \Phi} = 2\zeta(\nu - 2)\sigma \left( \frac{1}{2} m^2 + \frac{1}{4!} \lambda \Phi^4 \right). \tag{C·1}$$

Here, $m^2 = \xi^{-1} f^2$. The diagram (a) comes from the second term of (C·1), while the diagram (b) from the first. The 4-vertex in (b) represents the simple $\lambda/4! \Phi^4$ without $\sigma$ attached.

Considering $\sigma$ simply as a constant, we compute the amplitude for the diagram (a):

$$\mathcal{M}_a = -i(2\pi)^{-N} 2(\nu - 2)\zeta \sigma \lambda \int d^N k \frac{1}{(k^2 + m^2)}. \tag{C·2}$$

We use

$$\int d^N k \frac{1}{(k^2 + m^2)} = i\pi^2 m^2 \Gamma(1 - \nu), \tag{C·3}$$
where we have put $N = 4$, except inside the $\Gamma$ function. Substituting (C-3) into (C-2), and further using

\[(2 - \nu)\Gamma(1 - \nu) = \frac{1}{1-\nu}(2 - \nu)\Gamma(2 - \nu)\]

\[= \frac{1}{1-\nu}\Gamma(3 - \nu)\frac{\nu-2}{2} - 1, \quad (C-4)\]

we obtain

\[M_a = \zeta\sigma\frac{\lambda}{8\pi^2}m^2_{*0}. \quad (C-5)\]

Corresponding to (b), we find

\[M_b = i(2\pi)^{-N}2(\nu - 2)\zeta\sigma m^2_{*0}\lambda\int d^N k \frac{1}{(k^2 + m^2_{*0})^{2}}. \quad (C-6)\]

The integral is now obtained by operating with $(-\partial/\partial m^2_{*0})$ on the integral in (C-3), thus yielding

\[M_b = M_a. \quad (C-7)\]

Adding $M_a$ and $M_b$, we finally obtain

\[M = \zeta\sigma\frac{\lambda}{4\pi^2}m^2_{*0}, \quad (C-8)\]

from which follow (50) and (51).

References

15) C. Santos and R. Gregory, gr-qc/9611065.
16) Y. Fujii, gr-qc/9609044.
23) See, for example, E. Fischbach and C. Talmadge, Nature 356 (1992), 207.