

## Diverging Overland Flow

### 2. Application to Natural Watersheds

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The diverging overland flow model is utilized to predict surface runoff from several natural watersheds. The Lax – Wendroff scheme is used to obtain numerical solutions. This scheme is explicit, single step and second order. An analytical criterion is developed to determine time step for computation for a specified grid length in space. This criterion ensures numerical stability and convergence. The diverging geometry is specified from watershed topography. Thus the model contains one parameter which is optimized for each watershed by the Rosenbrock – Palmer optimization algorithm. The model results compare satisfactorily with runoff observations.

### Introduction

In part I of this series we developed analytical solutions for diverging overland flow for space – time invariant rainfall and discussed the effect of explicitly considering infiltration on these solutions. In this paper, the concluding part of the series, we apply the diverging overland flow model to several natural watersheds and examine its ability to predict surface runoff from these watersheds.

The diverging flow equation, obtained by coupling the continuity equation and the kinematic approximation to the momentum equation, can be written as

$$\frac{\partial \bar{h}}{\partial t} + n\alpha \bar{h}^{n-1} \frac{\partial \bar{h}}{\partial x} + \frac{\alpha \bar{h}^n}{r} = q(x, t) \quad (1)$$

where  $h$  is the depth of flow,  $q(r,t)$  the rate of effective lateral inflow per unit area varying in space  $r$  and time  $t$ ,  $\alpha$  the kinematic friction parameter and  $n$  is an exponent. This equation will be solved numerically subject to the initial conditions

$$h(aR,t) = 0, \quad 0 \leq t \leq T; \quad h(r,0) = 0, \quad aR \leq r \leq R \quad (2)$$

where  $R$  is the length of the diverging section,  $a$  is a divergence parameter and  $T$  is the duration of  $q(r,t)$ .

The Lax – Wendroff scheme (Houghton and Kasahara 1968) which has been successfully used in many investigations on kinematic wave modeling of watershed runoff (Kibler and Woolhiser 1970; Singh 1980; Agiralioglu and Singh 1980), is formulated to solve Eq. (1). Writing Eq. (1) as

$$\frac{\partial h}{\partial t} = -n\alpha h^{n-1} \frac{\partial h}{\partial r} - \frac{\alpha h^n}{r} + q \quad (3)$$

Expanding  $h(r,t + \Delta t)$  by Taylor Series,

$$h(r,t+\Delta t) = h(r,t) + \Delta t \frac{\partial h}{\partial t} + \frac{(\Delta t)^2}{2} \frac{\partial^2 h}{\partial t^2} + HOT \quad (4)$$

where *HOT* denotes higher order terms. Differentiating Eq. (3) with respect to  $t$ , we get

$$\frac{\partial^2 h}{\partial t^2} = -\alpha \frac{\partial}{\partial r} \left( n h^{n-1} \frac{\partial h}{\partial t} \right) - n \frac{\alpha}{r} h^{n-1} \frac{\partial h}{\partial t} + \frac{\partial q}{\partial t} \quad (5)$$

Inserting Eqs. (3) and (5) into Eq. (4), neglecting *HOT*, and simplifying,

$$h(r,t+\Delta t) = h(r,t) + \left[ q - n\alpha h^{n-1} \frac{\partial h}{\partial r} - \alpha \frac{h^n}{r} \right] \left[ \Delta t - \frac{(\Delta t)^2}{2} n\alpha \frac{h^{n-1}}{r} \right] + \frac{(\Delta t)^2}{2} \left\{ \frac{\partial q}{\partial t} - \alpha \frac{\partial}{\partial r} \left[ n h^{n-1} \left( q - n\alpha h^{n-1} \frac{\partial h}{\partial r} - \alpha \frac{h^n}{r} \right) \right] \right\} \quad (6)$$

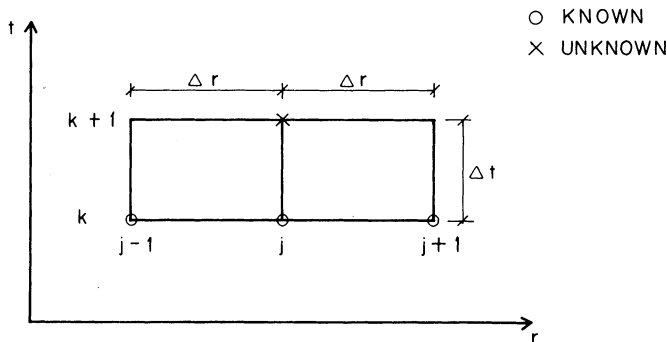


Fig. 1. Definition sketch for Lax-Wendroff scheme.

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Let there be  $j = 1, 2, \dots, M$  points at which the depth of flow is to be computed. The point  $j = M$  will correspond to the downstream point. Then, following the notation in Fig. 1, Eq. (6) can be written in finite difference form as

$$\begin{aligned}
 h_j^{k+1} = & h_j^k + \left[ q_j^k - \alpha \frac{(h_{j+1}^k)^n - (h_{j-1}^k)^n}{2\Delta r} - \alpha \frac{(h_j^k)^n}{r_j^k} \right] \\
 & \left[ \Delta t - \frac{(\Delta t)^2}{2} n \frac{(h_j^k)^{n-1}}{r_j^k} \right] \\
 & + \frac{(\Delta t)^2}{2} \left\{ \frac{q_j^{k+1} - q_j^k}{\Delta t} - \left[ \alpha \frac{n}{\Delta r} \frac{(h_{j+1}^k)^{n-1} + (h_j^k)^{n-1}}{2} \right. \right. \\
 & \left. \left. - \alpha \frac{(h_{j+1}^k)^n - (h_j^k)^n}{\Delta r} + \frac{q_{j+1}^k + q_j^k}{2} - \alpha \frac{(h_{j+1}^k)^n + (h_j^k)^n}{r_{j+1}^k + r_j^k} \right] \right\} \\
 & - \left[ \frac{n\alpha}{\Delta r} \frac{(h_j^k)^{n-1} - (h_{j-1}^k)^{n-1}}{2} \right. \\
 & \left. \left[ -\alpha \frac{(h_j^k)^n - (h_{j-1}^k)^n}{\Delta t} + \frac{q_j^k + q_{j-1}^k}{2} - \alpha \frac{(h_j^k)^n + (h_{j-1}^k)^n}{r_j^k + r_{j-1}^k} \right] \right\} \\
 j = & 1, 2, \dots, M-1 \tag{7}
 \end{aligned}$$

Eq. (7) gives a second order approximation for the depth of flow. We cannot use Eq. (7) for  $j = M$ . An alternative is to write a first order approximation of Eq. (6) for  $j = M$ . However, it was found that this would lead to numerical instability. A simple but more elegant procedure was therefore developed for computation at the downstream boundary.

### Computation at Downstream Boundary

The characteristic equations of Eq. (1) are

$$\frac{dh}{dr} = \frac{q}{n\alpha h^{n-1}} - \frac{h}{nr} \tag{8}$$

$$\frac{dt}{dr} = \frac{1}{n\alpha h^{n-1}} \tag{9}$$

We can solve these equations when  $q$  is or is not zero. First, consider the case when  $q \neq 0$ . Then the solution of Eq. (8), subject to the condition,  $r = r_{M-1}$ ,  $h = h_{M-1}$  with  $r = r_M$ ,  $h = h_M$  is

$$h_M = \left[ \frac{r_{M-1}^n}{r_M^n} h_{M-1}^n + \frac{q}{2\alpha r_M} (r_M^2 - r_{M-1}^2) \right]^{\frac{1}{n}} \tag{10}$$

Eq. (10) gives the depth at  $j = M$ . In order to find time we must substitute Eq. (10) into Eq. (9) to obtain,

$$\Delta t = \frac{1}{n\alpha} \int_{r_{M-1}}^{r_M} \left[ h_{M-1}^n \frac{r_{M-1}^{n-1}}{r^n} + \frac{q}{2\alpha r} (r^2 - r_{M-1}^2) \right]^{-(n-1)/n} dr \tag{11}$$

Eq. (11) can be easily solved by using a standard numerical integrator.

On the other hand, if  $q = 0$  then the solution of Eq. (8) is

$$h_M = h_{M-1} \left( \frac{r_{M-1}}{r_M} \right)^{\frac{1}{n}} \tag{12}$$

and  $\Delta t$  given by Eq. (9) is

$$\Delta t = \frac{(r_{M-1})^{-(n-1)/n}}{\alpha(2n-1)} (h_{M-1})^{1-n} (r_M^{(2n-1)/n} - r_{M-1}^{(2n-1)/n}) \tag{13}$$

### Computation Steps

It may be appropriate to briefly outline the steps involved in computation of  $h$  using Eq. (7) in conjunction with Eqs. (10)-(13).

1. Choose  $\Delta r$  appropriately. Normally  $M$  ranging anywhere between 8 and 15 will be appropriate.

2. Compute  $\Delta t$  using Eq. (11).

3. Compute  $h_M$  using Eq. (10).

4. Compare  $\Delta t$  with the first pulse of rainfall intensity lasting for  $d_1$  (see Fig. 2). If  $\Delta t$  is greater than or equal to  $d_1$ , compute  $h_j$ ,  $j = 1, 2, \dots, M-1$ , using Eq. (7).

If  $\Delta t$  greater than  $d_1$ , reduce this  $\Delta t$  to  $d_1$  and reduce  $h_M$  proportionately using linear interpolation. Using this reduced  $\Delta t$ , compute  $h_j$ ,  $j = 1, 2, 3, \dots, M-1$  using Eq. (7). This step is therefore based on the assumption that  $q$  is constant during  $d_1$  which it is. This is shown in Fig. 3.

5. Repeat the above steps until the duration of  $q$  equals the time of computation.

6. Repeat the above steps using Eqs. (12) – (13) in conjunction with Eq. (7) for  $q = 0$ .

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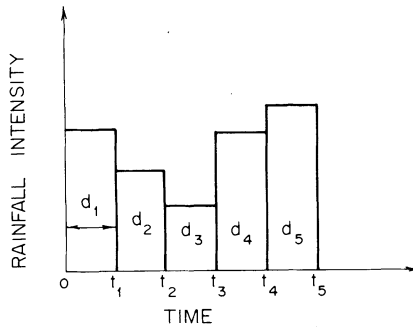


Fig. 2. Rainfall hyetograph

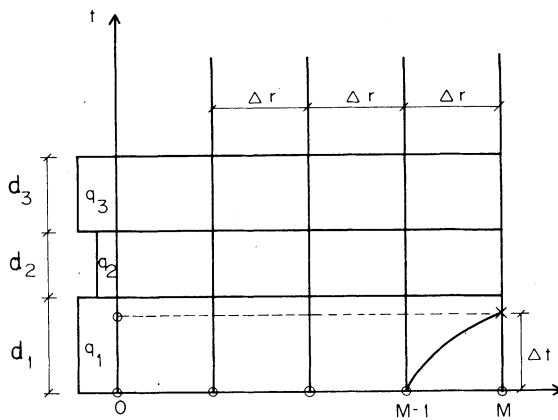


Fig. 3. Definition of the solution domain at the downstream boundary.

It may be noted that this procedure of apriori specifying  $\Delta t$  at each step of computation is very efficient, insures convergence and stability of the numerical scheme.

### Application to Natural Watersheds

The diverging overland flow model was applied to seven small natural watersheds in the U.S.A. Some pertinent characteristics of these watersheds are shown in Table 1. For a more complete discussion of these watersheds and rainfall – runoff data thereon see the U.S. Department of Agriculture publications entitled, »Hydrologic Data for Experimental Agricultural Watersheds in the United States.« These publications are released almost yearly and contain, on the average, one largest event a year on a watershed.

Table 1 – Characteristics of Agricultural Watersheds.

Serial Number	Watershed Name	Watershed Area (ha)	Weighted Average Slope (%)	Length of Flow (m)	Soil Characteristics
1	Coshocton, Ohio Watershed 111	0.48	7.52	144.8	Coshocton silt Loam
2	McCredie, Missouri Watershed W-1	61.92	3.42	1036.3	–
3	Stillwater, Oklahoma Watershed W-1	6.76	3.34	457.2	–
4	Riesel (Waco), Texas Watershed C	234.32	2.04	2438.4	Houston-Hunt clay
5	Riesel (Waco), Texas Watershed D	449.22	2.10	3657.6	Do
6	Riesel (Waco), Texas Watershed G	1772.59	2.06	6705.6	Houston-black clay
7	Hastings, Nebraska Watershed 2-H	1.21	6.10	182.9	Silt loam

**Geometric Representation**

The diverging geometry has 3 geometric parameters  $R$ ,  $a$  and  $\theta$  that need to be specified. In this study  $a$  was taken zero. This is reasonable for many watersheds.  $R$  was taken as the length of the watercourse from the most remote portion of the watershed to its outlet. Because the area must be preserved,  $\theta$  can be easily calculated. It is evident that the topographic map of a watershed will suffice to transform it in an equivalent diverging geometry (see Fig. 4).

**Rainfall Excess Determination**

Rainfall excess was determined using Philips equation (Philip 1957) which can be written as

$$f = A + St^{-.5} \tag{14}$$

where  $f$  is infiltration rate,  $t$  time and  $A$  and  $S$  are parameters dependent on Soil characteristics and initial moisture conditions.  $A$  was considered equivalent to steady infiltration. Thus it was determined from watershed soil characteristics.  $S$  was allowed to vary with each rainfall event, thus accounting for soil moisture conditions existing prior to its occurrence. It was estimated for each storm subject to preserving volume continuity. Agiralioglu and Singh (1980) have tabulated  $A$  and  $S$  for a number of rainfall-runoff events of a large number of small agricultural

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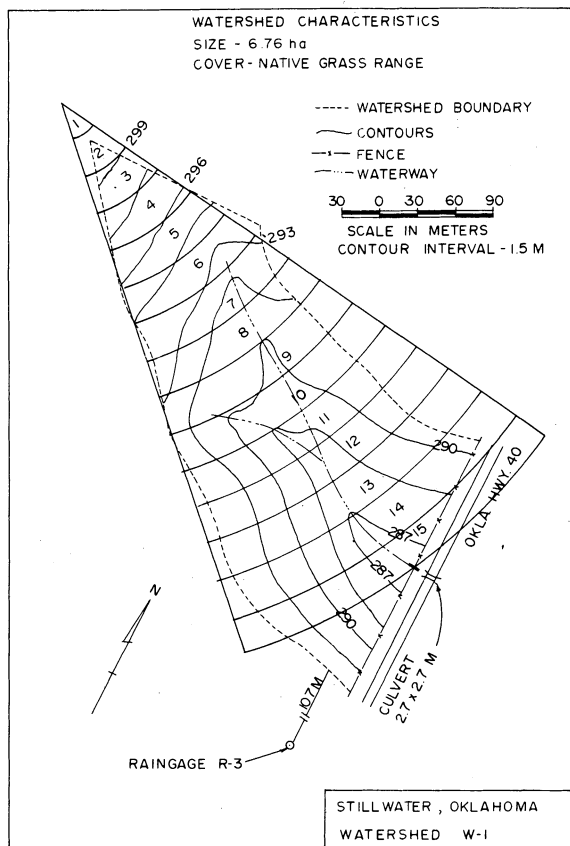


Fig. 4. Watershed W-1, Stillwater, Oklahoma.

watersheds in the U.S.A., the values used in this paper have been extracted from their study.

### Parameter Optimization

The diverging flow model has only one parameter  $\alpha$  which was estimated using the modified Rosenbrock optimization algorithm (Himmelblau 1972) subject to the objective function,

$$F = \sum_{j=1}^N (Q_{p0}(j) - Q_{pe}(j))^2 \Rightarrow \text{Min} \quad (15)$$

where  $F$  is the objective function or error criterion to be minimized,  $O_{p0}(j)$  the observed hydrograph peak for  $j$ th event,  $O_{pe}(j)$  the estimated hydrograph peak for  $j$ th event, and  $N$  the number of events in the optimization set. For each watershed

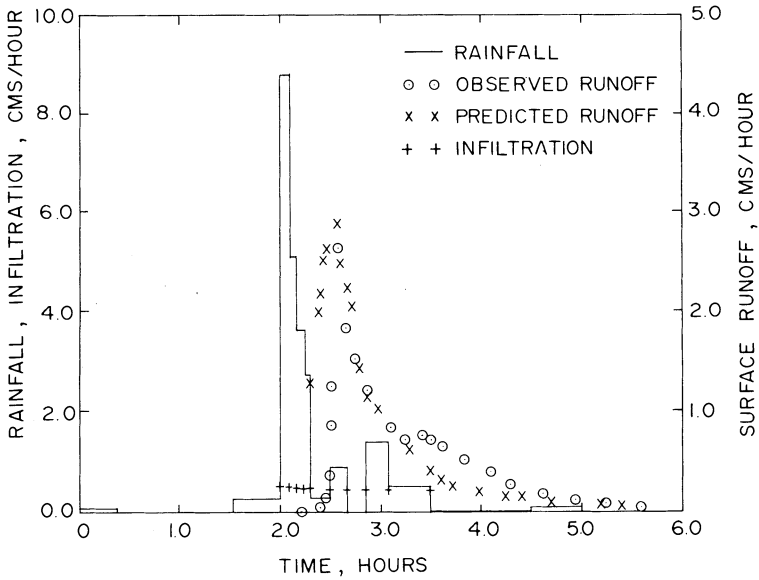


Fig. 5. Prediction of surface hydrograph for rainfall event of 3/15/1968 on watershed W-1, Stillwater, Oklahoma.

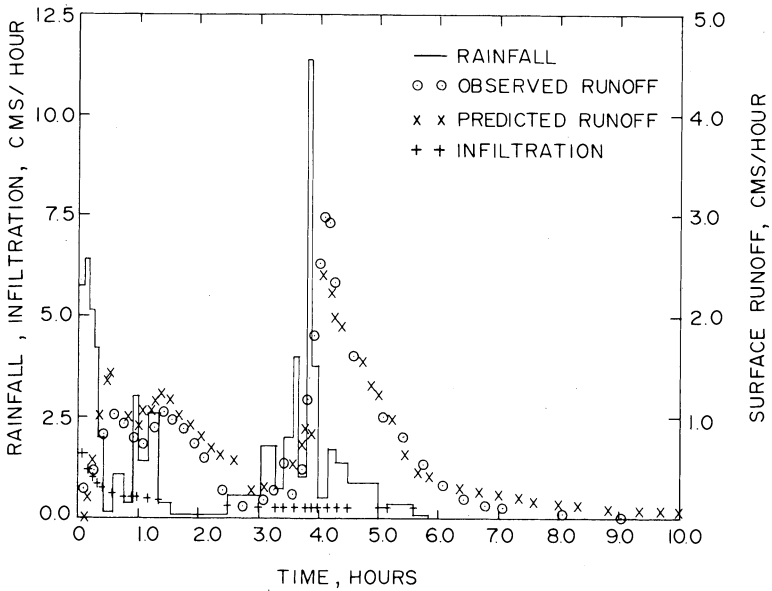


Fig. 6. Prediction of surface hydrograph for rainfall event of 6/7/1945, on watershed W-1, Mc. Credie, Missouri.



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rainfall-runoff events were divided into two sets: (1) optimization set, and (2) prediction set. These two sets did not have any events in common. The values of  $\alpha$  obtained for each watershed using its set of optimization events are given in Table 2.

Table 2 – Optimized Model Parameters.

Watershed Name	Optimized event number and date										Optimized Parameter $\alpha$ (cm <sup>3</sup> /sec)	n	
	1	2	3	4	5	6	7	8	9	10			
Coshocton, Ohio Watershed 111	9-23-45	6-12-57	6-28-57	4-25-61								3.9404	1.500
McCredie, Missouri Watershed W-1 Stillwater, Oklahoma Watershed W-1	7- 3-41	6-10-42	6- 8-43	5-14-45	5- 1-48	7-22-48	9-12-49	6-29-57	4- 5-65			12.5000	1.500
Riesel (Waco) Texas Watershed C	4-18-57	6-27-57	10- 1-59	10- 2-59	5-28-60	5-21-61	6- 7-62	5-10-64				15.000	1.500
Riesel (Waco) Texas Watershed D	4-24-57	5-13-57	7-16-61	6- 4-62	5- 9-57							13.7656	1.500
Riesel (Waco) Watershed G	5- 3-57	6-23-59	12-31-59	7-23-61	6- 4-62							21.8429	1.500
Hastings, Nebraska Watershed 2-H	2-14-59	6-23-59	12-31-59	7-16-61	7-23-61							12.5000	1.500
	7-13-52	6-20-42	8-29-44	6-12-58	5-22-54	7- 3-59	5-15-60	6- 1-51	6-26-52	8-23-62		12.1063	1.500

### Hydrograph Prediction

Utilizing  $\alpha$  obtained in the above manner runoff hydrograph was predicted for each event in the prediction set of each watershed. Sample predicted hydrographs are shown in Figs. 5-6. On comparing predicted hydrograph peaks with observed hydrograph peaks we found that they were in reasonable agreement. Hydrograph time and shape characteristics were predicted quite well. Although hydrograph timing was not involved in the procedure for estimating  $\alpha$ , the timing is well predicted.

A comparison of observed and computed hydrograph peak characteristics is shown in Table 3. It is seen that on the whole the relative error stays within 20% in prediction of hydrograph peak and within 35% in prediction of hydrograph peak time, although these are individual cases where the error goes very high. An examination of rainfall – runoff data for such cases revealed that the relatively high error was produced by inaccurate estimation of infiltration parameters and friction parameter  $\alpha$ . It must be kept in mind that the parameter  $\alpha$  was obtained by optimization using only a few events on a watershed. Therefore, it cannot realistically represent the true watershed conditions.

Table 3 – Comparison of Observed and Predicted Hydrograph Peak and its Timing for Agricultural Watersheds.

DIVERGING MODEL – PHILIP INFILTRATION PARAMETER									
Watershed Name	Observed Date	Predicted Hydrograph Peak (cm/hr)	Relative Hydrograph Peak (cm/hr)	Observed Error	Predicted Hydrograph Peak Time (min)	Relative Hydrograph Peak Time (Min)	Error	Remark	
Coshocton, Ohio Watershed 111	6-28-40	2.4130	1.6388	0.3208	84.0	86.0	-0.0238	evaluated	
Coshocton, Ohio Watershed 111	6-18-40	3.7338	1.6122	0.5685	18.0	16.0	0.1111		
Coshocton, Ohio Watershed 111	9- 1-50	3.9065	4.2381	-0.0849	46.0	80.0	-0.7391	evaluated	
McCredie, Missouri Watershed W-1	9-21-51	0.4648	0.7169	-0.5224	108.0	105.0	0.0278		
McCredie, Missouri Watershed 111	8-19-49	0.9119	1.4469	-0.5868	35.0	23.0	0.3429	evaluated	
McCredie, Missouri Watershed W-1	6- 7-45	2.9972	2.4299	0.1893	240.0	240.0	0.0000	evaluated	
McCredie, Missouri Watershed W-1	6-26-42	2.3978	2.2675	0.0543	60.0	66.5	-0.1081		
Stillwater, Oklahoma Watershed W-1	9-26-67	3.3782	3.1996	0.0529	65.0	91.2	-0.4034	evaluated	
Stillwater, Oklahoma Watershed W-1	6-24-67	2.7178	3.3318	-0.2259	50.0	62.0	-0.2400	evaluated	
Stillwater, Oklahoma Watershed W-1	3-18-68	2.6533	2.9126	-0.0978	21.0	20.0	0.0476	evaluated	
Stillwater, Oklahoma Watershed W-1	4- 3-68	3.4737	4.4805	-0.2298	23.0	16.0	0.3043	evaluated	
Riesel (Waco), Texas Watershed C	6-10-41	2.2403	1.9781	0.1170	81.0	54.0	0.3333		
Riesel (Waco), Texas Watershed C	6-23-59	1.5875	2.4656	-0.5531	107.0	38.0	0.6449		
Riesel (Waco), Texas Watershed C	7- 9-61	0.1265	0.1057	0.1641	30.0	10.0	0.6667		

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Riesel (Waco), Texas	5-10-65	3.5118	3.8724	-0.1027	45.0	28.0	0.3778	evaluated
Watershed C								
Riesel (Waco), Texas	5- 6-55	0.6934	0.1364	0.8033	33.0	5.0	0.8485	
Watershed D								
Riesel (Waco), Texas	7-16-61	0.4166	0.6627	-0.5930	86.0	35.0	0.5930	
Watershed D								
Riesel (Waco), Texas	5-10-65	2.2718	4.8601	-1.1393	41.0	28.0	0.3171	evaluated
Watershed D								
Riesel (Waco), Texas	7-14-41	0.2301	0.1854	0.1945	163.0	20.0	0.8773	
Watershed G								
Riesel (Waco), Texas	11-4-59	0.1887	0.1809	0.0413	198.0	135.0	0.3182	
Watershed D								
Riesel (Waco), Texas	3-29-65	2.4135	3.3469	-0.3867	145.0	128.0	0.1172	evaluated
Watershed G								
Hastings, Nebraska	8-11-39	2.8194	1.0350	0.6329	10.0	5.0	0.5000	
Watershed 2-H								
Hastings, Nebraska	5-20-49	0.7036	0.1494	0.7877	3.0	9.0	-2.0000	
Watershed 2-H								
Hastings, Nebraska	6-12-65	8.8138	7.9296	0.1003	9.0	23.0	-1.5556	evaluated
Watershed 2-H								
Hastings, Nebraska	6-12-65	2.1565	2.0110	0.0674	13.0	13.0	0.0000	evaluated
Watershed 2-H								
Hastings, Nebraska	6-29-65	2.0676	1.5612	0.2449	9.0	6.6	0.2639	
Watershed 2-H								
Hastings, Nebraska	8- 7-46	3.7592	1.5322	0.5924	7.0	7.0	0.0000	
Watershed 2-H								
Hastings, Nebraska	8- 7-42	2.5197	1.4870	0.4098	10.0	21.0	-1.1000	
Watershed 2-H								
Hastings, Nebraska	6-16-50	0.1735	0.0127	0.9269	2.0	1.0	0.5000	
Watershed 2-H								
Hastings, Nebraska	9- 7-42	3.5306	1.9168	0.4571	15.0	10.0	0.3333	
Watershed 2-H								
Hastings, Nebraska	9- 5-46	2.2885	1.4733	0.3562	9.0	15.0	-0.6667	
Watershed 2-H								

$$\text{Relative error} = \frac{Q_{p_0} - Q_{p_c}}{Q_{p_0}}$$

## Conclusions

On the basis of limited testing it can be reasonably argued that the diverging flow model is potentially promising and deserves further investigation. The model predicts hydrograph characteristics reasonably well. The parameter  $\alpha$  needs further investigation so that it can be estimated from watershed characteristics.

## Acknowledgement

The study was supported in part by funds provided by National Science Foundation under the project, »Free Boundary Problems in Water Resource Engineering,« NSF-ENG-79-23345.

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Received: 6 June, 1980

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