Note on the Bloch-Nordsieck’s Method

GYO TAKEDA, YASUTAKA TANIKAWA, TOSIYA TANIUTI

Physical Department, Kobe University

and

KEIITI SAeki

Faculty of Education, Kobe University

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The Bloch-Nordsieck’s method has been applied to mesonic systems by several authors. Here we examine the convergence character of this method and show that it is about the same with that of the current perturbation method. So the B-N’s method will be unsuccessful if the current perturbation treatment is incorrect for mesonic systems.

§ 1. The Bloch-Nordsieck’s method applied to mesonic systems

Since the Bloch-Nordsieck’s method was successfully applied to radiative processes in order to overcome the infrared catastrophe of electromagnetic systems, this same method has been applied also to mesonic systems. But, in latter cases, careful attention was never made about the applicability of it. So we want to inquire how is the case compared with the current perturbation method.

Here, as an illustration, we take up the symmetrical pseudoscalar meson theory with pseudovector coupling. Employing the same notations with those of the previous paper, the Hamiltonian becomes

\[ H = \mathbf{a} \cdot \mathbf{p} + m^2 + \frac{1}{2} \sum_{\mathbf{a}, \mathbf{k}} \omega_{\mathbf{a}} (P_{a, h} + Q_{a, h}^z) + (\frac{g}{\mu}) \sum_{\mathbf{a}, \mathbf{k}} (2\omega_{\mathbf{a}})^{-1/2} \tau_a (\mathbf{a} \cdot \mathbf{k} + \gamma_5 \omega_{\mathbf{k}}) (Q_{a, h} \cos \mathbf{k} \cdot \mathbf{r} - P_{a, h} \sin \mathbf{k} \cdot \mathbf{r}). \] (1)

The B-N’s method consists essentially in replacing uncommuting quantities \( a, \beta, \tau, \sigma \) by their classical representatives. To do this, we introduce three operators \( \Lambda, T \) and \( S \), given by

\[ \Lambda = \beta, \ T = t \cdot \tau, \ S = s \cdot \sigma \] (2)

where \( s \) and \( t \) are classical unit vectors. All of them have eigenvalues 1 and -1, so we can divide the wave function \( \mathcal{T} \) into eight parts according to signs of their eigenvalues

\[ \mathcal{T} = (\phi^+, \phi^-, \phi^+, \phi^-, \chi^+, \chi^-, \chi^+, \chi^-). \] (3)
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\( \phi \) and \( \chi \) belongs to \( A=1 \) and \(-1\) respectively. The first suffix on right shoulder of the wave function \( \phi \) and \( \chi \) corresponds to the sign of the eigenvalue of \( T \), and the second to that of \( S \).

From the Schrödinger equation

\[
(H-E)\Psi = 0 ,
\]

we can make eight equations

\[
\frac{1+A}{2} \cdot \frac{1+T}{2} \cdot \frac{1+S}{2} (H-E)\Psi = 0 .
\]

For example, preferring plus signs one and all, we have

\[
\langle m+(1/2) \sum_{a,k} A_{a,k} (P_{a,k} + Q_{a,k}) - E 
\]

\[+ (g/\mu) \sum_{a,k} (2Q_{a,k})^{-1/2} t_a (s \cdot k) (Q_{a,k} \cos k \cdot r - P_{a,k} \sin k \cdot r) \rangle \phi^{++} \]

\[= - (g/\mu) \sum_{a,k} (2Q_{a,k})^{-1/2} (Q_{a,k} \cos k \cdot r - P_{a,k} \sin k \cdot r) \]

\[\times \{ t_a (\sigma + s \cdot k) \phi^{+-} + (t_a + \tau_a) (s \cdot k) \phi^{+} + (t_a + \tau_a) (\sigma + s \cdot k) \phi^{-} \} \]

\[-i_B (s \cdot p) \chi^{+-} - i_B (s \cdot p) + (a \cdot p) \chi^{++} \]

\[= - (g/\mu) \sum_{a,k} Q_{a,k} (2Q_{a,k})^{-1/2} (Q_{a,k} \cos k \cdot r - P_{a,k} \sin k \cdot r) \]

\[\times \{ i_B t_a \chi^{++} + i_B (t_a + \tau_a) \chi^{--} \} .
\]

Other equations can be deduced from Eq. (6) by each combination of the following transformations:

(i) \( m \rightarrow -m \), \( \phi \rightarrow \chi \) and \( \chi \rightarrow \phi \).

(ii) \( t \rightarrow -t \) and change of the sign of first suffixes.

(iii) \( s \rightarrow -s \) and change of the sign of second suffixes.

When we attend to \( \phi^{++} \), the B-N's approximation is to omit other parts of \( \Psi \) and solve Eq. (6) without the right side.

In this approximation, a wave function and an proper energy corresponding to a single nucleon are as follows:

\[ \phi_0^{++} = \kappa^{++} Q^{-1/2} \exp \{ \sum_{a,k} A_{a,k} \sin k \cdot r (Q_{a,k} + (1/2) A_{a,k} \cos k \cdot r) \} \]

\[\times \Pi_{a,k} A_{a,k} (Q_{a,k} + A_{a,k} \cos k \cdot r) ,
\]

\[ E_0 = m + \sum_{a,k} (\omega_k / 2) - (1/2) \sum_{a,k} \omega_k A_{a,k}^2 ,
\]

where

\[ A_{a,k} = (g/\mu) (2Q_{a,k})^{-1/2} t_a (s \cdot k) \]

and \( \kappa^{++} \) is a unit spinor.

As an appraisal of errors introduced by this approximation, we shall adopt norms of other parts of \( \Psi \) and calculate them in the first approximation.

Then, solving Eq. (5), we obtain
\[ (1/12\pi)(g^2/4\pi)(K_c/\mu)^4 > |\phi^+|^2, |\phi^-|^2 \]
\[ > (1/6\pi)(g^2/4\pi)(K_c/\mu)^2 \exp \{ - (1/3\pi)(g^2/4\pi)(K_c/\mu)^2 \}, \]
\[ |\phi^-|^2 \sim (1/3\pi)(g^2/4\pi)(K_c/\mu)^2, \]
\[ |\chi^+|^2 \sim (1/24\pi)(g^2/4\pi)(K_c/\mu^2)(K_c/m)^2, \]
\[ |\chi^-|^2 \sim (1/16\pi)(g^2/4\pi)(K_c/\mu^2)(K_c/m)^2, |\chi^-|^2 = 0. \]

\( K_c \) is a cut-off meson momentum, and we have used a relation

\[ m \gg K_c \gg \mu, \]

which is not satisfied in practical cases but sufficient for rough estimation. Exact expressions for \(|\phi^+|^2\) and \(|\phi^-|^2\) are

\[ |\phi^+|^2 = \sum_{n_{sk}, n'_{sk}} |c^+(n_{sk}, n'_{sk}, \cdots)|^2, |\phi^-|^2 = \sum_{n_{sk}, n'_{sk}} |c^-(n_{sk}, n'_{sk}, \cdots)|^2 \]

with

\[ C^+(n_{sk}, n'_{sk}, \cdots) = - (\sum_{a,b} n_{a,k} \omega_k)^{-1} \sum_{a,b} (g/\mu)(2\omega_k)^{-1/2} \]
\[ \times t_a(s-s', k)u^+ K(-A_{sk}, n_{sk}; A_{sk}, I_{sk}) I_{sk} K(-A_{sk}, n_{sk}; O_{sk}), \]
\[ C^-(n_{sk}, n'_{sk}, \cdots) = - (\sum_{a,b} n_{a,k} \omega_k)^{-1} \sum_{a,b} (g/\mu)(2\omega_k)^{-1/2} \]
\[ \times (\tau_a - \tau_b)(s \cdot k)u^+ K(A_{sk}, n_{sk}; A_{sk}, I_{sk}) I_{sk} K(-A_{sk}, n_{sk}; A_{sk}, O_{sk}). \]

\( K(-A_{sk}, n_{sk}; A_{sk}, m_{sk}) \) is the same quantity used in B-N's paper.

If we replace \( \sum_{a,b} n_{a,k} \omega_k \) by \( \mu \) in Eqs. (12) and make summations of Eqs. (11), we obtain the upper limit of \(|\phi^+|^2\) and \(|\phi^-|^2\). And the lower one is obtained by limiting the summations to those terms which satisfy a condition

\[ n_{sk} + n'_{sk} + \cdots = 1. \]

We have already employed these limits in the first equation of (10).

§ 2. Comparison with perturbation method

Next we consider a nucleon state at rest, with positive energy, \( \sigma \)- and \( \tau \)-spin parallel to \( s \) and \( t \) respectively, by the usual perturbation method. Then, a norm of the first approximation is

\[ \sim (3/4\pi)(g^2/4\pi)(K_c/\mu)^2 + (3/32\pi)(g^2/4\pi)(K_c/\mu)^2(K_c/m)^2, \]

of which the second term comes from a negative energy part. Comparing with Eqs. (10), we shall find each of them has the same order of magnitude, irrespective of the magnitude of \( g^2/4\pi \).
Now, the wave function of the B-N's method including higher approximations can be written as follows:

$$\psi = \psi_0 + \psi_1 + \psi_2 + \cdots$$  \hspace{1cm} (15)

and if it is admitted to develop $\psi_i$ in $g$,

$$\begin{align*}
\psi_0 &= \psi_{0,0} + g \psi_{0,1} + g^2 \psi_{0,2} + \cdots , \\
\psi_1 &= g \psi_{1,1} + g^2 \psi_{1,2} + \cdots , \\
\psi_2 &= g^2 \psi_{2,2} + \cdots , \\
&\vdots
\end{align*}$$  \hspace{1cm} (16)

$\psi_{0,0}$ is just the same with the corresponding perturbational wave function of 0-th approximation. So both wave functions, Bloch-Nordsieck's and perturbational, which are solutions of Eq. (4), have about the same region of convergence including $g=0$ and coincide with each other in the limit of $g \to 0$. This means that the B-N's method cannot improve the perturbational one and is rather injurious in its complexity.

§ 3. Discussion of results

Before discussing the above results, we make similar estimation for an electron wave function. The results are

$$|\psi_1|^2 \sim f(\nu) \left( \frac{e^2}{4\pi} \right) \left( K_e/m \right)^2 + g(\nu) \left( \frac{e^2}{4\pi} \right)^2 \left( K_e/m \right)^2$$  \hspace{1cm} (17)

in B-N's method, where

$$f(\nu) = \frac{(1 - \nu^2)}{4\pi},$$

$$g(\nu) = \begin{cases} 
16\nu^6/9\pi^2 & \text{for } \nu < 1, \\
(1 - \nu^2)/\pi^2 \log(1 - \nu^2)^2 & \text{for } 1 - \nu < 1,
\end{cases}$$  \hspace{1cm} (18)

and, in the perturbation methods,

$$|\psi_1|^2 \sim \left( \frac{1}{4\pi} \right) \left( 1 - \nu^2/3 \right) \left( \frac{e^2}{4\pi} \right) \left( K_e/m \right)^2 + \left( 2\nu^2/3\pi \right) \left( \frac{e^2}{4\pi} \right) \log(K_e/K_{\text{min}}).$$  \hspace{1cm} (19)

$K_{\text{min}}$ is a cut-off momentum of the low frequency side, and the last term of (19) is related to the well known infrared catastrophe. Here we have considered a electron with an average velocity $\nu$, because radiation fields around it are produced principally by its mass motion.

But, in cases of mesonic fields, it is no matter whether a nucleon is at rest or in motion.

This infrared catastrophe destroys the convergency of perturbation method and is due to a strong coupling between an electron and low frequency photons.
That is, the Fourier decompose of interaction Hamiltonian contains a factor $\omega^{-1/2}$ and the energy difference between an initial and final state combined by this is $\omega$, if both of them belong to the positive or negative energy state. So transitions between positive energy states or those between negative energy states are produced very oftenly by low frequency photons and we cannot neglect them in 0–approximation.

In $B$-$N$'s method, by replacing $\alpha$ by its average $\nu$, we have succeeded in introducing these transitions in 0–approximation. So remaining transitions are only positive to negative, or negative to positive transitions, and energy differences as large as $2m$ eliminate the infrared catastrophe.

Now we are in a position to answer the question "Why the $B$-$N$'s method doesn't improve the perturbation method when applied to mesonic systems?" In the first $B$-$N$'s approximation we have neglected states $\phi^{+-}$, $\phi^{-+}$ etc., each of which contains many different states characterized by numbers of unbound mesons. We shall denote them by $\phi^{+-}(n_{ak})$, $\phi^{-+}(n_{ak})$ and so on. And in the next approximation transitions to these neglected states can occur, probabilities of which depend on (I) the magnitude of the energy differences between initial and final states and (II) the aspect of mesonic bound fields around a nucleon.

As for the latter point, the origin of the bound fields $Q_{ak}$ of $\phi^{+-}$ are shifted by $A_{ak}\cos k r$ as seen in Eq. (7) and those of $\phi^{+-}$, $\chi^{+-}$ and $\chi^{-+}$ by $-A_{ak}\cos k r$. This difference brings a damping factor $\exp\{-\left[\frac{1}{3\pi}\right]\left(\frac{\kappa^2}{4\pi}\right)(K\mu)^2\}$ to the corresponding probabilities between the former and the latters, for example, $\phi^{+-}(n_{ak})$ to $\phi^{+-}(n_{ak})$, while many states with different $n_{ak}$ can be excited.

These two competing effects are just compensated in electro-magnetic cases, but this time circumstances will not be so simple.

In Table I we tabulate transition schemes appearing in the next approximation with the differences of energy and those of the aspect of bound field between initial and final states. Even if the damping factor reduces transition probabilities from $\phi^{+-}$ to $\phi^{-+}$ and $\phi^{-+}$, we can easily see that the presence of $\phi^{+-}$ transition

<table>
<thead>
<tr>
<th>Initial state</th>
<th>Final state</th>
<th>Energy difference</th>
<th>Aspect of bound field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi^{+-}(n_{ak})$</td>
<td>$\phi^{-+}(n_{ak})$</td>
<td>$\sum a_{ak} n_{ak} \omega_{k}$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$\phi^{-+}(n_{ak})$</td>
<td>$\chi^{+-}(1_{ak})$</td>
<td>$\sum a_{ak} \omega_{k}$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$\chi^{+-}(1_{ak})$</td>
<td>$\chi^{+-}(n_{ak})$</td>
<td>$\sim 2m$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$\chi^{-+}(n_{ak})$</td>
<td>$\chi^{-+}(n_{ak})$</td>
<td>$\sim 2m$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$\chi^{-+}$ (forbidden)</td>
<td>$\chi^{-+}$ (forbidden)</td>
<td>$\sim 2m$</td>
<td>$\times$</td>
</tr>
</tbody>
</table>

The origin of bound field of a final state is shifted to the same or opposite direction with that of a initial state $\phi^{+-}$, according as its corresponding transition scheme has a sign $\circ$ or $\times$. 


makes B-N's method as worse as the perturbation method.

Until now, we only consider the symmetrical pseudoscalar theory with pseudo-vector coupling. But B-N's method is unsuccessful for any of other meson theories except the neutral scalar and pseudoscalar ones in which cases mesonic radiations are only produced by mass motions of a nucleon.

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