NOTES AND CORRESPONDENCE

The Deep-Atmosphere Euler Equations in a Generalized Vertical Coordinate

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ABSTRACT

Previous analysis of the hydrostatic primitive equations using a generalized vertical coordinate is extended to the deep-atmosphere nonhydrostatic Euler equations, and some special vertical coordinates of interest are noted. Energy and axial angular momentum budgets are also derived. This would facilitate the development of conserving finite-difference schemes for deep-atmosphere models. It is found that the implied principles of energy and axial angular momentum conservation depend on the form of the upper boundary. In particular, for a modeled atmosphere of finite extent, global energy conservation is only obtained for a rigid lid, fixed in space and time. To additionally conserve global axial angular momentum, the height of the lid cannot vary with longitude.

1. Introduction

The hydrostatic primitive equations were reviewed and analyzed in Kasahara (1974, hereafter K74) using a generalized vertical coordinate, defined to be any variable that is a single-valued monotonic function of geometric height. This influential review has proven to be a valuable reference, much cited by atmospheric modelers. In particular, some mathematical subtleties, encountered when deriving energetics and easy to overlook, are well expounded.

Almost all of today's global atmospheric models use the hydrostatic primitive equations in conjunction with a terrain-following vertical coordinate—this greatly simplifies the application of lower boundary conditions—and the formulations of their underlying dynamical cores fall within K74's framework. However, those of several recent models do not. These can be grouped into two classes.

First, the Qian et al. (1998) and Yeh et al. (2002) models employ the somewhat more general "nonhydrostatic primitive equations" [i.e., the hydrostatic primitive equations but with the vertical acceleration term, $\frac{Dw}{Dt}$, restored in the vertical momentum equation (see Tanguay et al. 1990)], together with Laprise's (1992) terrain-following "hydrostatic pressure" as vertical coordinate. These equations (as well as the hydrostatic primitive equations) make the shallow-atmosphere approximation. [The shallow-atmosphere equations may be obtained from the deep-atmosphere Euler ones by (a) setting the spherical polar coordinate $r$ to the earth's mean radius $a$ wherever it appears undifferentiated and (b) dropping all of the $2\Omega \cos \phi$ terms in the components of the momentum equation, as well as all metric terms not involving $\tan \phi$. The first step corresponds to the shallow-atmosphere approximation per se, whereas step b corresponds to the "traditional" approximation (e.g., Phillips 1973; White and Bromley 1995) and is necessary in order to obtain conservation principles for energy, axial angular momentum, and potential vorticity, analogous to those of the deep-atmosphere equations.]

Second, the dynamical core of the Met Office's recent global unified model [see Cullen et al. (1997) for an early overview of its formulation] does not make the shallow-atmosphere assumption but is instead based on the more complete deep-atmosphere Euler equations.

To put the above discussion into better perspective, note that there are four combinations of hydrostatic/nonhydrostatic and shallow-/deep-atmosphere approximations [see Staniforth (2001) for a comparative discussion of their virtues and vices], namely, the unapproximated Euler equations (nonhydrostatic deep), the nonhydrostatic primitive equations of Tanguay et al. (1990; nonhydrostatic shallow), the quasihydrostatic equations of White and Bromley (1995; hydrostatic deep), and the hydrostatic primitive equations of, for example, K74 (hydrostatic shallow). Numerical weather and climate prediction models generally employ one of these four equation sets.
With the recent trend toward the use of more complete equation sets, it is therefore timely to extend K74’s analysis of the shallow-atmosphere hydrostatic primitive equations to the deep-atmosphere nonhydrostatic Euler equations using a generalized vertical coordinate that can, as a special case, be terrain following. This is the purpose of the present work.

The plan of the paper is as follows. In section 2, the dry nonhydrostatic Euler equations for a rotating deep atmosphere are first given in spherical polar coordinates. They are then transformed using a generalized vertical coordinate so that a variety of vertical coordinate systems then ensue as special cases. As in K74, moisture is neglected from the analysis since there are many different ways of incorporating it, and it does not directly affect the choice of vertical coordinate anyways. The equations governing the evolution of total energy and axial angular momentum are derived (in sections 3 and 4, respectively) and the requirements on the upper boundary for conservation principles for each quantity to exist are determined. (Conservation of a quantity is taken here to mean that the global integral of that quantity is invariant.) The shallow-atmosphere equivalents of these results are also given (section 5) and related to those of K74. A summary and conclusions are given in section 6. Finally, to avoid overburdening the mathematical developments in the remainder of the paper, some useful identities are given in the appendix.

2. Deep-atmosphere governing equations

a. Deep-atmosphere equations in spherical polar coordinates

In standard notation, the dry equations for a rotating spherical deep atmosphere are (Daley 1988; Thuburn et al. 2002):

\[
\begin{align*}
\frac{Du}{Dt} - \frac{\nu w \tan \phi}{r} + \frac{\nu w}{r} - 2\Omega w \sin \phi + 2\Omega w \cos \phi \\
+ \frac{1}{\rho r \cos \phi} \frac{\partial \rho}{\partial \phi} &= F^e, \\
\frac{Du}{Dt} + \frac{u^2 \tan \phi}{r} + \frac{\nu w}{r} + 2\Omega u \sin \phi + \frac{1}{\rho r \cos \phi} \frac{\partial \rho}{\partial \phi} &= F^v, \\
\delta \frac{Dw}{Dt} - \frac{(u^2 + v^2)}{r} - 2\Omega u \cos \phi + g + \frac{1}{\rho} \frac{\partial \rho}{\partial \phi} &= \delta_v F^e, \\
\frac{D\rho}{Dt} + \rho \left[ \frac{1}{r \cos \phi} \frac{\partial u}{\partial \phi} + \frac{1}{r \cos \phi} \frac{v}{\partial \phi} \cos \phi \right] + \frac{1}{\rho r^2 \sin \phi} \frac{\partial (r^2 w)}{\partial r} &= 0,
\end{align*}
\]

\[
\frac{D(c_T)}{Dt} - \frac{1}{\rho} \frac{D\rho}{Dt} = F^T, \\
p = \rho RT,
\]

where

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{r \cos \phi} \frac{\partial}{\partial \phi} + \frac{v}{r} \frac{\partial}{\partial \phi} + w \frac{\partial}{\partial r},
\]

\[
R = c_p - c_v,
\]

\((F^e, F^v, F^\theta)\) and \(F^\theta\) are any parametrized source/sink terms, \(g = \frac{\partial \Phi}{\partial r}\) is the gravitational acceleration, and \(\Phi = \Phi(r)\) is the geopotential [the “normal” geopotential in the terminology of Phillips (1973)].

Equations (2.1)–(2.6) are, respectively, three components of the momentum equation, the continuity equation, the thermodynamic equation, and the equation of state. A “vertical acceleration” switch has been introduced: vertical acceleration is retained or dropped according to whether \(\delta_v\) is, respectively, set to unity or zero. The nonhydrostatic primitive equation set of the new dynamical core of the Met Office’s unified model [see Cullen et al. (1997) for an early overview] corresponds to the former choice. The quasihydrostatic set, given in section 4a of White and Bromley (1995), corresponds to the latter; both \(Dw/Dt\) and \(F^\theta\) are neglected such that the remaining terms are in quasihydrostatic balance, analogous to imposing hydrostatic balance for shallow atmospheres (cf. Laprise 1992).

b. Transformation to generalized vertical coordinate \(s\)

Following K74 for the shallow-atmosphere hydrostatic equations, here the deep-atmosphere nonhydrostatic equation set (2.1)–(2.6) is transformed using a generalized vertical coordinate \(s\) such that \((\lambda, \phi, z, t) \rightarrow (\lambda, \phi, s, t)\). The following transformation relations hold for \(\psi = \lambda, \phi, \text{or } r\):

\[
\begin{align*}
\frac{\partial}{\partial s} &= \frac{\partial}{\partial r} \left( \frac{\partial}{\partial s} \right), \\
\frac{\partial}{\partial r} &= \frac{\partial}{\partial s} \left( \frac{\partial}{\partial r} \right) \frac{\partial}{\partial s}.
\end{align*}
\]

where it is assumed that the transformation is monotonic; that is, \(\partial \psi/\partial r\) is nonzero and single signed. Subscripts signify what is held constant while carrying out the operation contained within the associated parenthases.

Equations (2.1)–(2.6) thus transform to

\[
\begin{align*}
\frac{Du}{Dt} - \frac{\nu w \tan \phi}{r} + \frac{\nu w}{r} - 2\Omega w \sin \phi + 2\Omega w \cos \phi \\
+ \frac{1}{\rho r^2 \sin \phi} \frac{\partial (r^2 w)}{\partial r} &= F^v,
\end{align*}
\]

\[
\begin{align*}
\frac{Du}{Dt} - \frac{\nu w \tan \phi}{r} + \frac{\nu w}{r} - 2\Omega w \sin \phi + 2\Omega w \cos \phi \\
+ \frac{1}{\rho r \cos \phi} \frac{\partial \rho}{\partial \phi} &= F^v, \\
\frac{Du}{Dt} - \frac{\nu w \tan \phi}{r} + \frac{\nu w}{r} - 2\Omega w \sin \phi + 2\Omega w \cos \phi \\
+ \frac{1}{\rho r \cos \phi} \frac{\partial \rho}{\partial \phi} &= F^v,
\end{align*}
\]
\[
\frac{Dv}{Dt} + \frac{u^2 \tan \phi}{r} + \frac{vw}{r} + 2\Omega u \sin \phi
\]
\[
+ \frac{1}{\rho} \left( \frac{\partial p}{\partial s} - \frac{\partial r}{\partial s} \frac{\partial s}{\partial r} \right) = F^r, 
\]
\[
(2.12)
\]
\[
\frac{Dw}{Dt} = \frac{(u^2 + v^2)}{r} - 2\Omega u \cos \phi + g + \frac{1}{\rho} \frac{\partial r}{\partial s} \frac{\partial s}{\partial r} = \delta_s F^r, 
\]
\[
(2.13)
\]
\[
\frac{\partial}{\partial t} \left( \rho \cos \phi \frac{\partial z}{\partial s} \right) + \frac{\partial}{\partial \lambda} \left( \frac{u}{r \cos \phi} \rho \cos \phi \frac{\partial z}{\partial r} \right) + \frac{\partial}{\partial s} \left( \dot{\rho} \cos \phi \frac{\partial z}{\partial s} \right) = 0,
\]
\[
(2.14)
\]
\[
\frac{D(c_s T)}{Dt} - \frac{1}{\rho} \frac{Dp}{Dt} = F^r, 
\]
\[
(2.15)
\]
\[
\frac{Dp}{Dt} = \frac{\partial}{\partial t} + \frac{u}{r \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{r \cos \phi} \frac{\partial}{\partial r} + \frac{s}{\partial s}, 
\]
\[
(2.16)
\]
\[
\delta_s = D_s \frac{D}{Dt} = \frac{\partial s}{\partial t} \left( w - \frac{\partial r}{\partial s} - \frac{u}{r \cos \phi} \frac{\partial r}{\partial s} \right),
\]
\[
(2.17)
\]
\[
D = \frac{\partial}{\partial t} + \frac{u}{r \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{r \cos \phi} \frac{\partial}{\partial r} + \frac{s}{\partial s},
\]
\[
(2.18)
\]
\[
and \text{partial derivatives with respect to } \lambda, \phi, \text{and } t \text{ are evaluated holding } s \text{ constant. The transformation of the equations is straightforward with the possible exception of (2.14). This is derived by (a) rewriting (2.4) in flux form using (2.7) [which results in (A.1) with } s \text{ set identically equal to } r]; \text{ (b) transforming this using (2.9)–(2.10); (c) multiplying the result through by } \partial r/\partial s \text{ before rewriting in flux form using (A.10); and (d) applying (2.17) to eliminate } w \text{ in favor of } s. \]
\[
c. \text{ Vertical boundary conditions}
\]
\[
\text{To conserve mass, both the lower and upper boundaries, } s = s_H(\lambda, \phi, t) \text{ and } s = s_T(\lambda, \phi, t), \text{ respectively, are assumed to be material surfaces with no mass transport across them. Thus, }
\]
\[
\dot{s}_H = \frac{\partial s_H}{\partial t} + \frac{u_H}{r_H \cos \phi} \frac{\partial s_H}{\partial \lambda} + \frac{v_H}{r_H \cos \phi} \frac{\partial s_H}{\partial \phi}, \quad \text{and (2.19)}
\]
\[
\dot{s}_T = \frac{\partial s_T}{\partial t} + \frac{u_T}{r_T \cos \phi} \frac{\partial s_T}{\partial \lambda} + \frac{v_T}{r_T \cos \phi} \frac{\partial s_T}{\partial \phi}, \quad \text{(2.20)}
\]
\[
\text{where subscripts } \text{"H" and "T" denote values at } s = s_H \text{ and } s_T, \text{ respectively. In particular, } r_H(\lambda, \phi, t) \text{ and } r_T(\lambda, \phi, t) \text{ are the heights of the lower and upper boundaries, respectively. The lower boundary is usually assumed stationary, that is, } r_H = r_H(\lambda, \phi). \text{ The upper boundary condition (2.20) is a generalization of that assumed by K74: it reduces to his condition when } s_T \text{ coincides with a coordinate surface since then } s_T = 0. \text{ This is in fact the case for most atmospheric models since they employ a terrain-following vertical coordinate designed to respect}
\]
\[
\left. \frac{\partial}{\partial \lambda} (s_T) \right|_{ss} = \left. \frac{\partial}{\partial \lambda} (s_T) \right|_{ss} = 0, \quad \text{(2.21)}
\]
\[
\text{where } s_H \text{ and } s_T \text{ are constants, often normalized to zero and unity. The lower and upper material surfaces then also correspond to constant coordinate surfaces. However, the generality of (2.19) and (2.20) is retained herein.}
\]
\[
d. \text{ Some possible choices of vertical coordinate}
\]
\[
\text{When } s \text{ is chosen to be identically equal to } r, \text{ then the equation set (2.11)–(2.16) trivially reduces to the original equations (2.1)–(2.6). On the other hand, and as just mentioned, terrain-following coordinates are widely used in atmospheric models. They are usually either height based or pressure based.}
\]
\[
\text{A simple example (K74; Gal-Chen and Somerville 1975) is the height-based coordinate}
\]
\[
s = \xi = \frac{r - r_T}{r_T - r_H(\lambda, \phi)}, \quad \text{(2.22)}
\]
\[
\text{where } r_T \text{ is constant and, in this case (or, strictly, a nonlinear form of this), the equation set yields the formulation of the Met Office’s new dynamical core (Cullen et al. 1997). The } \xi \text{ coordinate is valid for both hydrostatic and nonhydrostatic equation sets.}
\]
\[
\text{A popular choice of vertical coordinate for shallow-atmosphere hydrostatic primitive equation models is (Phillips 1957; K74) the pressure-based coordinate}
\]
\[
s = \sigma = \frac{p - p_T}{p_H(\lambda, \phi) - p_T}, \quad \text{(2.23)}
\]
\[
\text{where } p_T \text{ is constant. Laprise (1992) extended this to include nonhydrostatic but still-shallow atmospheres, by introducing a family of terrain-following hydrostatic-pressure coordinates. This has been further generalized to the deep-atmosphere Euler equations by Wood and Staniforth (2003), via the introduction of a mass-based terrain-following vertical coordinate. With this choice for } s, \text{ (2.11)–(2.16) reduce to those derived therein.}
\]
\[
3. \text{ Energetics}
\]
\[
a. \text{ Kinetic energy evolution}
\]
\[
\text{Multiplying each of the momentum equations (2.11)–(2.13) by their respective velocity components, summing the results, multiplying by } g, \text{ and then using (A.5) with } g = K, \text{ (2.17) and (A.10) yield the following equation for the evolution of kinetic energy:}
\]
\[ \frac{\partial (K\mathcal{F})}{\partial t} = p \left( \frac{\partial (\mathcal{F})}{\partial t} + \frac{\partial}{\partial \lambda} \left( \frac{u}{r \cos \phi} \mathcal{F} \right) + \frac{\partial}{\partial \phi} \left( \frac{v}{r} \mathcal{F} \right) \right) + \frac{\partial (\mathcal{F})}{\partial s} \left( K + \frac{p}{\rho} \mathcal{F} \right) \]

\[ - \frac{\partial}{\partial s} \left( \frac{pr^2 \cos \phi \mathcal{F}}{s} \right) - gw \mathcal{F} + (uF^v + vF^w + \delta_w F^*) \mathcal{F}, \quad (3.1) \]

where

\[ K = \frac{u^2 + v^2 + \delta_w w^2}{2} \]

is the kinetic energy per unit mass, and \( \mathcal{F} \) is defined by (A.1). Note that when \( \delta_w = 0 \), that is, for the quasihydrostatic equations of White and Bromley (1995), the \( w^2/2 \) contribution to the kinetic energy is absent.

b. Potential gravitational energy evolution

Multiplying Eq. (A.2) through by \( \Phi \), noting that \( \Phi \) is a function only of \( r \), exploiting Eq. (2.17), and recalling that \( g = d\Phi/dr \) yields

\[ \frac{\partial}{\partial t} (\Phi \mathcal{F}) = gw \mathcal{F} - \frac{\partial}{\partial \lambda} \left( \frac{u}{r \cos \phi} \Phi \mathcal{F} \right) - \frac{\partial}{\partial \phi} \left( \frac{v}{r} \Phi \mathcal{F} \right) \]

\[- \frac{\partial}{\partial s} (s \Phi \mathcal{F}), \quad (3.3)\]

c. Internal energy evolution

Applying (A.5) with \( \mathcal{G} = c_wT \), using (2.15), (2.16), and (2.18), subtracting \( \partial (RT \mathcal{F})/\partial t \) and using (2.8) yields the following equation for the evolution of internal energy:

\[ \frac{\partial}{\partial t} (c_wT \mathcal{F}) = - p \left[ \frac{\partial (\mathcal{F})}{\partial t} + \frac{\partial}{\partial \lambda} \left( \frac{u}{r \cos \phi} \mathcal{F} \right) \right] + \frac{\partial}{\partial \phi} \left( \frac{v}{r} \mathcal{F} \right) \]

\[ - \frac{\partial}{\partial \lambda} \left( \frac{u}{r \cos \phi} c_wT \mathcal{F} \right) - \frac{\partial}{\partial \phi} \left( \frac{v}{r} c_wT \mathcal{F} \right) \]

\[ - \frac{\partial}{\partial s} (s c_wT \mathcal{F}) + F \mathcal{F}, \quad (3.4) \]

where, by definition, \( c_wT \) is the internal energy per unit mass.

d. Global conservation of total energy

Summing (3.1), (3.3), and (3.4) gives the evolution equation for total energy, that is,

\[ \frac{\partial}{\partial t} (E \mathcal{F}) = - \frac{\partial}{\partial \lambda} \left( \frac{u}{r \cos \phi} \left( E + \frac{p}{\rho} \right) \mathcal{F} \right) \]

\[- \frac{\partial}{\partial \phi} \left( \frac{v}{r} \left( E + \frac{p}{\rho} \right) \mathcal{F} \right) - \frac{\partial}{\partial s} \left( s \left( E + \frac{p}{\rho} \right) \mathcal{F} \right) \]

\[ - \frac{\partial}{\partial s} \left( pr^2 \cos \phi \mathcal{F} \right) + gw \mathcal{F} + (uF^v + vF^w + \delta_w F^*) \mathcal{F}, \quad (3.5) \]

where

\[ E = K + \Phi + c_wT \]

is the total energy per unit mass.

Integrating (3.5) with respect to \( s \) from \( s_\mu \) to \( s_T \), applying (A.8) with \( \tau = E \mathcal{F} \) and (A.9) with \( \tau = (p/\rho) \mathcal{F} \), and using Eq. (A.1) then leads to

\[ \frac{\partial}{\partial t} \int_{s_\mu}^{s_T} E \mathcal{F} \, ds \]

\[ = - \frac{\partial}{\partial \lambda} \int_{s_\mu}^{s_T} \frac{u}{r \cos \phi} \left( E + \frac{p}{\rho} \right) \mathcal{F} \, ds \]

\[- \frac{\partial}{\partial \phi} \int_{s_\mu}^{s_T} \frac{v}{r} \left( E + \frac{p}{\rho} \right) \mathcal{F} \, ds \]

\[- \left( pr^2 \cos \phi \frac{d^2 \phi}{ds^2} \right) \frac{d^2 \phi}{ds^2} + \left( pr^2 \cos \phi \frac{d^2 \phi}{ds^2} \right) \frac{d^2 s}{ds^2} \]

\[- \left( pr^2 \cos \phi \frac{d^2 \phi}{ds^2} \right) \frac{d^2 \phi}{ds^2} + \left( pr^2 \cos \phi \frac{d^2 \phi}{ds^2} \right) \frac{d^2 s}{ds^2} \]

\[ + \int_{s_\mu}^{s_T} (uF^v + vF^w + \delta_w F^* + F^\tau) \mathcal{F} \, ds. \quad (3.7) \]

Since \( s_T \) and \( s_\mu \) are generally functions of time, then, as noted by K74,

\[ \left[ \frac{\partial r}{\partial t} \right] = \frac{\partial r_T}{\partial t} - \frac{\partial r_T}{\partial s_T} \frac{d s_T}{d t}, \quad \text{and} \quad \left[ \frac{\partial s}{\partial t} \right] = \frac{\partial s_\mu}{\partial t} - \frac{\partial s_\mu}{\partial s_\mu} \frac{d s_\mu}{d t}. \quad (3.8) \]

Hence, using (3.8) and (3.9), (3.7) reduces to

\[ \frac{\partial}{\partial t} \int_{s_\mu}^{s_T} E \mathcal{F} \, ds \]

\[ = - \frac{\partial}{\partial \lambda} \int_{s_\mu}^{s_T} \frac{u}{r \cos \phi} \left( E + \frac{p}{\rho} \right) \mathcal{F} \, ds \]
Finally, integrating (3.10) along constant $s$ surfaces over the entire globe, with respect to $\lambda$ and $\phi$, from $\lambda = 0$ to $2\pi$ and $\phi = -\pi/2$ to $+\pi/2$, applying periodicity in the $\lambda$ direction and noting that $\cos\phi$ is independent of $s$ and that $\cos(\pm\pi/2) = 0$, gives

$$\frac{\partial}{\partial t} \int_{\lambda} \rho E \ d\lambda' = \int_{\lambda} \rho (u F_x + v F_y + \delta_v F w' + F T) \ d\lambda'$$
$$- \int_{s_H} r \frac{\partial r}{\partial t} dA_T + \int_{s_H} H \frac{\partial H}{\partial t} dA_H. \quad (3.11)$$

where, for arbitrary $g$, the respective global volume and lower surface area integrals of $g$ are defined by

$$\int_{\lambda} g \ d\lambda' = \int_{-\pi/2}^{+\pi/2} \int_{0}^{2\pi} \int_{r} r^2 \cos\phi \frac{\partial r}{\partial s} ds \ d\lambda \ d\phi,$$
$$\int_{s_H} g \ dA_T = \int_{-\pi/2}^{+\pi/2} \int_{0}^{2\pi} G r^2 \cos\phi \ d\lambda \ d\phi. \quad (3.12)$$

and similarly for the upper surface area integral $\int_{s_H} g \ dA_H$. With the exception of studies of flow over water waves, the lower surface is generally assumed to be stationary so that $\partial r_{y}/\partial t = 0$ and the last term on the right-hand side of (3.11) vanishes. Then (3.11) implies that, in the absence of forcing (i.e., when $F_x = F_y = \delta_v F w' = F T = 0$), the globally integrated total energy is only conserved if

$$\int_{s_T} p r \frac{\partial r}{\partial t} dA_T = 0. \quad (3.14)$$

If this is not the case, then energy is lost or gained by the system through the work done by or against the pressure at the upper boundary in displacing it.

Therefore, for (3.14) to hold generally, and hence for the globally integrated total energy to be conserved, the upper boundary needs to be either (a) at $p_T = 0$, which requires inclusion of the entire atmosphere; or (b) fixed in space and time; that is, the boundary needs to be specified as $r_T = r_T(\lambda, \phi)$, so that $\partial r_T/\partial t = 0$. Choice $b$ is natural for models employing a height-based vertical coordinate and, therefore, such models conserve total energy (in the absence of forcing).

4. Axial angular momentum conservation

a. Evolution

Using the identity

$$\frac{DA}{Dt} = r \cos\phi \frac{Du}{Dt} + (u + 2\Omega r \cos\phi) \frac{D}{Dt} (r \cos\phi), \quad (4.1)$$

substituting (2.11), applying (A.5) with $g = A$, and using definition (A.1) for $g$ and (A.10) yields

$$\frac{\partial}{\partial t} (A T) = - \frac{\partial}{\partial \lambda} \left( \frac{u}{r \cos\phi} A T \right) + \frac{\partial}{\partial \phi} \left( \frac{u \cos\phi}{r} A T \right) - \frac{\partial}{\partial s} (s A T)$$
$$\frac{\partial}{\partial \lambda} \left( pr^2 \cos\phi \frac{\partial r}{\partial s} \right) + \frac{\partial}{\partial s} \left( pr^2 \cos\phi \frac{\partial r}{\partial \lambda} \right) + r \cos\phi F w', \quad (4.2)$$

where

$$A = (u + \Omega r \cos\phi) \ r \cos\phi \quad (4.3)$$

is the axial angular momentum per unit mass.

b. Global conservation

Integrating (4.2) with respect to $s$ from $s_H$ to $s_T$, and applying (A.6) with $\tau = p T / \rho$ and (A.8) with $\tau = A T$, leads to

$$\frac{\partial}{\partial t} \int_{s_H} A_T \ ds$$
$$= - \frac{\partial}{\partial \lambda} \int_{s_H} \frac{u}{r \cos\phi} A_T \ ds - \frac{\partial}{\partial \phi} \int_{s_H} \frac{u \cos\phi}{r} A_T \ ds$$
$$- \frac{\partial}{\partial \lambda} \int_{s_H} pr^2 \cos\phi \frac{\partial r}{\partial s} \ ds$$
$$\int_{s_H} r \cos\phi F w' \ ds + \left( pr^2 \cos\phi \frac{\partial r}{\partial s} \right)_T$$
$$- \left( pr^2 \cos\phi \frac{\partial r}{\partial \lambda} \right)_H + \left( pr^2 \cos\phi \frac{\partial r}{\partial s} \right)_T \frac{\partial s_T}{\partial \lambda}$$
$$- \left( pr^2 \cos\phi \frac{\partial r}{\partial \lambda} \right)_H \frac{\partial s_H}{\partial \lambda}. \quad (4.4)$$

Similarly as for (3.8) and (3.9),

$$\left( pr^2 \cos\phi \frac{\partial r}{\partial \lambda} \right)_T - \left( pr^2 \cos\phi \frac{\partial r}{\partial \lambda} \right)_H$$

$$= (pr^2 \cos\phi) \left[ \frac{\partial r}{\partial \lambda} \right]_T - \left( \frac{\partial r}{\partial s} \right)_T \frac{\partial s_T}{\partial \lambda}$$
$$- (pr^2 \cos\phi) \left[ \frac{\partial r}{\partial \lambda} \right]_H - \left( \frac{\partial r}{\partial s} \right)_H \frac{\partial s_H}{\partial \lambda}. \quad (4.5)$$

and so (4.4) reduces to
Finally, integrating (4.6) along constant \( s \) surfaces over the entire globe, with respect to \( \lambda \) and \( \phi \), from \( \lambda = 0 \) to \( 2\pi \) and from \( \phi = -\pi/2 \) to \( +\pi/2 \), applying periodicity in the \( \lambda \)-direction and noting that \( \cos \phi \) is independent of \( s \) and that \( \cos(\pm \pi/2) = 0 \), gives

\[
\frac{\partial}{\partial t} \left[ \int_{s_H}^{s_T} A \, dv \right] = \int_{s_H}^{s_T} \rho F^* r \cos \phi \, dv + \int_{s_T}^{s_H} p_T \frac{\partial r_T}{\partial \lambda} \, dA_T - \int_{s_H}^{s_T} p_T \frac{\partial r_H}{\partial \lambda} \, dA_H. \tag{4.7}
\]

where the volume and area integrals with respect to \( dv \) and \( dA \) are defined by (3.12) and (3.13).

The first term on the right-hand side of (4.7) is the net torque due to zonal mechanical forcing while the last one is the mountain torque due to the surface pressure field acting on any variation of the height of the lower boundary in the \( \lambda \) direction. The second term similarly represents the torque of the pressure field acting on any variation of the height of the upper boundary in the \( \lambda \) direction. Therefore, the globally integrated axial angular momentum will be conserved, in the absence of any surface torque and zonal mechanical forcing, only if

\[
\int_{s_H}^{s_T} p_T \frac{\partial r_H}{\partial \lambda} \, dA_H = 0. \tag{4.8}
\]

Equation (4.8) can only be guaranteed to hold if the upper boundary is either (a) at \( p_T = constant \), since the integral of \( r_T \frac{\partial r_T}{\partial \lambda} \) with respect to \( \lambda \) around a latitude circle vanishes; or (b) uniform in the \( \lambda \) direction; that is, the boundary needs to be specified as \( r_T = r_T(\phi, t) \), so that \( \frac{\partial r_T}{\partial \lambda} = 0 \).

As seen in section 3, for \( p_T \neq 0 \), global energy conservation requires \( r_T = r_T(\lambda, \phi) \) and so for such a model to conserve both energy and axial angular momentum, the upper boundary must be fixed in space and time and can only be a function of latitude, \( r_T = r_T(\phi) \). For models employing a height-based vertical coordinate, \( r_T \) is usually a constant in both space and time and, therefore, such models conserve both total energy and axial angular momentum (in the absence of zonal mechanical forcing and mountain torque). As a corollary, models that impose a material surface with constant (nonzero) pressure at the upper boundary do not conserve total energy and axial angular momentum; they may however possess an energy-like invariant [see, e.g., Eq. (5.18) of K74 and preceding discussion, and (Eq. 49) of Laprise (1992) and preceding discussion].

The final term on the right-hand side of (4.7) is one form of the mountain torque. However, since

\[
\int_{s_H}^{s_T} \frac{\partial (p_H r_H)}{\partial \lambda} \, dA_H = 0, \tag{4.9}
\]

integration by parts shows that the mountain torque term can be equivalently written as

\[
-\int_{s_H}^{s_T} p_H \frac{\partial r_H}{\partial \lambda} \, dA_H + \int_{s_H}^{s_T} r_H \frac{\partial p_H}{\partial \lambda} \, dA_H = +\int_{s_H}^{s_T} \frac{\partial p_H}{\partial \lambda} \, dA_H, \tag{4.10}
\]

which is the preferred form of several authors, for example, Simmons and Burridge (1981) and Laprise and Girard (1990).

5. Shallow-atmosphere conservation

Following a similar procedure to that given above and in K74, the shallow-atmosphere analogues to (3.11) and (4.7) for the conservation of global total energy and axial angular momentum, respectively, can be obtained.

For total energy, the deep-atmosphere result (3.11) is recovered except that the global volume and surface integrals (3.12)–(3.13) are redefined as

\[
\int_{s_H}^{s_T} G \, dv = \left[ +\pi/2 \right] \int_{-\pi/2}^{\pi/2} \int_{s_H}^{s_T} G a^2 \cos \phi \frac{\partial r_H}{\partial s} \, ds \, d\lambda \, d\phi, \tag{5.1}
\]

\[
\int_{s_H}^{s_T} G \, dA_H = \left[ +\pi/2 \right] \int_{-\pi/2}^{\pi/2} \int_{s_H}^{s_T} G a^2 \cos \phi \, dA_H. \tag{5.2}
\]

where \( a \) is earth’s mean radius, and similarly for the upper surface area integral \( \int_{s_H}^{s_T} G \, dA_H \). This result is somewhat more general than the corresponding Eq. (5.11) given in K74, for two reasons. First, it is valid for both the hydrostatic and nonhydrostatic primitive equations, whereas K74’s result is restricted to the former. Note that the fundamental difference between the hydrostatic and nonhydrostatic results is that—see section 3a—the nonhydrostatic one includes an additional contribution of \( \omega^2/2 \) in the definition of kinetic energy. Second, K74’s Eq. (5.11) is for the evolution of \( K + c T \), which is a pseudoenergy, rather than for the evolution of true total energy \( E = K + \Phi + c T \). However, when K74 applies his Eq. (5.11) to the geometric height case, it results [after some manipulation using his identity (5.14)] in his Eq. (5.15).
then exactly agrees, for this case, with the shallow-atmosphere conservation law (with \( \delta \), set identically to zero) of the present work since his measure of energy then agrees with that of total energy employed herein.

For axial angular momentum the shallow-atmosphere analogue of (4.7) is

\[
\frac{\partial}{\partial t} \int_\nu \rho \, d\mathcal{V} = \int_\nu \rho \Phi a \cos \phi \, d\mathcal{V} + \int_{\partial \mathcal{V}} \rho \frac{\partial r_u}{\partial \lambda} \, d\mathcal{A},
\]

\[
- \int_{\partial \mathcal{V}} \rho \frac{\partial r_u}{\partial \lambda} \, d\mathcal{A}, \quad \text{(5.3)}
\]

where the volume and surface integrals are again redefined by (5.1)–(5.2), and \( A \) and \( u \) are also, respectively, redefined as

\[
A = (u + \Omega a \cos \phi) a \cos \phi, \quad \text{(5.4)}
\]

\[
u = \alpha \cos \phi \frac{DA}{Dt}. \quad \text{(5.5)}
\]

Thus, the shallow-atmosphere approximation affects the global constraints on energy and axial angular momentum by fundamentally changing the definitions of the wind components, kinetic energy, and axial angular momentum.

Because, for both total energy and axial angular momentum, the upper boundary contributions have the same functional form regardless of whether the atmosphere is shallow or deep, the conclusions given at the ends of sections 3d and 4b for a deep atmosphere also hold for a shallow one.

6. Summary and conclusions

K74’s analysis for the hydrostatic primitive equations using a generalized vertical coordinate has been extended in four ways:

- the hydrostatic constraint has been relaxed to permit nonhydrostatic effects;
- the shallow-atmosphere assumption has been removed to permit application to deep atmospheres;
- the upper boundary is no longer constrained to be a coordinate surface to permit more general upper boundary conditions; and
- axial angular momentum conservation has been examined to determine its sensitivity to the choice of upper boundary condition.

This leads to a formulation of the deep-atmosphere nonhydrostatic Euler equations using a generalized vertical coordinate. It includes, as a special case, the formulation of the Met Office’s new dynamical core (Cullen et al. 1997) in a height-based terrain-following coordinate. As a further special case, it also includes the Wood and Staniforth (2003) generalization, to the deep-atmosphere Euler equations, of Laprise’s (1992) formulation for the shallow-atmosphere nonhydrostatic primitive equations, using a mass-based terrain-following vertical coordinate.

For an atmosphere of finite extent and regardless of the precise choice of vertical coordinate, energy and axial angular momentum conservation (in the absence of zonal mechanical forcing and mountain torque) are only obtained when the upper boundary is fixed in both space and time. It can then only be a function of latitude that includes, as a special case, being located at constant height. This result has been shown to be independent of whether the atmosphere is shallow or deep, and hydrostatic or nonhydrostatic. In particular, models that impose a material surface with constant (nonzero) pressure at the upper boundary, do not conserve total energy and axial angular momentum, although they may possess an energy-like invariant. This is consistent with, and generalizes, K74’s analysis for a shallow hydrostatic atmosphere. There it was shown that total energy is conserved for a rigid lid in height coordinates, but that pseudoenergy is an invariant for an isobaric lid in pressure coordinates.

Considerable effort has been made by various authors to design finite-difference schemes for the hydrostatic primitive equations that ensure that there are no numerical internal sources of energy and axial angular momentum (Konor and Arakawa 1997, and references therein). Of particular importance for the development of such schemes are the detailed budgets of energy and axial angular momentum; the (discrete) flux terms that result from the discretization of the continuous equations must exactly balance. An essential tool for devising such schemes is the insight provided by the budgets of the continuous equations. If one wishes to extend Arakawa’s conserving finite-difference schemes for the hydrostatic primitive equations to the discretization of the deep-atmosphere Euler equations, then the methodology and results of the present work could provide such an insight.

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APPENDIX

Useful Identities

Defining \( \mathcal{J} \) to be the product of density with the Jacobian of the transformation from Cartesian to generalized terrain-following spherical polar coordinates, that is,

\[
\mathcal{J} = \rho \cos \phi r^2 \partial r \partial \lambda \partial \phi,
\]

(A.1)

(2.14) may be rewritten as
\[
\frac{\partial f}{\partial t} + \frac{\partial}{\partial \lambda} \left( \frac{u}{r \cos \phi} \right) + \frac{\partial}{\partial \phi} \left( \frac{v f}{r} \right) + \frac{\partial}{\partial s} (s f) = 0. \tag{A.2}
\]

From (2.18),
\[
\frac{D G}{Dt} = \frac{\partial G}{\partial t} + \frac{u}{r \cos \phi} \frac{\partial G}{\partial \lambda} + \frac{v}{r} \frac{\partial G}{\partial \phi} + s \frac{\partial G}{\partial s},
\]
for any scalar \( G \), and so
\[
\psi \frac{D G}{Dt} = \frac{\partial}{\partial t} (\psi G) + \frac{u}{r \cos \phi} \frac{\partial (\psi G)}{\partial \lambda} + \frac{v}{r} \frac{\partial (\psi G)}{\partial \phi} + \frac{\partial}{\partial s} (\psi G) \tag{A.3}
\]
Using (A.2), the right-hand side may be rewritten in flux form to yield identity A:
\[
\psi \frac{D G}{Dt} = \frac{\partial}{\partial t} (\psi G) + \frac{u}{r \cos \phi} \frac{\partial (\psi G)}{\partial \lambda} + \frac{v}{r} \frac{\partial (\psi G)}{\partial \phi} + \frac{\partial}{\partial s} (\psi G). \tag{A.4}
\]
Leibniz’ theorem for differentiation of an integral gives identity B:
\[
\int_{s_1}^{s_2} \frac{\partial \tau}{\partial \lambda} \, ds = \frac{\partial}{\partial \lambda} \int_{s_1}^{s_2} \tau \, ds - \left( \tau \frac{\partial s_1}{\partial \lambda} - \tau \frac{\partial s_2}{\partial \lambda} \right), \tag{A.6}
\]
where \( \tau \) is a general variable and \( x \) represents any of \( \lambda \), \( \phi \), and \( t \).

It follows from (A.6) that
\[
\int_{s_1}^{s_2} \left[ \frac{\partial \tau}{\partial t} + \frac{u \tau}{r \cos \phi} + \frac{\partial}{\partial \lambda} \left( \frac{v \tau}{r} \right) + \frac{\partial}{\partial s} (s \tau) \right] \, ds = \frac{\partial}{\partial t} \int_{s_1}^{s_2} \tau \, ds + \frac{\partial}{\partial \lambda} \int_{s_1}^{s_2} \frac{u \tau}{r \cos \phi} \, ds + \frac{\partial}{\partial \phi} \int_{s_1}^{s_2} \frac{v \tau}{r} \, ds - \tau \frac{\partial s_1}{\partial \lambda} + \tau \frac{\partial s_2}{\partial \lambda} - \frac{\partial s_1}{\partial \phi} + \frac{\partial s_2}{\partial \phi} - \frac{\partial s_1}{\partial s} + \frac{\partial s_2}{\partial s}. \tag{A.7}
\]
Invoking the lower and upper boundary conditions (2.19)–(2.20), this reduces to identity C:
\[
\int_{s_1}^{s_2} \left[ \frac{\partial \tau}{\partial t} + \frac{u \tau}{r \cos \phi} + \frac{\partial}{\partial \lambda} \left( \frac{v \tau}{r} \right) + \frac{\partial}{\partial s} (s \tau) \right] \, ds = \frac{\partial}{\partial t} \int_{s_1}^{s_2} \tau \, ds + \frac{\partial}{\partial \lambda} \int_{s_1}^{s_2} \frac{u \tau}{r \cos \phi} \, ds + \frac{\partial}{\partial \phi} \int_{s_1}^{s_2} \frac{v \tau}{r} \, ds. \tag{A.8}
\]
Using \( x = t \) in (A.6), (A.8) is rewritten as identity D:
\[
\int_{s_1}^{s_2} \left[ \frac{\partial}{\partial \lambda} \left( \frac{u \tau}{r \cos \phi} \right) + \frac{\partial}{\partial \phi} \left( \frac{v \tau}{r} \right) \right] \, ds
\]

\[
= \left( \frac{\partial s_T}{\partial t} + \tau \frac{\partial s_T}{\partial \phi} \right) - \left[ \frac{\partial s_T}{\partial \phi} \phi \right]_{\tau} + \frac{\partial}{\partial \lambda} \int_{s_1}^{s_2} \frac{u \tau}{r \cos \phi} \, ds + \frac{\partial}{\partial \phi} \int_{s_1}^{s_2} \frac{v \tau}{r} \, ds. \tag{A.9}
\]
For \( \psi = \lambda, \phi, \) or \( t \), and arbitrary \( n \), the following holds as identity E:
\[
\frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial \lambda} \left( r^n \frac{\partial v}{\partial \phi} \right) \right) \equiv \frac{\partial}{\partial \phi} \left( r^n \frac{\partial v}{\partial \phi} \right). \tag{A.10}
\]

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