Curvature Singularity of Space Curves and Its Relationship to Computational Mechanics

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Curve geometry plays a fundamental role in many aspects of analytical and computational mechanics, particularly in developing new data-driven science (DDS) approaches. Furthermore, curvature and torsion of space curves serve as deformation measures that need to be properly interpreted, shedding light on the significance of relationship between differential-geometry curve framing methods and computational-mechanics motion description. Alternate space-curve framing methods were proposed to address existence of Frenet frame at isolated zero-curvature points. In this paper, both mechanics and differential-geometry approaches are used to establish Frenet-frame continuity and existence of Serret-Frenet equations at curvature-vanishing points for curves with arbitrary parameterization. Frenet–Euler angles, referred to for brevity as Frenet angles, are used to define curve geometry, with particular attention given to definition of Frenet bank angle used to prove existence of curve normal and binormal vectors at curvature-vanishing points. Solving curvature-singularity problem and using mechanics description based on Frenet angles contributes to successful development and computer implementation of new DDS approaches based on analysis of recorded motion trajectories (RMT). Centrifugal-inertia force is always in direction of curve normal vector, and in most applications, this force is continuous and approaches zero value as curve curvature approaches zero. Discontinuity in definition of Frenet frame can negatively impact quality of numerical results that define RMT curves. The study also demonstrates that Frenet-frame curvature singularity can be solved without need for integrating curve torsion, which is not, in general, an exact differential. Differences between Frenet-angle and conventional definitions of normal and binormal vectors are highlighted to shed light on significance of Frenet-angle geometry description in achieving continuity and existence of Frenet frame in neighborhoods of curvature-vanishing points. [DOI: 10.1115/1.4053339]

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1 Introduction

Curve geometry plays fundamental role in wide range of physics and engineering disciplines that include mechanics, transportation, astronomic, and computer sciences [1–6]. Curve geometry is also central in the interpretation of inertia forces, motion and deformation descriptions, and motion-trajectory analysis, which is expected to be crucial in developing algorithms for autonomous vehicles and positive train control (PTC). Curvature of space curves, for example, enters into definition of strain-energy function of Euler–Bernoulli beams. For this reason, accurate definitions of curvature and torsion, which represent geometric invariants of space curves, are necessary for wide range of engineering and physics applications. The relationship between mechanics and differential geometry is clear from the fact that curve geometry in its most general form can be represented by two in-plane and out-off-plane bending modes that can be conveniently described using angles. Furthermore, curve frames have been extensively used in mechanics to describe the orientation of beam cross sections and associate the orientation of these cross-section frames with deformation modes.

Space-curve *Frenet frame*, in particular, has been used widely in mechanics because of its clear geometric interpretation and its direct relationship to kinematic variables and inertia forces that influence motion. Nonetheless, Frenet frame suffers from singularity at curvature-vanishing points; at which two of the frame axes, normal and binormal vectors, cannot be defined if classical differential-geometry approaches are used. This singularity problem motivated researchers to seek alternative frames to define curve geometry [1–6]. This paper addresses this fundamental issue and demonstrates that Frenet-frame singularity problem can be solved using simple procedure based on curve-coordinate derivatives or alternatively *Frenet angles*, and consequently, use of alternative curve frames that may lack the clear Frenet-frame geometric interpretations may not be necessary if the objective is to solve curvature-singularity problem.

Frenet frame defines the geometry of a curve, described in its parametric form by the equation $\mathbf{r}(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix}^T$, where t is the curve parameter [7–14]. Frenet frame is defined by three orthogonal unit vectors that represent curve tangent vector $\mathbf{t}$, normal vector $\mathbf{n}$, and binormal vector $\mathbf{b}$. Transformation matrix that defines the orientation of Frenet frame is written as $\mathbf{A}_f = \begin{bmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \end{bmatrix}$. Unit vector tangent to the curve is defined by differentiating curve equation with respect to curve arc length s, that is, $\mathbf{t} = d\mathbf{r}/ds = \mathbf{r}_s$, where $a_s = da/ds$. Norm of curvature vector $\mathbf{r}_{ss} = d\mathbf{t}/ds$ defines curve *curvature* κ as $\kappa = |\mathbf{r}_{ss}| = 1/R$, where $R = R(s)$ is curve *radius of curvature*. Unit normal vector $\mathbf{n}$ is defined along curvature vector $\mathbf{r}_{ss}$ according to $\mathbf{n} = \mathbf{r}_{ss}/|\mathbf{r}_{ss}|$. While curvature vector can also be written in the form $\mathbf{r}_{ss} = (\mathbf{r}_{tt} - (\mathbf{r}_{tt}^T\mathbf{r}_t)\mathbf{t})/|\mathbf{r}_t|^2$, this form will not be used in this study since it defines curvature vector as linear combination of two vectors, one of them is function of second derivatives of

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coordinates. It will be shown that a curvature vector can be expressed as linear combination of two orthogonal vectors expressed in terms of only first derivatives of curve coordinates. First derivative vectors are well-defined at isolated zero-curvature points. Binormal vector $\mathbf{b}$ is defined using cross product $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ [7–14].

When using computational-mechanics data-driven science (DDS) approaches in the analysis of *recorded motion trajectories* (RMT), differential-geometry procedure described previously fails, at zero-curvature points, to define normal vector $\mathbf{n}$ and curve Frenet frame at these zero-curvature points. Consequently, at these points, binormal vector $\mathbf{b}$ and its derivative $\mathbf{b}_s$ cannot be defined if a procedure based on the definition of curvature vector is used. Consequently, at points of curvature singularity, Serret-Frenet equations $\mathbf{t}_s = \kappa\mathbf{n}$, $\mathbf{n}_s = -\kappa\mathbf{t} + \tau\mathbf{b}$, and $\mathbf{b}_s = -\tau\mathbf{n}$, where τ is curve torsion, fail to exist if the classical differential-geometry approach that depends on curvature vector is used; and in this case, unit vectors that define Frenet frame cannot be determined, except for the tangent vector $\mathbf{t}$. Furthermore, curve torsion τ at zero-curvature points cannot be determined as the norm of vector $\mathbf{b}_s$. This curvature singularity creates problems in the analysis of RMT curves which can be determined using experimental measurements or computer simulations of detailed models. Centrifugal-inertia force is always in direction of curve normal vector $\mathbf{n}$, and in most applications, this force is continuous and approaches zero value as curve curvature approaches zero. Discontinuity in the definition of Frenet frame can negatively impact the quality of numerical results that define RMT curves.

Crucial to the analysis of this paper, and in distinguishing this analysis from previously published work [1], is the fact that curve curvature κ and torsion τ are not, in general, *exact differentials* and cannot be integrated to determine angles. Resorting to numerical integration of non-exact differentials does not lead to unique solutions or definition of specific curve frames whose geometries are defined in terms of non-commutative finite rotations [19]. Exceptions to this rule are planar curves with zero torsion. In this special case, curvature κ can be written as derivative of an angle, and therefore, can be integrated to determine this angle.

This paper introduces a method based on Frenet angles or curve derivatives to prove that Frenet frame and Serret-Frenet equations are well defined at curvature-vanishing points without need for integrating curve torsion τ or introducing other framing methods. Results of the analysis of this study show that curvature vector $\mathbf{r}_{ss}$ should be viewed as a vector along well-defined normal vector $\mathbf{n}$ and such a curvature vector has a norm defined by curve curvature κ that varies continuously and can assume zero value without affecting the definition of normal vector $\mathbf{n}$ or continuity of Frenet frame. Consequently, all derivatives that appear in Serret-Frenet equations are well-defined at all points at which tangent vector $\mathbf{t}$ can be defined, which is the assumption of a *regular curve*. The provided proof, established using curve derivatives or *Frenet angles*, which facilitate proving the existence of Frenet frame and Serret-Frenet equations, also demonstrates continuity of centrifugal force $m\dot{s}^2/R$ of a vehicle or a particle with mass m at curvature-vanishing points [15–18]. The difference between Frenet bank angle and torsion angle used in the definition of *Bishop frame* is also discussed [1]. Unlike other framing methods, the analysis of this paper is focused on Frenet-frame definition and does not require evaluation and integration of curve torsion. Continuity of normal vector at curvature-vanishing points is demonstrated numerically using two different curve examples: one with zero torsion and the other has non-zero torsion.

At zero-curvature points, conventional normal vector can flip over, as in cases of switching from left to right curve. This change can be associated with instantaneous 180-degree rotation of Frenet normal plane about curve tangent vector, resulting in abrupt direction change in conventional normal and bi-normal vectors. Frenet-angle approach introduced in this paper alleviates this discontinuity by defining normal and bi-normal vectors that can assume opposite directions to conventional normal and bi-normal vectors. That is, within some curve segments, Frenet-angle and conventional normal and bi-normal vectors may coincide; while within other curve segments, Frenet-angle normal and bi-normal vectors can be opposite to conventional vectors used in differential geometry. In such latter curve segments, Frenet-angle Serret-Frenet equations lead to skew-symmetric Cartan matrix based on sign convention different from conventional Cartan matrix. Such different sign-conventions are used in classical differential-geometry literature [7–14].

Furthermore, the approach used in this paper differs from the approach used in [20], which is based on improvement made in Bishop frame. The approach in this investigation does not employ Bishop's two normal-plane vectors whose derivatives are along curve tangent vector, employs curvature components different from those previously used in the literature, is based on using Frenet bank angle that differs from Bishop torsion angle, does not define curve torsion as angle derivative, does not employ interpolation or extrapolation techniques to solve curvature singularity problem, and preserves structure of curve skew-symmetric Cartan matrix which is not preserved using approaches based on alternate curve framing methods.

2 Tangent Vector and Frenet Angles

The geometry of general three-dimensional curves can be represented by in-plane and out-of-plane bending modes that can be described using two independent angles. This geometry/mechanics relationship allows clear interpretation of motion and forces that can be obtained using DDS approaches based on RMT analysis. Curve *tangent vector*, defined as $\mathbf{r}_t = d\mathbf{r}/dt = \begin{bmatrix} x_t & y_t & z_t \end{bmatrix}^T$ where $a_t = da/dt$, has norm $|\mathbf{r}_t| = \sqrt{x_t^2 + y_t^2 + z_t^2}$. Therefore, curve unit tangent is defined as $\mathbf{t} = \mathbf{r}_s = d\mathbf{r}/ds = \begin{bmatrix} x_t & y_t & z_t \end{bmatrix}^T / |\mathbf{r}_t|$. The unit tangent $\mathbf{t}$ can always be written in terms of two angles which define curve in-plane and out-of-plane bending as [17]

$$\mathbf{t} = \mathbf{r}_s = \frac{1}{|\mathbf{r}_t|}\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta \\ \sin\psi\cos\theta \\ \sin\theta \end{bmatrix} \tag{1}$$

where angles ψ and θ are defined in terms of curve coordinate derivatives according to

$$\left.\begin{array}{ll} \cos\psi = x_t/\sqrt{x_t^2 + y_t^2}, & \sin\psi = y_t/\sqrt{x_t^2 + y_t^2}, \\ \cos\theta = \sqrt{x_t^2 + y_t^2}/|\mathbf{r}_t|, & \sin\theta = z_t/|\mathbf{r}_t|, \end{array}\right\} \tag{2}$$

Angles ψ and θ are referred to as *Frenet horizontal curvature angle* and *Frenet vertical-development angle*, respectively [15–18]. They can be used in three-angle Euler sequence of rotations to define Frenet frame that does not suffer from discontinuities and also define derivatives that appear in Serret-Frenet equations. The third angle ϕ, called *Frenet bank angle*, is used in mechanics applications to measure the deviation of centrifugal-force vector from the horizontal plane. The sequence of rotation used in this study for angles ψ, θ, and ϕ is $Z, -Y, -X$, with negative signs used for consistency with procedures used in the construction of highway roads and railroad tracks [15–18]. Unit-tangent equation shows that space curve is completely defined in terms of two angles, while a planar curve can always be defined in terms of one angle, curvature angle ψ.

Regarding singularity associated with Euler angles, it is to be noted that a transformation matrix written in terms of Euler angles is never singular. Such a matrix is orthogonal and has a well-defined inverse. Euler-angle singularity is associated with the singularity of the matrix that relates angular velocity to angle derivatives. Such a singularity becomes an issue only when interest is in determining angle derivatives. As discussed in the literature [15,16], Frenet bank angle ϕ for a space curve is dependent on the other two angles and is introduced for purpose of avoiding derivatives in the definition of normal and binormal vectors. As it will be shown in a later section, when $\theta=\pi/2$, the curve bends vertically with curvature $\kappa=\theta_s$ and Frenet-frame transformation matrix is well-defined and does not suffer from Euler-angle singularity problem because of dependence of bank angle ϕ on Frenet angles ψ and θ. When $\theta=\pi/2$, ϕ is well-defined and is equal to $\pi/2$ for $\theta_s\neq 0$.

3 Curvature Vector

Zero curvature corresponds to infinite radius of curvature that characterizes straight segments of motion-trajectory curves. In the neighborhood of zero-curvature points, centrifugal-inertia force, which is in direction of curve normal vector, approaches zero. Curvature vector of classical differential-geometry approaches is used to define curve curvature and radius of curvature. This approach, however, fails to determine curve normal vector resulting in a discontinuity in definition of Frenet frame widely used in mechanics to define deformation modes. For a general three-dimensional curve, one has differential relationship $ds=|\mathbf{r}_t|dt$, or $ds/dt=|\mathbf{r}_t|=\sqrt{x_t^2+y_t^2+z_t^2}$. It follows that $d(1/|\mathbf{r}_t|)/dt=-(x_tx_{tt}+y_ty_{tt}+z_tz_{tt})/|\mathbf{r}_t|^3$. *Curvature vector* $\mathbf{r}_{ss}=d^2\mathbf{r}/ds^2=\left(d\left(\begin{bmatrix}x_t & y_t & z_t\end{bmatrix}^T/|\mathbf{r}_t|\right)/dt\right)(dt/ds)$ can be written using equation $dt/ds=1/|\mathbf{r}_t|$ as $\mathbf{r}_{ss}=(1/|\mathbf{r}_t|)\left(d\left(\begin{bmatrix}x_t & y_t & z_t\end{bmatrix}^T/|\mathbf{r}_t|\right)/dt\right)$. This equation yields

$$\begin{aligned}\mathbf{r}_{ss}&=\frac{\mathbf{r}_{tt}-(\mathbf{r}_{tt}^T\mathbf{t})\mathbf{t}}{|\mathbf{r}_t|^2}=\frac{1}{|\mathbf{r}_t|^4}\left(-(x_tx_{tt}+y_ty_{tt}+z_tz_{tt})\begin{bmatrix}x_t\\y_t\\z_t\end{bmatrix}+|\mathbf{r}_t|^2\begin{bmatrix}x_{tt}\\y_{tt}\\z_{tt}\end{bmatrix}\right)\\&=\frac{1}{|\mathbf{r}_t|^4}\left(-(\mathbf{r}_t^T\mathbf{r}_{tt})\begin{bmatrix}x_t\\y_t\\z_t\end{bmatrix}+(\mathbf{r}_t^T\mathbf{r}_t)\begin{bmatrix}x_{tt}\\y_{tt}\\z_{tt}\end{bmatrix}\right)=\frac{1}{|\mathbf{r}_t|^2}\begin{bmatrix}x_{tt}-\alpha_cx_t\\y_{tt}-\alpha_cy_t\\z_{tt}-\alpha_cz_t\end{bmatrix}\end{aligned}\tag{3}$$

where $\alpha_c=(x_tx_{tt}+y_ty_{tt}+z_tz_{tt})/|\mathbf{r}_t|^2=\mathbf{r}_t^T\mathbf{r}_{tt}/|\mathbf{r}_t|^2$. *Curvature* κ is magnitude of curvature vector defined as

$$\begin{aligned}\kappa=\kappa(t)&=\sqrt{(x_{tt}-\alpha_cx_t)^2+(y_{tt}-\alpha_cy_t)^2+(z_{tt}-\alpha_cz_t)^2}/|\mathbf{r}_t|^2\\&=\sqrt{(x_{tt}^2+y_{tt}^2+z_{tt}^2)+(\alpha_c)^2|\mathbf{r}_t|^2}/|\mathbf{r}_t|^2=|\mathbf{r}_t\times\mathbf{r}_{tt}|/|\mathbf{r}_t|^3\end{aligned}\tag{4}$$

Term at the end of the right side of the preceding equation is the result of the identity $\mathbf{r}_s\times\mathbf{r}_{ss}=\mathbf{r}_s\times\kappa\mathbf{n}$ which shows that $\kappa=|\mathbf{r}_s\times\mathbf{r}_{ss}|=|\mathbf{r}_t\times\mathbf{r}_{tt}|/|\mathbf{r}_t|^3$. At zero-curvature points, $x_{tt}=y_{tt}=z_{tt}=\alpha_c=0$.

Two orthogonal unit vectors $\mathbf{n}_h$ and $\mathbf{n}_v$, orthogonal to tangent vector $\mathbf{t}$, can be defined as

$$\left.\begin{aligned}\mathbf{n}_h&=(\mathbf{k}\times\mathbf{r}_t)/|\mathbf{k}\times\mathbf{r}_t|=\left(1/\sqrt{x_t^2+y_t^2}\right)\begin{bmatrix}-y_t & x_t & 0\end{bmatrix}^T\\&=\begin{bmatrix}-\sin\psi & \cos\psi & 0\end{bmatrix}^T,\\\mathbf{n}_v&=(\mathbf{r}_t\times\mathbf{n}_h)/|\mathbf{r}_t\times\mathbf{n}_h|=\left(1/|\mathbf{r}_t|\sqrt{x_t^2+y_t^2}\right)\begin{bmatrix}-x_tz_t & -y_tz_t & x_t^2+y_t^2\end{bmatrix}^T\\&=\begin{bmatrix}\cos\psi\sin\theta & \sin\psi\sin\theta & \cos\theta\end{bmatrix}^T\end{aligned}\right\}\tag{5}$$

where $\mathbf{k}=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$. It is to be noted that two vectors $\mathbf{n}_h$ and $\mathbf{n}_v$ are defined in terms of first derivatives, and consequently, they are well-defined at curvature-vanishing points. Projection of curvature vector $\mathbf{r}_{ss}$ along these two vectors leads to $\mathbf{r}_{ss}=\alpha_h\mathbf{n}_h+\alpha_v\mathbf{n}_v=\kappa\mathbf{n}$, which can be written as

$$\mathbf{n}=(\alpha_h/\kappa)\mathbf{n}_h+(\alpha_v/\kappa)\mathbf{n}_v=(\cos\phi)\mathbf{n}_h-(\sin\phi)\mathbf{n}_v\tag{6}$$

where $\kappa=\sqrt{\alpha_h^2+\alpha_v^2}$, $\tan\phi=-\alpha_v/\alpha_h$ and

$$\left.\begin{aligned}\alpha_h&=\mathbf{r}_{ss}^T\mathbf{n}_h=(y_{tt}x_t-x_{tt}y_t)/|\mathbf{r}_t|^2\sqrt{x_t^2+y_t^2},\\\alpha_v&=\mathbf{r}_{ss}^T\mathbf{n}_v=(z_{tt}(x_t^2+y_t^2)-z_t(x_tx_{tt}+y_ty_{tt}))\Big/\left(|\mathbf{r}_t|^3\sqrt{x_t^2+y_t^2}\right)\end{aligned}\right\}\tag{7}$$

Angle ϕ, called *Frenet bank angle*, defines super-elevation of curve *osculating plane* (OP), which contains normal vector as well as velocity, acceleration, and centrifugal-force vectors. For a motion-trajectory curve with zero Frenet bank angle, centrifugal force lies in a plane parallel to the horizontal plane. This is clear from the equation $\mathbf{n}=(\cos\phi)\mathbf{n}_h-(\sin\phi)\mathbf{n}_v$, which shows that if $\phi=0$, $\mathbf{n}=\mathbf{n}_h=\left(1/\sqrt{x_t^2+y_t^2}\right)\begin{bmatrix}-y_t & x_t & 0\end{bmatrix}^T$. Because the general definition of centrifugal-force vector of vehicle or particle with mass m is $\mathbf{F}_c=(m\dot{s}^2/R)\mathbf{n}$, zero Frenet bank angle implies that centrifugal force remains in a plane parallel to the horizontal plane, and therefore, the OP is not Frenet super-elevated. It is important, however, to distinguish between Frenet super-elevation of motion-trajectory curves and railroad super-elevation used in the construction of railway tracks; the former is motion-dependent, while the latter is motion-independent [15].

The fundamental difference between Frenet bank angle ϕ and Bishop-torsion angle will be explained in later sections. Nonetheless, equation $\mathbf{n}=(\cos\phi)\mathbf{n}_h-(\sin\phi)\mathbf{n}_v$ should not be confused with an equation used in development of Bishop frame [1] in which normal vector is written as linear combination of two vectors $\mathbf{m}_1$ and $\mathbf{m}_2$, which have derivatives along tangent vector $\mathbf{t}$ as will be discussed in a later section. Derivatives of vectors $\mathbf{n}_h$ and $\mathbf{n}_v$ do not in general define vectors parallel to tangent vector $\mathbf{t}$. Because vectors $\mathbf{n}_h$ and $\mathbf{n}_v$ are functions of first derivatives only, their existence is ensured based on the assumption of existence of tangent vector $\mathbf{t}$, as previously mentioned. Therefore, existence of angle ϕ at zero-curvature points ensures existence of normal vector and derivatives that appear in Serret-Frenet equations despite $\kappa=\alpha_h=\alpha_v=0$ at these points. It is also important to note that curvature vector $\mathbf{r}_{ss}$ can be written in terms of other orthogonal vectors, and consequently, there are an infinite number of choices of such orthogonal vectors as it is clear from Eq. (3). Existence of ϕ and $\mathbf{n}$ at zero-curvature points is discussed in the following section.

4 Existence of ϕ and n

Existence of normal vector $\mathbf{n}$ at curvature-vanishing points can be proven using space-curve coordinates. To this end, *L'Hopital rule* is used to derive an expression for Frenet bank angle at curvature-vanishing points. While use of Frenet angles is not necessary, these angles provide clear interpretation of space-curve geometry as a combination of in-plane and out-of-plane bending modes.

4.1 Existence of Normal Vector.

Frenet bank angle, as previously defined, can be computed using ratio α_v/α_h, which can be written using arbitrary curve parameter t, as

$$\tan\phi=-\frac{\alpha_v}{\alpha_h}=-\frac{(z_{tt}(x_t^2+y_t^2)-z_t(x_tx_{tt}+y_ty_{tt}))}{|\mathbf{r}_t|(y_{tt}x_t-x_{tt}y_t)}\tag{8}$$

At points of curvature singularity, $\alpha_h = \alpha_v = 0$, and Frenet bank angle ϕ is not defined. To define ratio α_v/α_h, *L'Hopital rule* is used with curvature-singularity conditions $\kappa = x_{tt} = y_{tt} = z_{tt} = 0$. Using this procedure, one has

$$\lim_{\kappa\to 0}\left(\frac{d\alpha_v/dt}{d\alpha_h/dt}\right) = \frac{z_{ttt}(x_t^2 + y_t^2) - z_t(x_t x_{ttt} + y_t y_{ttt})}{|\mathbf{r}_t|(y_{ttt}x_t - x_{ttt}y_t)} \tag{9}$$

This limit defines Frenet bank angle ϕ at zero-curvature points. It is clear that the limit of the preceding equation is written in terms of first and third derivatives and does not involve second derivatives which appear in curvature expression. At isolated zero-curvature points, curve torsion is not in general equal to zero. Furthermore, limit of the preceding equation is expressed in terms of derivatives of curve coordinates, and therefore, it is independent of any angle representation. In case of RMT curve obtained using computer simulations or measurements, *numerical differentiation* can be used to define limit of the preceding equation. Information from previous time-steps can be stored and used in simple derivative expressions to determine this limit.

To demonstrate existence of curve normal vector and Serret-Frenet equations, the tangent vector expressed in terms of Frenet angles is differentiated with respect to arc length s. This leads to

$$\mathbf{r}_{ss} = \psi_s \cos\theta \begin{bmatrix} -\sin\psi \\ \cos\psi \\ 0 \end{bmatrix} + \theta_s \begin{bmatrix} -\cos\psi\sin\theta \\ -\sin\psi\sin\theta \\ \cos\theta \end{bmatrix} \tag{10}$$

Comparing this equation with previous development, one has

$$\alpha_h = \psi_s\cos\theta, \quad \alpha_v = \theta_s \tag{11}$$

Furthermore, the normal vector is well defined as

$$\mathbf{n} = \begin{bmatrix} -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi \\ \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi \\ -\cos\theta\sin\phi \end{bmatrix} = \cos\phi \begin{bmatrix} -\sin\psi \\ \cos\psi \\ 0 \end{bmatrix} - \sin\phi \begin{bmatrix} -\cos\psi\sin\theta \\ -\sin\psi\sin\theta \\ \cos\theta \end{bmatrix} \tag{12}$$

Therefore, Frenet frame is defined by the equation

$$\mathbf{A}_f = \begin{bmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & -\sin\psi\sin\phi - \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & \cos\psi\sin\phi - \sin\psi\sin\theta\cos\phi \\ \sin\theta & -\cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix} \tag{13}$$

This equation defines Frenet frame everywhere including at points with curvature singularities. According to this description, Frenet-frame vectors **t**, **n**, and **b** are all differentiable, and therefore, derivatives of tangent, normal, and binormal vectors that appear in Serret-Frenet equations exist at curvature-vanishing points.

Normal and bi-normal vectors defined, respectively, by second and third columns of the preceding equations can be opposite to conventional normal and bi-normal vectors used in classical differential geometry literature. This sign change is necessary to alleviate discontinuity problems and avoid software crashing in neighborhood of curvature singularities. In curve segments for which Frenet-angle and conventional-Frenet normal and bi-normal vectors do not coincide, Frenet-angle skew-symmetric Cartan matrix can be based on different sign convention. Curve segments for which Frenet-angle normal and bi-normal vectors are opposite to conventional ones can be easily identified by performing dot product of Frenet-angle normal vector and centripetal part of curve absolute acceleration vector. A positive dot product indicates that two sets of vectors have same direction, while negative dot product indicates that two sets of vectors have opposite directions. In numerical implementation, normal vector defined by second column of the matrix in the preceding equation must be used to ensure continuity of Frenet-angle normal and bi-normal vectors.

4.2 Euler-Angle Singularity. Curve angle representation described in this section is not affected by Euler-angle singularity, and therefore, such kinematic singularity is not an issue in Frenet-frame definition. For example, if $\theta = \pi/2$, one has tangent, normal, and binormal vectors given, respectively, by $\mathbf{t} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, $\mathbf{n} = -\begin{bmatrix} \cos\psi & \sin\psi & 0 \end{bmatrix}^T$, and $\mathbf{b} = \begin{bmatrix} \sin\psi & -\cos\psi & 0 \end{bmatrix}^T$. Derivative of tangent vector, when $\theta = \pi/2$, shows that curve curvature is defined as $\kappa = \theta_s$. That is, at $\theta = \pi/2$ configuration, the curve bends vertically and curvature κ is described by a single bending mode along vector $\mathbf{n}_v$. Furthermore, using equations and identities given in Refs. [15,16], one can show that at this configuration, Frenet bank angle is $\phi = -\pi/2$, and Frenet frame is well-defined.

4.3 Special Case. Not every planar curve has one of its coordinates constant as demonstrated in numerical-example section. Nonetheless, without any loss of generality, a coordinate transformation can be used to make one of the planar curve coordinates constant. For example, in the special case of planar curve with constant z coordinate, normal vector **n** is well-defined since Frenet vertical-development angle θ is equal to zero, and tangent and normal vectors can be written in this case, respectively, as $\mathbf{t} = \mathbf{r}_s = \begin{bmatrix} \cos\psi & \sin\psi & 0 \end{bmatrix}^T$ and $\mathbf{n} = \begin{bmatrix} -\sin\psi & \cos\psi & 0 \end{bmatrix}^T$. As previously shown, angle ψ depends on first derivatives of curve coordinates because $\cos\psi = x_t/\sqrt{x_t^2 + y_t^2}$ and $\sin\psi = y_t/\sqrt{x_t^2 + y_t^2}$. Therefore, in this case of planar curve, normal vector **n** is well-defined and there is no need for using limit defined by the L'Hopital rule.

5 Comparison with Other Framing Methods

Frenet frame, particularly its osculating plane which is *plane of motion*, has unique geometric properties that can be fully exploited in motion-trajectory analysis. Elements of its skew-symmetric *Cartan matrix* are curvature and torsion widely used in analytical and computational mechanics. The curvature-singularity problem, however, was the main motivation for introducing other framing methods which lack geometric interpretations offered by Frenet frame [1–6]. Important for analysis and discussion presented in this section is the fact that elements of *Cartan matrix* defined using orthogonality condition of the matrices that define frame orientations are not in general exact differentials, as it is the case with elements of angular velocity vectors [19].

5.1 Normal Vector. Unit vector $\mathbf{n}$ normal to the curve can always be written as a linear combination of two other orthogonal unit vectors that lie in curve *normal plane*. The choice of two orthogonal unit vectors in this linear combination is not unique. That is, one can always write $\mathbf{n} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, where c_1 and c_2 can always be expressed in terms of angle β and must satisfy condition $\sqrt{c_1^2 + c_2^2} = 1$ because $\mathbf{n}$ is a unit vector. There are an infinite number of choices of orthogonal unit vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, and angle β. To define unique value of β, a condition must be imposed. In the approach used in this paper to overcome curvature singularity, a normal vector is written as $\mathbf{n} = (\alpha_h/\kappa)\mathbf{n}_h + (\alpha_v/\kappa)\mathbf{n}_v$ which upon using definition $\tan\phi = -\alpha_v/\alpha_h$ can be written as $\mathbf{n} = (\cos\phi)\mathbf{n}_h - (\sin\phi)\mathbf{n}_v$. In this case, the condition imposed on choice of angle ϕ is $\psi_s \sin\phi\cos\theta + \theta_s\cos\phi = 0$ [16]. The advantage of using this condition, in addition to its consistency with geometric description of railroad tracks, is avoiding integration of curve torsion τ, as it is required when *Bishop frame* is used.

5.2 Bishop Framing Method. One of the framing methods which became popular for alleviating curvature singularity is *Bishop frame* [1] which replaces normal and binormal vectors $\mathbf{n}$ and $\mathbf{b}$ by two other orthogonal unit vectors $\mathbf{m}_1$ and $\mathbf{m}_2$, respectively. Bishop frame, defined by three orthogonal unit vectors $\mathbf{t}$, $\mathbf{m}_1$ and $\mathbf{m}_2$, shares tangent vector $\mathbf{t}$ with Frenet frame, and the two frames differ by single rotation about $\mathbf{t}$. In the definition of Bishop frame, normal vector is written as $\mathbf{n} = (\cos\theta_B)\mathbf{m}_1 + (\sin\theta_B)\mathbf{m}_2$, where $\cos\theta_B = k_1/\kappa$, $\sin\theta_B = k_2/\kappa$, and θ_B is referred to in this paper as *Bishop-torsion angle*. In defining Bishop frame, it is assumed that $\mathbf{m}_{1s} = -k_1\mathbf{t}$ and $\mathbf{m}_{2s} = -k_2\mathbf{t}$. Therefore, Bishop frame is defined by the following equations [1]:

$$\left.\begin{aligned} &\mathbf{t}_s = k_1\mathbf{m}_1 + k_2\mathbf{m}_2, \\ &\mathbf{m}_{1s} = -k_1\mathbf{t}, \quad \mathbf{m}_{2s} = -k_2\mathbf{t} \end{aligned}\right\} \tag{14}$$

In these equations, $\kappa(s) = \sqrt{k_1^2 + k_2^2}$. Bishop introduced angle $\theta_B(s)$ such that $\tau(s) = d\theta_B/ds$ and $\tan\theta_B = k_2/k_1$. Therefore, angle θ_B can be determined by integrating curve torsion τ as $\theta_B = \int\tau ds$. Depending on initial value of θ_B, orientation of Bishop frame is defined. Nonetheless, initial value of θ_B does not have an effect on torsion τ due to differentiation. The condition used by Bishop to define angle θ_B is to equate derivative of normal vector as defined by the equation $\mathbf{n} = (\cos\theta_B)\mathbf{m}_1 + (\sin\theta_B)\mathbf{m}_2$ to second Serret-Frenet equation $\mathbf{n}_s = -\kappa\mathbf{t} + \tau\mathbf{b}$. Noting that $\mathbf{n}_s = -\kappa\mathbf{t} + \theta_{Bs}[(-\sin\theta_B)\mathbf{m}_1 + (\cos\theta_B)\mathbf{m}_2]$, one can write $\theta_{Bs} = \partial\theta_B/\partial s = \tau$. This condition differs from the condition used in this study and previous investigations [15–18], and therefore, Frenet bank angle ϕ differs from Bishop-torsion angle θ_B. Furthermore, Bishop frame is based on definitions $\mathbf{m}_{1s} = -k_1\mathbf{t}$ and $\mathbf{m}_{2s} = -k_2\mathbf{t}$, which show differences between two vectors $\mathbf{m}_1$ and $\mathbf{m}_2$, and two vectors $\mathbf{n}_\text{h}$ and $\mathbf{n}_\text{v}$ whose derivatives are not in general along tangent vector $\mathbf{t}$.

It is important to point out that the Bishop frame, referred to as *parallel transport frame*, was introduced as an alternative frame well-defined at the vanishing-curvature points. This frame, however, is not unique and does not include curve normal vector, and therefore, its axes do not define osculating plane which is plane of motion. Definition of Bishop fame also requires integration of curve torsion τ and based on the assumption of smooth θ_B. At curvature-vanishing points, $\kappa = k_1 = k_2 = 0$, and torsion τ is not defined according to classical differential geometry.

5.3 Bishop-Torsion and Frenet Bank Angles, and Exact Differentials. Bishop-torsion angle θ_B is not Frenet bank angle ϕ despite the fact that both angles are rotations about tangent vector $\mathbf{t}$. This fact is clear from the definition of the two angles in terms of curve torsion τ, and from the fact that bank angle ϕ is a member of three-angle sequence that defines curve geometry regardless of whether Bishop frame is introduced. Bishop-torsion angle θ_B is defined in terms of curve torsion as $\theta_B = \int\tau ds$, while Frenet bank angle is defined in terms of torsion as $\phi_s = \psi_s\sin\theta - \tau$ [16]. These two definitions define the relationship between two angles as

$$\theta_{Bs} = \tau = \psi_s\sin\theta - \phi_s \tag{15}$$

In case of Frenet frame, if sequence of finite rotations is used to define frame geometry, such finite rotations cannot all be directly associated with bending and twisting modes as discussed in the literature [19]. This fact is clear from the preceding equation since curvature and torsion of a space curve are not, in general, *exact differentials*. If $\theta_{Bs} = \tau = \psi_s\sin\theta - \phi_s$ were an exact differential, one must be able to write $\theta_{Bs} = (\partial\theta_B/\partial\psi)\psi_s + (\partial\theta_B/\partial\phi)\phi_s$, with $(\partial\theta_B/\partial\psi) = \sin\theta$ and $(\partial\theta_B/\partial\phi) = -1$. These two equations show that $(\partial^2\theta_B/\partial\psi\partial\phi) \neq (\partial^2\theta_B/\partial\phi\partial\psi)$, demonstrating that $\theta_{Bs} = \tau = \psi_s\sin\theta - \phi_s$ is not an exact differential and cannot be integrated as is the case with the angular velocity vector of rigid bodies in space. Condition $\theta_{Bs} = \tau = \psi_s\sin\theta - \phi_s$ resembles non-integrable nonholonomic constraints encountered in mechanics. These nonholonomic constraints impose restrictions on derivatives but do not impose restrictions on coordinates. That is, number of independent derivatives is less than number of independent coordinates.

Nonetheless, in computational mechanics, some investigators used finite rotations that follow a sequence of three Euler angles to describe bending and torsion modes. When these angles are used, a distinction needs to be made between twist of beam centerline which is due to out-of-plane bending and torsion due to shear. In case of Euler–Bernoulli beams, centerline geometry can be described by only two independent angles, and therefore, a condition must be imposed on the angles as previously discussed in this paper and in the literature [19].

5.4 Definition of the Torsion. In Bishop's framing method, curve torsion τ is used to define Bishop's torsion angle θ_B from the equation $\theta_B = \int\tau ds$. The general expression of torsion, however, is defined by the equation

$$\begin{aligned} \tau &= (\mathbf{r}_t \times \mathbf{r}_{tt}) \cdot \mathbf{r}_{ttt}/|\mathbf{r}_t \times \mathbf{r}_{tt}|^2 \\ &= \frac{x_{ttt}(y_t z_{tt} - y_{tt} z_t) + y_{ttt}(z_t x_{tt} - x_t z_{tt}) + z_{ttt}(x_t y_{tt} - y_t x_{tt})}{(y_t z_{tt} - y_{tt} z_t)^2 + (z_t x_{tt} - x_t z_{tt})^2 + (x_t y_{tt} - y_t x_{tt})^2} \end{aligned} \tag{16}$$

This equation shows that torsion is not defined at curvature-vanishing points at which $x_{tt} = y_{tt} = z_{tt} = 0$, leading to zero values for the numerator and denominator in the preceding equation. Therefore, at curvature-vanishing points, angle θ_B cannot be defined unless a procedure similar to the L'Hopital rule, adopted to define curvature, is also used to define torsion. An alternative to applying L'Hopital rule to define torsion is to use Frenet angles to define torsion using equation $\tau = \psi_s\sin\theta - \phi_s$ [16]. Since ϕ is a well-defined smooth function, use of Frenet bank angle ϕ to define torsion can be more convenient. It is also to be noted that ϕ can have non-zero values at curvature-vanishing points and can oscillate around non-zero constant nominal value [18].

In case of zero torsion, $\phi_s = \psi_s\sin\theta$, and the curve is planar curve that can lie in a plane rotated with respect to the global system. In this case, zero Frenet bank angle ($\phi = 0$) implies that normal vector lies in a plane parallel to the horizontal plane and normal vector $\mathbf{n}$, which defines direction of centrifugal force, is the intersection of this horizontal plane with the osculating plane.

6 Frenet Super-Elevation

Having mechanics interpretation of curve geometry contributes to meaningful analysis of motion trajectories recorded experimentally or obtained using computer simulations of detailed vehicle models. Frenet bank angle ϕ defines super-elevation of osculating plane (OP), which contains absolute velocity, acceleration, and centrifugal-force vectors, as previously mentioned. Projection of gravity force onto OP defines amount of centrifugal-force balancing. Therefore, definition of Frenet bank angle can play important role in RMT analysis and accurate definitions of balance speeds that ensure safe vehicle operations [18].

To further demonstrate that torsion angle θ_B does not have the same interpretation as Frenet bank angle ϕ which enters into definition of Frenet frame, vectors that define axes of Bishop frame are expressed in terms of vectors that define axes of Frenet frame. Angle θ_B, performed about local axis $\bar{\mathbf{t}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, defines a simple rotation matrix when pre-multiplied by transformation matrix $\mathbf{A}_f$ that defines Frenet frame leads to matrix $\mathbf{A}_B$ that defines axes of Bishop frame as

$$\begin{aligned}\mathbf{A}_B &= \begin{bmatrix} \mathbf{t} & \mathbf{m}_1 & \mathbf{m}_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{t} & (\mathbf{n}\cos\theta_B + \mathbf{b}\sin\theta_B) & (-\mathbf{n}\sin\theta_B + \mathbf{b}\cos\theta_B) \end{bmatrix}\end{aligned} \tag{17}$$

Upon using Serret-Frenet equations, $\mathbf{t}_s = \kappa\mathbf{n}$, $\mathbf{n}_s = -\kappa\mathbf{t} + \tau\mathbf{b}$, and $\mathbf{b}_s = -\tau\mathbf{n}$; skew-symmetric Cartan matrix $\mathbf{A}_B^T\mathbf{A}_{Bs}$, where $\mathbf{A}_{Bs} = d\mathbf{A}_B/ds$, can be written as

$$\begin{aligned}\mathbf{A}_B^T\mathbf{A}_{Bs} &= \begin{bmatrix} \mathbf{t}^T \\ (\mathbf{n}^T\cos\theta_B + \mathbf{b}^T\sin\theta_B) \\ (-\mathbf{n}^T\sin\theta_B + \mathbf{b}^T\cos\theta_B) \end{bmatrix} \\ &\quad\times \begin{bmatrix} \mathbf{t}_s & (\mathbf{n}_s\cos\theta_B - \tau\mathbf{n}\sin\theta_B) & (-\mathbf{n}_s\sin\theta_B - \tau\mathbf{n}\cos\theta_B) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\kappa\cos\theta_B & \kappa\sin\theta_B \\ \kappa\cos\theta_B & 0 & 0 \\ -\kappa\sin\theta_B & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -k_1 & -k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix}\end{aligned} \tag{18}$$

This equation defines same curvature components as obtained by Bishop [1]. The analysis presented in the preceding sections showed that normal vector $\mathbf{n}$ can be written as $\mathbf{n} = \mathbf{n}_h\cos\phi - \mathbf{n}_v\sin\phi$. This equation defines binormal vector $\mathbf{b}$ as $\mathbf{b} = \mathbf{t}\times\mathbf{n} = \mathbf{n}_h\sin\phi + \mathbf{n}_v\cos\phi$. Using these definitions of vectors $\mathbf{n}$ and $\mathbf{b}$, two Bishop vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ can be written as

$$\left.\begin{aligned}\mathbf{m}_1 &= \mathbf{n}_h\cos(\theta_B - \phi) + \mathbf{n}_v\sin(\theta_B - \phi) \\ \mathbf{m}_2 &= -\mathbf{n}_h\sin(\theta_B - \phi) + \mathbf{n}_v\cos(\theta_B - \phi)\end{aligned}\right\} \tag{19}$$

This equation shows the difference between two vectors $\mathbf{m}_1$ and $\mathbf{m}_2$, and two vectors $\mathbf{n}_h$ and $\mathbf{n}_v$.

Based on the analysis presented in this section and preceding sections, the following observations can be made:

(1) Condition of Eq. (15) is of nonholonomic type due to the fact that curve curvature and torsion, when written in terms of angle derivatives, are not *exact differentials*. Therefore, Eq. (15) imposes constraint on derivatives, but does not impose constraint on angles; that is, angle θ_B can assume any value. Non-integrability of torsion explains the non-uniqueness of Bishop frame.

(2) Torsion angle θ_B used in definition of Bishop frame has nothing to do with super-elevation of Frenet-frame osculating plane defined solely by Frenet bank angle ϕ. This fact will be demonstrated by an example which shows that angle θ_B remains constant or zero based on assuming arbitrary initial value, while Frenet bank angle ϕ is oscillatory.

(3) Therefore, angle θ_B does not enter into definition of absolute velocity, acceleration, and centrifugal-force vectors, while Frenet bank angle ϕ enters into definition of curve geometry. Condition $\theta_B = 0$ does not necessarily imply that normal vector $\mathbf{n}$ lies in a plane parallel to the horizontal plane.

(4) Because torsion τ can be written in terms of Frenet–Euler angles as $\tau = \psi_s\sin\theta - \phi_s$, the definition of angle ϕ at curvature-vanishing points ensures proper definition of torsion τ at those points.

The fundamental difference between Frenet bank angle ϕ and Bishop-torsion angle θ_B can be further made clear by considering *helix curve* which has constant torsion. For the helix curve, Frenet bank angle ϕ is identically zero, while Bishop-torsion angle θ_B is linear function of curve arc length s, and therefore, θ_B increases as curve arc length increases.

7 Numerical Results

In this section, two curve examples are considered to demonstrate application of the equations developed in this paper. In the first example, the curvature and Frenet angles are computed and presented. As previously discussed, Frenet bank angle defines super-elevation of the osculating plane which contains tangent and normal vectors.

The first curve used in this section is shown in Fig. 1 and is defined in its parametric form as $\mathbf{r} = \begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} t & Y_o\sin\omega t & t \end{bmatrix}^T$ m, where $Y_o = 0.02$ m, $\omega = 6$ rad/s are constants, and t is time, considered as curve parameter. Curve tangent vector is $\mathbf{r}_t = \begin{bmatrix} x_t & y_t & z_t \end{bmatrix}^T = \begin{bmatrix} 1 & \omega Y_o\cos\omega t & 1 \end{bmatrix}^T$ m/s, and norm of this tangent vector is $|\mathbf{r}_t| = \sqrt{2 + (\omega Y_o\cos\omega t)^2}$. In this case, Frenet curvature angle ψ can be determined from the equation $\tan\psi = y_t/x_t = \omega Y_o\cos\omega t$, and the vertical-development angle θ can be determined from the equation $\tan\theta = z_t/\sqrt{x_t^2 + y_t^2} = 1/\sqrt{1 + (\omega Y_o\cos\omega t)^2}$. To determine Frenet bank angle ϕ, derivative of vector $\mathbf{r}_t$ is determined as $\mathbf{r}_{tt} = \begin{bmatrix} x_{tt} & y_{tt} & z_{tt} \end{bmatrix}^T = \omega^2 Y_o\begin{bmatrix} 0 & -\sin\omega t & 0 \end{bmatrix}^T$ m/s^2. Using this vector, α_h and α_v can be evaluated, respectively, as

$$\begin{aligned}\alpha_h &= (y_{tt}x_t - x_{tt}y_t)/|\mathbf{r}_t|^2\sqrt{x_t^2 + y_t^2} \\ &= (-\omega^2 Y_o\sin\omega t)/(2 + (\omega Y_o\cos\omega t)^2)\sqrt{1 + (\omega Y_o\cos\omega t)^2}\end{aligned} \tag{20}$$

and

$$\begin{aligned}\alpha_v &= (z_{tt}(x_t^2 + y_t^2) - z_t(x_t x_{tt} + y_t y_{tt}))/\left(|\mathbf{r}_t|^3\sqrt{x_t^2 + y_t^2}\right) \\ &= (\sin\omega t\cos\omega t)/(2 + (\omega Y_o\cos\omega t)^2)^{3/2}\sqrt{1 + (\omega Y_o\cos\omega t)^2}\end{aligned} \tag{21}$$

Curve curvature κ is determined using the equation $\kappa = \sqrt{\alpha_h^2 + \alpha_v^2}$ 1/m. Frenet bank angle can be determined from the equation $\tan\phi = -\alpha_v/\alpha_h$. In case of zero curvature, angle ϕ can be determined using ratio $[z_{ttt}(x_t^2 + y_t^2) - z_t(x_t x_{ttt} + y_t y_{ttt})]/|\mathbf{r}_t|(y_{ttt}x_t - x_{ttt}y_t)$, which can be determined using the third derivative of $\mathbf{r}$ with

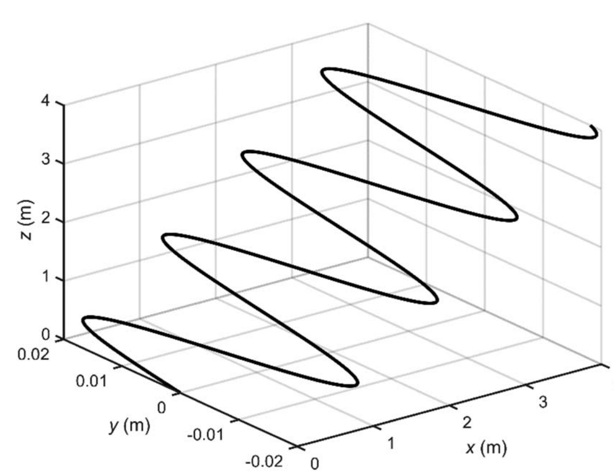

Fig. 1 Three-dimensional plot of the curve

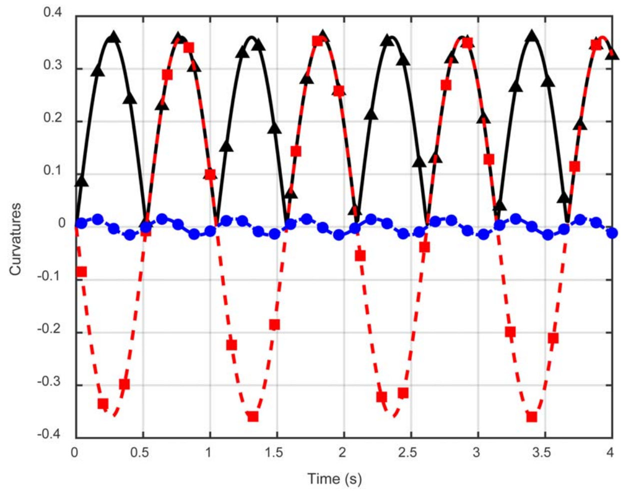

Fig. 2 Curve curvatures (1/m): black-triangle, curvature κ; red-square, curvature component α_h; and blue-circle, curvature component α_v

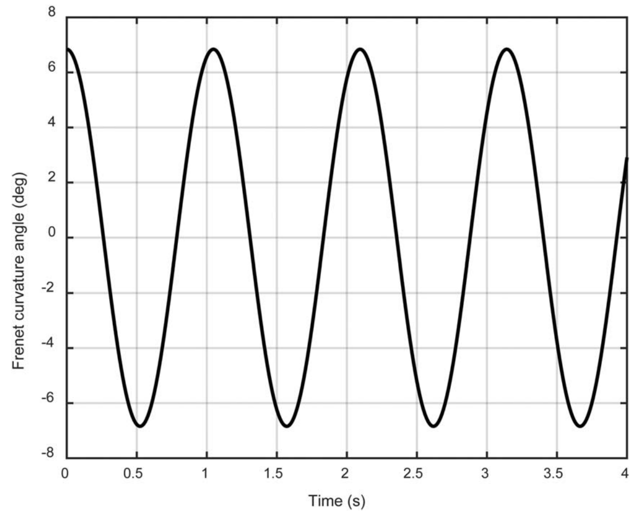

Fig. 4 Frenet curvature angle ψ

respect to t as

$$\mathbf{r}_{ttt} = \left[x_{ttt} \quad y_{ttt} \quad z_{ttt} \right]^T = \omega^3 Y_o \left[0 \quad -\cos \omega t \quad 0 \right]^T \text{ m/s}^3 \tag{22}$$

Curve curvature results of Fig. 2 show repeated zero-curvature points. Figure 2 also shows curvature components α_h and α_v, which are oscillatory and assume positive and negative values. While curve curvature κ is always positive, that is, $\kappa \geq 0$, curvature components α_h and α_v can assume negative values. At curvature-vanishing points, curvature vector $\mathbf{r}_{ss}$ is a zero vector. The results shown in Figs. 3–5, however, show that all Frenet angles are well defined, and therefore, normal vector $\mathbf{n} = \left[n_1 \quad n_2 \quad n_3 \right]^T$ is defined at zero-curvature points. Components of the normal vector, defined by Eq. (12), are shown in Fig. 6. The results shown in this figure and Eq. (12) demonstrate that components of normal vector $\mathbf{n}$ are smooth, and binormal vector $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is well-defined and smooth, demonstrating existence of Frenet frame and derivatives of vectors $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ at points of zero curvature. Existence of derivatives of these vectors provides proof of existence of Serret-Frenet equations at curvature-singularity points.

The example considered in this section was for a planar curve which is super-elevated and vertically developed. This planar curve has zero torsion despite the fact that the three Frenet angles can vary nonlinearly. Another example with non-zero torsion is

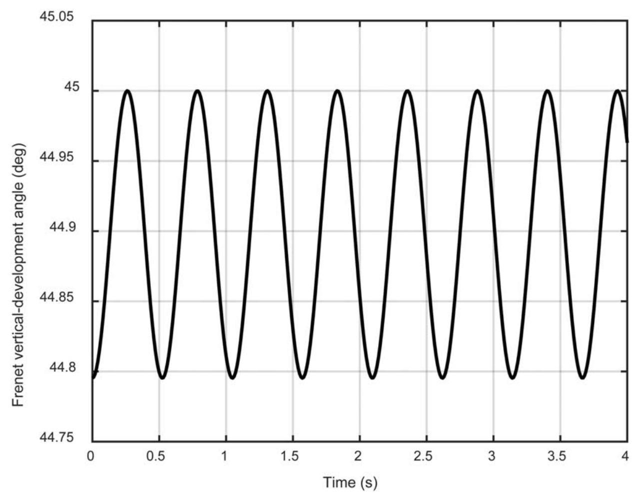

Fig. 5 Frenet vertical-development angle θ

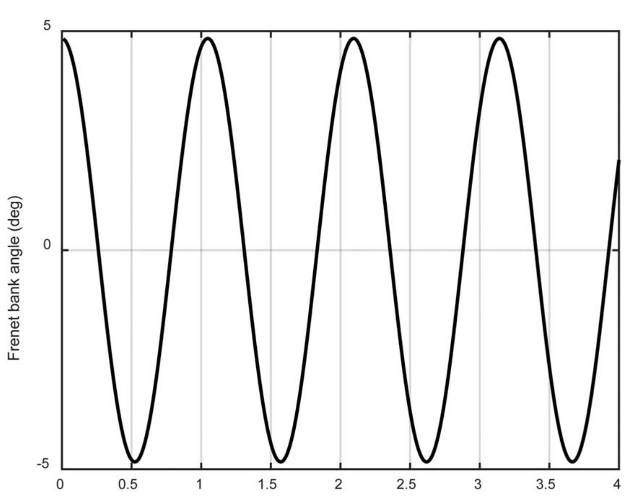

Fig. 3 Frenet bank angle ϕ

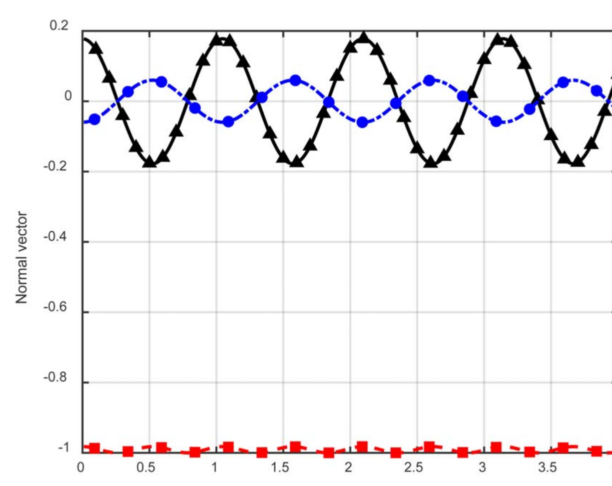

Fig. 6 Normal vector components: black-triangle, n_1; red-square, n_2; and blue-circle, n_3

used in this section to shed light on Bishop frame. The curve considered is defined by the equation $\mathbf{r}(t) = \left[\cos t \quad \sin t \quad bt^2/2\right]^T$ m. Tangent vector is $\mathbf{t} = (1/|\mathbf{r}_t|)\left[-\sin(t) \quad \cos(t) \quad bt\right]^T$, where $|\mathbf{r}_t| = \sqrt{1+(bt)^2}$. This tangent vector can be written in terms of Frenet angles as

$$\begin{aligned}\mathbf{t} &= (1/|\mathbf{r}_t|)\left[-\sin t \quad \cos t \quad bt\right]^T \\ &= \left[\cos\psi\cos\theta \quad \sin\psi\cos\theta \quad \sin\theta\right]^T\end{aligned} \tag{23}$$

Using this equation, one has $\cos\psi = -\sin t$, $\sin\psi = \cos t$, $\cos\theta = 1/|\mathbf{r}_t|$, and $\sin\theta = bt/|\mathbf{r}_t|$. The torsion τ is $\theta_{Bs} = \tau = \psi_s \sin\theta - \phi_s$. This equation shows that the torsion is not, in general, an exact differential.

8 Conclusions

Curve geometry plays a fundamental role in analytical and computational mechanics, particularly in analysis of motion trajectories that can be recorded experimentally or using computer simulations of detailed vehicle models. RMT curves and their geometry are central in new DDS computer algorithms that can be used for developing safety and operation guidelines of transportation systems, particularly in algorithms designed for autonomous vehicles and positive train control (PTC). Furthermore, curvature and torsion of space curves are widely used in deformation analysis of structural systems. For these reasons, addressing singularity problems of RMT curves is necessary for proper interpretation of experimentally or numerically recorded motion trajectories.

Nonetheless, because curve curvature and torsion are not, in general, exact differentials, they cannot be integrated to define angles. Consequently, an approach different from what has been previously proposed in the literature to alleviate curvature singularity is needed. This paper introduces a procedure to establish the existence of Frenet frame and Serret-Frenet equations at curvature-vanishing points for general three-dimensional curve with arbitrary parameterization. The paper demonstrates that it is not necessary to determine a curve normal vector from a curvature vector which can approach zero at vanishing-curvature points. *Frenet curvature*, *vertical-development*, and *bank angles* are used to define curve geometry. Particular attention is given to definition of Frenet bank angle, used to prove existence of curve normal and binormal vectors at curvature-vanishing points. Difference between Frenet bank angle and torsion angle used to define *Bishop frame* is discussed. Because of the unique features of Frenet frame and its clear geometric interpretation, the analysis of this study is focused on addressing and providing a solution to the curvature-singularity problem. The procedure described in this paper for the definition of Frenet frame at zero-curvature points is developed for arbitrary curve parameters and does not require evaluation and integration of curve torsion. Numerical differentiation can be used to determine L'Hopital limit if the curve cannot be defined analytically. Results of the analysis demonstrate that Frenet frame and Serret-Frenet equations are well defined at curvature-vanishing points. While use of Frenet angles in the analysis of curve geometry is not necessary, these angles give a clear geometric interpretation for curve geometry and allow obtaining simple expressions for curve geometric invariants.

As discussed previously, it is important to distinguish between space-curve torsion resulting from out-of-plane bending and torsion due to shear. In computational-mechanics literature, torsion due to shear is often described using an angle, considered as degree of freedom [21]. Shear-torsion angle used in some beam formulations is distinguished from Bishop torsion angle which is not unique [1]. Shear-torsion angle is independent from space-curve geometry and its magnitude depends on structural material properties, and consequently, distinction is made between geometric analysis and frames used in [1] and in this paper, and analysis introduced in [21] which defines a frame independent of Frenet-frame geometry. Furthermore, geometry approach presented in this study for addressing curvature singularity needs to be further tested using challenging problems discussed in differential geometry literature [22]. This can further shed light on fundamental geometry/analysis issues [23,24].

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Conflict of Interest

There are no conflicts of interest. This article does not include research in which human participants were involved. Informed consent is not applicable. This article does not include any research in which animal participants were involved.

Data Availability Statement

The data sets generated and supporting the findings of this article are obtainable from the corresponding author upon reasonable request. The authors attest that all data for this study are included in the paper. Data provided by a third party are listed in Acknowledgment. No data, models, or code were generated or used for this paper.

References

[1] Bishop, L. R., 1975, "There Is More Than One Way to Frame a Curve," Amer. Math. Monthly, **82**(3), pp. 246–251.
[2] Bahaddin Bukcu, B., and Karacan, M. K., 2008, "Special Bishop Motion and Bishop Darboux Rotation Axis of the Space Curve," J. Dyn. Syst. Geom. Theor., **6**(1), pp. 27–34.
[3] Bahaddin Bukcu, B., and Karacan, M. K., 2009, "The Slant Helices According to Bishop Frame," World Acad. Sci. Eng. Technol., **35**, pp. 1039–1042.
[4] Andrew, J. H., and Ma, H., 1995, "Quaternion Frame Approach to Streamline Visualization," IEEE Trans. Vis. Comput. Graph., **1**(2), pp. 164–174.
[5] Yung-chow Wong, Y. C., 1963, "A Global Formulation of the Condition for a Curve to Lie on a Sphere," Monatsh. Fur Math., **67**(4), pp. 363–365.
[6] Yung-chow Wong, Y. C., 1972, "On an Explicit Characterization of Spherical Curves," Proc. Am. Math. Soc., **34**(1), pp. 239–242.
[7] Guggenheimer, H. W., 1963, *Differential Geometry*, McGraw-Hill, New York.
[8] O'Neill, B., 1966, *Elementary Differential Geometry*, Academic Press, New York.
[9] Farin, G., 1999, *Curves and Surfaces for CAGD: A Practical Guide*, 5th ed., Morgan Kaufmann Publishers, San Francisco, CA.
[10] Gallier, J., 2011, *Geometric Methods and Applications: For Computer Science and Engineering*, Springer, New York.
[11] Goetz, A., 1970, *Introduction to Differential Geometry*, Addison Wesley, Reading, MA.
[12] Kreyszig, E., 1991, *Differential Geometry*, Dover Publications.
[13] Piegl, L., and Tiller, W., 1997, *The NURBS Book*, 2nd ed., Springer, Berlin.
[14] Rogers, D. F., 2001, *An Introduction to NURBS With Historical Perspective*, Academic Press, San Diego, CA.
[15] Shabana, A. A., and Ling, H., 2021, "Characterization and Quantification of Railroad Spiral-Joint Discontinuities," Mech. Based Des. Struct. Mach., **50**, pp. 1–26.
[16] Ling, H., and Shabana, A. A., 2021, "Euler Angles and Numerical Representation of the Railroad Track Geometry," Acta Mech., **232**(8), pp. 3121–3139.
[17] Shabana, A. A., 2021, *Mathematical Foundation of Railroad Vehicle Systems: Geometry and Mechanics*, John Wiley & Sons.
[18] Shabana, A. A., 2021, "Frenet Oscillations and Frenet-Euler Angles: Curvature Singularity and Motion-Trajectory Analysis," Nonlinear Dyn., **106**(1), pp. 1–19.
[19] Shabana, A. A., and Ling, H., 2019, "Noncommutativity of Finite Rotations and Definitions of Curvature and Torsion," ASME J. Comput. Nonlinear Dyn., **14**(9), p. 091005.
[20] Carroll, D., Kose, E., and Sterling, I., 2013, "Mathematical Foundation of Railroad Vehicle Systems: Geometry and Mechanics," J. Math. Res., **5**, pp. 97–106.
[21] von Dombrowski, S., 2002, "Analysis of Large Flexible Body Deformation in Multibody Systems Using Absolute Coordinates," Multibody System Dynamics, **8**, pp. 409–432.
[22] Spivak, M., 1999, *A Comprehensive Introduction to Differential Geometry*, Vol. 1, 3rd ed., Publish or Perish, Inc., Houston, TX.
[23] Lan, P., and Shabana, A. A., 2010, "Integration of B-Spline Geometry and ANCF Finite Element Analysis," Nonlinear Dyn., **61**(1), pp. 193–206.
[24] Shabana, A. A., 2022, "Space-Curve Cartan Matrix and Exact Differentiability of the Curvature and Torsion," Mechanics-Based Design of Structures and Machines.