$$\frac{d^2Z}{dy^2} = \frac{M}{EI} = \frac{1}{2}\,\rho Ag(l - y)^2/(EI)$$

$$\frac{d\theta}{dy} = \frac{T}{C} = \rho Ag(l - y)e/C$$

The two energy expressions may be integrated over the length of the beam and expressed as functions of the static terminal displacement Z_l and terminal rotation θ_l which are themselves related by the equation

$$\theta_l = \frac{4eEI}{l^2C}\,Z_l$$

The resulting energy equation is

$$\frac{8EI}{5l^3}\left(1 + \frac{20}{3}\,\alpha\right)Z_l^2 = \rho Al\left[\frac{52}{405} + \frac{148}{105}\,\alpha + \frac{64}{15}\left(1 + \frac{J}{e^2A}\right)\alpha^2\right]\left(\frac{dZ_l}{dt}\right)^2$$

where $\alpha = (e^2EI)/(l^2C)$.

Taking the dimensions given in the paper, the frequency figures out as 1006.3 per min when $e = 0$, and 721.5 per min when $e = 0.86$ in. Both results agree well with the values quoted by the author. It is suggested that, when it is known that the fundamental frequency alone is the critical one, this method may be adequate for the purpose. It entails considerably less labor than the more accurate method used by the author. When higher frequencies are critical, the Ritz method or its equivalent is probably best.

Author's Closure

The author wishes to express his thanks to Dr. Eksergian and to Professor Waters for their appropriate and well-prepared discussions.

The discussion by Dr. Eksergian provides an interesting illustration of the relationships which exist among the several energy methods.

Professor Waters discusses an important point which the author neglected to make clear in the paper. As he has demonstrated, the Rayleigh method serves quite adequately when only the fundamental frequency is desired.

To clarify the relationship which exists between the energy equations presented by Professor Waters and those used by the author it may be pointed out that when the static deflection curve is employed a ratio of $e\varphi_1/a_1$ is defined automatically. If a similar value for this ratio is assumed in writing Equations [1] in the paper, only one unknown will appear in Equation [7], and thus Equation [7] will yield the fundamental frequency directly.

Assuming equal accuracies in the performance of numerical work, the slight difference in the values of the fundamental frequencies computed by the two methods can be attributed to the difference in the assumed elastic curves. This demonstration of the accuracy of the Rayleigh method may be added to the many others to be found in the literature.

Analysis of Clamped Rectangular Plates[1]

Stewart Way.[2] The analysis of clamped rectangular plates with various loadings, on the basis of the theory of Lagrange and Kirchhoff, is a worth-while undertaking in its own right; but it also contributes to our understanding of certain more complicated problems. For example, if we are interested in the large-deflection problem of a rectangular plate for the condition of no boundary-membrane stresses, we have to deal with the equation $\Delta\Delta F = E(w_{xy}{}^2 - w_{xx}w_{yy})$ where F is an Airy stress function that satisfies the boundary conditions $F = 0, \dfrac{\partial F}{\partial n} = 0$. Thus, the stress function becomes analogous to the deflection of a clamped rectangular plate with load proportional to $w_{xy}{}^2 - w_{xx}w_{yy}$. In solving large-deflection problems, a deflection form with unknown parameters is usually assumed, so it is very useful to be able to calculate readily the membrane stresses (defined by F), corresponding to any assumed deflection.

The writer would like to make one remark in connection with the superposition method. As originally outlined by Timoshenko this method involves the superposition of two solutions w_1 and w_2 satisfying the equations

$$D\Delta\Delta w_1 = q(x, y)$$
$$\Delta\Delta w_2 = 0$$

where w_1 and w_2 each satisfies the boundary condition $w_1 = w_2 = 0$, and where their sum satisfies the other condition, $\dfrac{\partial}{\partial n}(w_1 + w_2) = 0$. The solution w_1 is conveniently taken as the deflection of a simply supported plate with load $q(x, y)$. A variation of this procedure is to make w_1 and w_2 separately satisfy boundary conditions $\dfrac{\partial w_1}{\partial n} = 0, \dfrac{\partial w_2}{\partial n} = 0$ and make the sum satisfy the other boundary condition, $w_1 + w_2 = 0$. The latter method is preferable for certain types of load distribution, as for example $q = \dfrac{q_0}{2}\left(\cos\dfrac{2\pi x}{a} + \cos\dfrac{2\pi y}{a}\right)$ on a square plate. In that case w_1 is at once $\dfrac{q_0}{2D}\left(\dfrac{a}{2\pi}\right)^4\left(\cos\dfrac{2\pi x}{a} + \cos\dfrac{2\pi y}{a}\right)$ which satisfies $\dfrac{\partial w_1}{\partial n} = 0$ at the boundary.

The cases of loading treated by the author are well chosen and will be quite useful to the designer.

S. P. Timoshenko.[3] This paper is an important addition to the theory of bending of clamped rectangular plates. The writer's only criticism is that the paper is so condensed that it can be read without difficulty only by the persons who themselves are working on the theory of plates. In the introductory historical part of the paper, it would seem desirable to add the name of B. M. Kojalovich, who was the first to solve the problem of bending of clamped rectangular plates under uniformly distributed load.[4] He applied his theoretical solution to a particular case of a plate and showed how the deflections and the bending stresses can be calculated with sufficient accuracy.

Kojalovich's solution was put in a form more suitable for calculation by I. G. Boobnov, who prepared the table of deflections and moments for various shapes of clamped rectangular plates carrying uniform load.[5] Until recently this table has been the most complete source of information on clamped rectangular plates.

The paper will be more complete if the author will add a discussion on the convergence of his method of calculation and also if he will deduce some conclusions regarding the accuracy of the

[1] By Dana Young, published in the December, 1940, issue of the Journal of Applied Mechanics, Trans. A.S.M.E., vol. 62, p. A-139.

[2] Westinghouse Research Laboratories, East Pittsburgh, Pa. Jun. A.S.M.E.

[3] Professor of Theoretical and Applied Mechanics, Stanford University, Calif. Fellow A.S.M.E.

[4] Doctor's Thesis, St. Petersburg, 1902.

[5] "Theory of Structures of Ships," by I. G. Boobnov, St. Petersburg, vol. 2, 1914, p. 465.

results obtained. From the calculations already made, it can be stated how the value of the maximum bending moment changed with the increase of the number of terms in the series used. It can also be shown with what accuracy are satisfied the load conditions and the boundary conditions if the number of terms in the series is such as taken in the paper. A comparison of the results obtained by using series with those obtained by using equations of finite differences also will be of interest.

In conclusion, the writer would like to express the hope that the author continue his investigations of clamped rectangular plates and that he make calculations for other forms of loading. The general solution in the form of a series is quite complicated and it can be used by practical engineers only if the tables for moments and deflections are calculated for various loading conditions.

Author's Closure

The discussions by Professor Timoshenko and Dr. Way have brought out several interesting points in connection with the clamped-plate problem and have raised a question concerning the convergence of the method. With regard to this question of accuracy, it may be shown, as suggested by Professor Timoshenko, how closely the conditions of the problem are satisfied by the results given in the paper which were calculated by taking only the first eight terms in each series. The deflection functions used satisfy exactly the differential equation and the condition for zero deflection at the boundaries. Due to taking a finite number of terms in the series expressions for the edge slopes, the actual slope at the boundaries will not quite be zero. The amount of this deviation from zero edge slope may be calculated for any of the plates given, and this will give an idea of the error involved in the method.

For example, consider the case of the square plate with the loading, shown in Fig. 1 of the paper, the results for which are given in Table 2 of the paper. Due to the form of the deflection functions, the edge slopes are exactly zero at the corners and at the center of the sides parallel to the y axis. At other points along the edges, the actual slope may be calculated using the values of A_k and B_m as given in Table 2. Thus, it is found that the edge slope at $x = 0$, $y = {}^1/_2 b$ is 2 per cent of the slope for a similarly loaded simply supported plate. The slope at $x = {}^1/_2 a$, $y = {}^1/_4 b$ is less than 1 per cent of the slope for a simply supported plate. An additional small error arises from the fact that the load on the plate is represented in the function w_1 by a series expansion, only a finite number of terms of which are taken in the calculation of moments. However, the number of terms that may be used is independent of the number of terms in the series for A_k and B_m and so this source of inaccuracy may be made as small as we please with very little extra work.

Professor Timoshenko has suggested that a comparison be made of the results obtained by this method and by the method of finite differences. Calculations by the latter method have been made by A. Smotrow[6] for clamped plates carrying a triangular hydrostatic load. These calculations were carried out for three plates with side ratios of $\frac{b}{a} = 1.5$, $\frac{b}{a} = 1$, and $\frac{b}{a} = \frac{2}{3}$, respectively. In each case, a network of 36 rectangles was used. The moments so determined by Smotrow are summarized in Table 1 of this closure, together with the corresponding values found in this investigation and given in Table 4 of the paper. While there is a general correspondence in the results, it will be noted that the moments found by the method of finite differences are smaller than the others. This is to be expected, and it is believed that a considerably closer network of points would have to be taken in the finite-differences method to obtain the accuracy of the superposition method.

[6] "Berechnung von Platten," by A. Smotrow (in Russian), Staats-Verlag für Literatur des Bauwesens, Moskau und Leningrad, 1936.

Notes on the Dynamics of Electric Locomotives[1]

B. F. Langer.[2] In this paper, the author has given some interesting extensions of his earlier work on locomotive dynamics. The writer is particularly interested in his discussion of the conditions existing at speeds higher than the critical.

He bases his energy calculations on the assumption that the vectors representing angular and lateral motion are at right angles to each other, and the energy balance comes about as a result of a change in the value of the ratio between the angular and lateral amplitudes. It is also interesting to see what results can be obtained from the assumption that the energy balance at speeds above the critical comes primarily from a change in the phase angle between the angular and lateral motions. Suppose we do not assume that $\alpha = \pi/2$, then the author's Equation [12], giving the conditions for energy balance, becomes

$$\frac{\theta}{2\omega Y} = \frac{v \sin \alpha \pm \sqrt{v^2 \sin^2 \alpha - V^2}}{V^2}$$

which is the same as the original except that v is replaced by $v \sin \alpha$. Thus as the speed v increases, the energy balance can be maintained by shifting α away from $\pi/2$. To say that the phase angle changes from $\pi/2$ to some other value is the same as saying that the center of rotation of the wheel base shifts from the center of gravity of the creepage forces to some other location, either forward or back. The writer has made observations on rigid-frame locomotives which indicate that just such a shift occurs.

As an example, two types of 2-D-2 locomotive were observed during tests made by M. Mauzin, research engineer of the S.N.C.F. in France. One type had a natural period of roll much longer than its nosing period, and the other had a period of roll about the same as its nosing period. Both locomotives, however, actually oscillated at their nosing frequencies, which were about the same. At top speed, the locomotive with the long

[1] By B. S. Cain, published in the March, 1941, issue of the Journal of Applied Mechanics, Trans. A.S.M.E., vol. 63, p. A-30.

[2] Research Laboratories, Westinghouse Electric & Manufacturing Company, East Pittsburgh, Pa.

TABLE 1 MOMENTS IN CLAMPED RECTANGULAR PLATE WITH TRIANGULAR LOAD; COMPARISON OF SUPERPOSITION AND FINITE-DIFFERENCES METHODS

	$\frac{b}{a} = 1.5$		$\frac{b}{a} = 1.0$		$\frac{b}{a} = \frac{2}{3}$	
	This paper	Smotrow	This paper	Smotrow	This paper	Smotrow
M_y at $x = 0$, $y = \frac{1}{2}b$	$-0.0187qb^2$	$-0.0129qb^2$	$-0.0334qb^2$	$-0.0281qb^2$	$-0.0462qb^2$	$-0.0419qb^2$
M_y at $x = 0$, $y = 0$	0.0045	0.0036	0.0115	0.0107	0.0184	0.0184
M_y at $x = 0$, $y = -\frac{1}{2}b$	−0.0066	−0.0055	−0.0179	−0.0166	−0.0295	−0.0287
M_x at $x = 0$, $y = 0$	0.0082	0.0081	0.0115	0.0107	0.0102	0.0082
M_x at $x = \frac{1}{2}a$, $y = 0$	−0.0168	−0.0157	−0.0257	−0.0223	−0.0285	−0.0208