the method as due to H. Liebmann.[7] Dr. Richardson's papers, antedating Liebmann's treatment by some nine years, clearly establish his priority.

A Theory of Flexure for Beams With Nonparallel Extreme Fibers[1]

J. N. Goodier.[2] The stress distributions discussed by the author are of the plane-stress type. They are based on the supposition that the three components of stress (σ_z, τ_{rz}, $\tau_{\theta z}$), which might be found on the plane faces of the uniform plate are zero throughout. This will, as a rule, be reasonably close to the truth only for thin plates. It has been widely supposed that Filon's "generalized plane stress" offers some escape from this restriction, but it can be shown that generalized plane stress has actually no wider range than simple plane stress.[3] This question of thickness arises in the paper in the application to the bend of a rigid frame, with a curved inner flange. The flanged part cannot be regarded as thin. Nevertheless we should not expect the three stresses which are neglected to have appreciable magnitude when the loads merely cause bending, and the agreement of the author's results with test results appears to confirm this point of view. His success with this problem seems to dispose of a structural element which has hitherto been difficult to deal with in a convincing manner.

F. C. Roop.[4] In connection with the author's example of the application of his theory to a plate girder with nonparallel flanges, it is of interest to examine the values of the principal stresses and of the maximum shearing stresses. Since a force applied in any direction at the edge of the wedge produces no shearing stresses on radial and circumferential elements, it is only in the case of loading by a couple that the radial stresses given by the author are not themselves principal stresses.

For couple loading, expressions for the algebraically greatest principal stress and the maximum shearing stress are found by substituting from the author's Equations [20] and [21] (together with $\sigma_{\theta 3} = 0$) in the well-known formulas for these quantities. By differentiating these expressions with respect to θ, equating the derivatives to zero, solving the resulting equations, and discarding those solutions which either correspond to minima or give values of θ greater than α, expressions are arrived at which give the value of the coordinate θ at which the greatest values of the quantities in question occur. The algebraically least principal stress for any θ is the negative of the greatest principal stress for $-\theta$, because of the symmetrical nature of the stress distribution about the polar axis.

The analysis, according to the foregoing procedure, of the expression for the greatest principal stress σ_{p3} shows that no solution of $\partial\sigma_{p3}/\partial\theta = 0$ corresponding to a maximum of σ_{p3} can be found which gives θ less than α unless $\alpha > \alpha_c$, where

$$\alpha_c = \frac{1}{4}\tan^{-1}\left(-\frac{2A_f}{tr}\right);\quad 22^1/_2 \text{ deg} \leq \alpha_c \leq 45 \text{ deg} \ldots [1]$$

Consequently, when $\alpha \leq \alpha_c$, the greatest value on any one circular-cylindrical section occurs at $\theta = \alpha$. When $\alpha \geq \alpha_c$, the analysis shows that the value ψ of the coordinate θ at which the maximum of σ_{p3} occurs is given by

$$\psi = \frac{1}{2}\cos^{-1}\frac{\sqrt{1+8s^2}-1}{4s};\quad 22^1/_2 \text{ deg} \leq \psi \leq 45 \text{ deg} \ldots [2]$$

where $s = (2A_f/tr)\sin 2\alpha - \cos 2\alpha$.

Substituting in the expression for σ_{p3} the value of θ at which the maximum of σ_{p3} occurs gives the maximum principal stress $\sigma_{p3\,\max}$. The resulting expressions for $\sigma_{p3\,\max}/\sigma_{r3\,\max} = B$ (see following the author's Equation [20]) are

For $\alpha \leq \alpha_c$

$$B = \frac{1}{2}\left[1 + \sqrt{\left\{1 + \left(\frac{2A_f}{tr}\right)^2\right\}}\right] \ldots\ldots\ldots [3a]$$

For $\alpha_c \leq \alpha \leq 45$ deg

$$B = \frac{1}{2}\frac{\sin 2\psi}{\sin 2\alpha}(1 - \sec 4\psi) \ldots\ldots\ldots\ldots [3b]$$

For 45 deg $\leq \alpha \leq$ 90 deg

$$B = \frac{1}{2}\sin 2\psi(1 - \sec 4\psi) \ldots\ldots\ldots\ldots\ldots [3c]$$

These results are shown in Fig. 1 of this discussion, in which the coordinates α (half wedge angle) and $\frac{A_f}{A_f + tr\alpha}$ (relative flange area) should be regarded as two independent variables. Each point in the figure represents a plate girder having values of α and

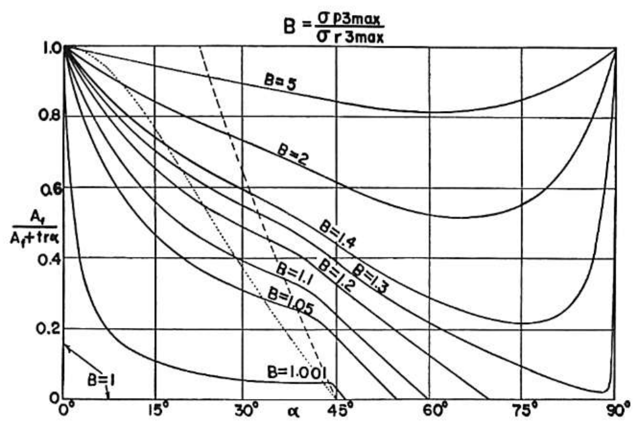

Fig. 1 Maximum Principal-Stress Curves

(Values of relative flange area and of half wedge angle for different values of the ratio of maximum principal stress to maximum radial stress for a wedge-shaped plate girder loaded by a couple.)

$A_f/(A_f + tr\alpha)$ given by the coordinates of the point, and lies on some curve B = const, which gives the value of B for this plate girder. The dashed line represents plate girders for which $\alpha = \alpha_c$ (see Equation [1] of this discussion).

As an example, a plate girder of half wedge angle 22.92 deg ($\alpha = 0.4$), with 55 per cent of its area in the flanges $\left(\frac{A_f}{A_f + tr\alpha} = 0.55\right)$ is seen to have a maximum principal stress 20 per cent in excess of the maximum radial stress ($B = 1.2$); furthermore, this maximum principal stress occurs at $\theta = \alpha$, just inside the junction between the (concentrated) flange and the web, since the representative point for this girder lies to the left of the dashed line.

[7] "Die Angenaherte Ermittelung Harmonischer Funktionen und Konformer Abbildungen," by H. Liebmann, Sitzungberichte der Bayerischen Akademie der Wissenschaften zu München, Mathematische Physikalische Klasse, 1918, p. 385.

[1] By William R. Osgood. Published in the September, 1939, issue of the Journal of Applied Mechanics, Trans. A.S.M.E., vol. 61, 1939, p. A-122.

[2] Professor of Mechanics, Cornell University, Ithaca, N. Y.

[3] "On the Problems of the Beam and the Plate in the Theory of Elasticity," by J. N. Goodier, Transactions of the Royal Society of Canada, vol. 32, 1938, pp. 65–88.

[4] Assistant Physicist, National Bureau of Standards, Washington, D. C.

For $\alpha \geq 45$ deg, the curves B = const in Fig. 1 also represent ψ = const, the correspondence being as follows:

B	ψ, deg	B	ψ, deg
1.001	43.78	1.3	29.98
1.050	36.82	1.4	28.93
1.100	34.30	2.0	26.05
1.200	31.57	5.0	23.63

Dividing the values of $f_3/\sigma_{r3\,max}$ (see third equation following the author's Equation [20]) by B gives values of $f_3/\sigma_{p3\,max}$.

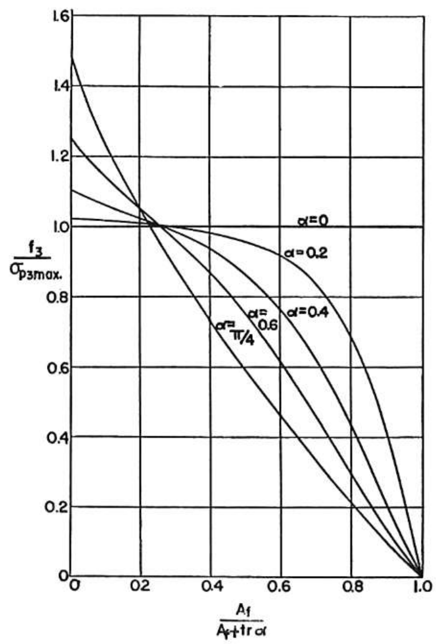

FIG. 2 COUPLE-LOADING CURVES

(Ratio of maximum stress by the ordinary beam theory to maximum principal stress by the wedge theory for all values of relative flange area and for different values of the half wedge angle of a wedge-shaped plate girder loaded by a couple.)

These are plotted in Fig. 2 of this discussion. By taking the same example previously given in this discussion, it is seen that while the maximum fiber stress by the ordinary theory is only 2.2 per cent (see Fig. 4 of the paper) less than the maximum radial stress by the present theory, it is 18.5 per cent less than the maximum principal stress by this theory, shown in Fig. 2 of this discussion. The concurrence of the curves in Fig. 2 at the point (1, 0) is due to the fact that, as the relative flange area approaches 100 per cent, the web becomes vanishingly thin, and the shearing stresses increase without limit.

In the case of the maximum shearing stress τ_{p3}, an analysis similar to the foregoing one in this discussion shows that the greatest value ($\tau_{p3\,max}$) of τ_{p3} on any one circular-cylindrical section occurs at the polar axis ($\theta = 0$) in case $\alpha \geq \alpha_s$, where, for B = const (see Fig. 1 of this discussion), α_s is related to α_c (Equation [1] of this discussion), by the expression

$$\alpha_s = 2\alpha_c - 45 \text{ deg} \qquad [4]$$

In this case, the value of $\tau_{p3\,max}$ is equal to $\tau_{r\theta 3\,max}$ (see following the author's Equation [21]), since no normal stresses act at points on the polar axis (Equation [20]). When $\alpha \leq \alpha_s$ the value of $\tau_{p3\,max}$ is given by

$$S = \frac{\tau_{p3\,max}}{\sigma_{p3\,max}} = 1 - \frac{1}{2B} \qquad [5]$$

and occurs at $\theta = \alpha$; see Equation [3a] of this discussion.

The dotted line in Fig. 1 of this discussion represents plate girders for which $\alpha = \alpha_s$, and in the region to the left of this line, the curves B = const are also curves for S = const, the correspondence being as follows:

B	S	B	S
1.000	0.500	1.3	0.615
1.001	0.501	1.4	0.643
1.050	0.524	2.0	0.750
1.100	0.545	5.0	0.900
1.200	0.583		

It is interesting to note that when $\alpha = \alpha_s$, τ_{p3} is independent of θ.

It must be remarked that in the case of a plate girder, the load on which is equivalent to forces at the edge of the wedge in addition to a couple, the foregoing results cannot be directly applied. This is true because the principal stresses will not in general act in the same directions in the combined case as in pure couple loading, and consequently they cannot be computed for the combined case by simple addition of their values for the cases which are superposed to produce the combined case.

AUTHOR'S CLOSURE

The author is indebted to Mr. Goodier for pointing out explicitly some of the limitations of the theory of plane stress when applied to cases in which the components of stress in a direction normal to the "plane of stress" are not all zero. These components were considered implicitly, but it would have been well to mention them.

Mr. Roop has gone as far as it is practicable to go in considering maximum normal and shearing stresses in the literal solution. His discussion adds welcomed completeness to the paper.

Clamped Rectangular Plates With a Central Concentrated Load[1]

EVERETT O. WATERS.[2] The author has made a very interesting use of successive-approximation methods in solving a set of linear simultaneous equations. During the last year, the writer was confronted with the task of solving sets of such equations in connection with the general problem of pressure distribution in a journal-bearing oil film, and found that when the set contained more than four equations he could get better accuracy and speed by successive approximation than by the "precise" methods usually taught in algebra textbooks.

TABLE 1 SUCCESSIVE-APPROXIMATION EQUATIONS

FIRST APPROX.	r	+ 12s	+ 13t	+ 14u	+ 15v	+	= 01
SECOND APPROX.	21r	+ s	+ 23t	+ 24u	+ 25v	+	= 02
THIRD APPROX.	31r	+ 32s	+ t	+ 34u	+ 35v	+	= 03
FOURTH APPROX.	41r	+ 42s	+ 43t	+ u	+ 45v	+	= 04
FIFTH APPROX.	51r	+ 52s	+ 53t	+ 54u	+ v	+	= 05
	⋮	⋮	⋮	⋮	⋮		⋮

Briefly, the equations had the form given in Table 1 of this discussion. They are characterized by unity coefficients on a major

[1] By Dana Young. Published in the September, 1939, issue of the JOURNAL OF APPLIED MECHANICS, Trans. A.S.M.E., vol. 61, 1939, p. A-114.

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