Fermi-Bose Similarity

Minoru OMOTE, Yoshio OHNUKI* and Susumu KAMEFUCHI**

Institute of Physics, University of Tsukuba, Ibaraki 300-31
*Department of Physics, Nagoya University, Nagoya 464
**Department of Physics, Tokyo University of Education, Tokyo 112

(Received June 26, 1976)

The usual theory of the Lie algebras and the Lie groups is shown to be formally extended to the cases in which the group parameters commute and/or anticommute in the most general manner. It is then proved that the parafermi and parabose algebras for \( f \) degrees of freedom correspond to the Lie algebras of \( SO(2f+1) \) and of a graded version of \( Sp(2f, R) \), respectively. Further the trilinear and bilinear commutation relations for a general system comprising parafermi and parabose fields are shown to coincide with the Lie commutation relations of a certain group in the above-mentioned sense. Throughout our argumentation parafermi and parabose fields can formally be treated in an analogous manner. It is thus concluded that the fermi-bose similarity that is known to hold in the ordinary field theory persists also in parafield theory.

§ 1. Introduction

Although fermi and bose fields possess quite different properties in regard especially to the spectra of the number operators \( N_k \) (with \( k \) to denote modes), the underlying formalisms for the respective fields manifest a striking similarity in various other respects. For example, the bilinear commutation relations for annihilation operators \( a_k \) and creation operators \( a_k^\dagger \) differ only in the signs attached to the commutator brackets: \( [a_k, a_k^\dagger] = \delta_{kl} \) and \( [a_k, a_k^\dagger] = 0 \). The so-called fermi-bose symmetry or supersymmetry\(^3\) which has recently been calling attention of theoretical physicists also exhibits another facet of the close link between fermi and bose fields. The purpose of the present paper is to pursue this kind of similarity, or fermi-bose similarity, within a wider framework of quantum field theory, i.e., parafield theory. (Throughout the present paper the word 'parafields' is to be taken to include the ordinary fermi or bose fields as well.)

In such an attempt we find it necessary to make use of a formally generalized version of the Lie algebras and the Lie groups. For a Lie algebra of the usual type the Lie commutation relations involve only commutators, and when related to a Lie group the corresponding group parameters \( x_i \) (\( i = 1, 2, \cdots \)) are just ordinary numbers, i.e., \( [x_i, x_j] = 0 \). In the following discussions, on the other hand, we have to be concerned with both commutators and anticommutators simultaneously. In order to interpret them as 'Lie commutators' it is necessitated to consider a group whose group parameters \( x_i \) have a more general commutation property such that
Fermi-Bose Similarity

\[ [x_i, x_j]_{(\omega)} = 0, \]

where the symbol \((ij) = (ji)\) denotes a signature \(\pm\) that depends on a pair of indices \(i\) and \(j\). Recently some authors have studied, under the name of graded Lie algebras and graded Lie groups, the cases of \(Z\)-grading and in particular of \(Z_2\)-grading. The latter corresponds in our notations to \((ij) = (-)^{g'_i g'_j}\) where \(g'_i\) are some integers. In other words \(x_i\)'s are split into two sets \(I\) and \(II\) such that \((ii') = (ij) = +\) and \((jj') = -\) for \(i, i' \in I\) and \(j, j' \in II\), or \(x_i\)'s are just ordinary parameters and \(x_j\)'s form a Grassmann algebra. To our knowledge, however, the general cases with arbitrary signatures \((ij)\) have not been studied in the literature (of physics, at least). In view of this we shall discuss such general cases at some length with the result that a formal generalization of the routine of usual Lie algebras and Lie groups is in fact possible. Since, however, this is not the main purpose of the present paper, the discussions are all relegated to the Appendix.

In addition to the set of operators \(a_k\) and \(a_k^\dagger\) for \(f \) modes of a parafermi or parabose field \((k = 1, 2, \ldots, f)\), let us consider another set of operators quadratic in \(a_k\)'s and \(a_k^\dagger\)'s, i.e., \(N_{kl}, L_{kl}\) and \(M_{kl}\) such as defined by \((2\cdot10)\) below. Then, as shown in a previous paper,\(^9\) the set of the latter operators form the Lie algebras of \(SO(2f)\) and of \(Sp(2f, R)\) for the parafermi and parabose cases, respectively. Further it has been shown by Ryan and Sudarshan\(^6\) that in the parafermi case the entire set of operators, i.e., \(a_k, a_k^\dagger, N_{kl}, L_{kl}\) and \(M_{kl}\), form the Lie algebra of \(SO(2f+1)\). Now, by invoking graded Lie algebras we shall show in §2 that the corresponding set of operators for the parabose case form, in fact, the Lie algebra of a graded version of \(Sp(2f, R)\). In this way the fermi-bose similarity is seen to persist in this respect also.

In §3 we proceed to consider a more general system which comprises a number of parafermi and parabose fields. The corresponding operators are then shown to form the Lie algebra of a generalized group in the above mentioned sense, where the group parameters have the most general commutation properties. In this case both the trilinear commutation relations for individual fields and the same relations between different fields are combined into a single framework of the Lie commutation relations of the group. Incidentally it is known that such a set of parafields must have the same (parastatistics) order. Here parafermi and parabose fields are placed on an equal footing, so that the difference between the two types of fields completely disappears.

In §4 we discuss the most general case of a system of parafields where all fields are classified into families such that fields are mutually transformed only within each family. In a way similar to the preceding sections it can be shown that parafields belonging to the same family satisfy trilinear commutation relations, whereas those belonging to different families satisfy bilinear commutation or anti-commutation relations. This enables us to define families of parafields in an alternative manner by referring to the types of commutation relations. The Appendix is devoted to a formal generalization of the Lie algebras and the Lie groups in
the sense already mentioned above.

§ 2. Lie-algebraic structure of parafield operators

To begin with, let us investigate the Lie-algebraic properties of annihilation and creation operators for a single parafermi or paraboze field. We denote these operators again by \(a_k\) and \(a_k^\dagger\) and restrict ourselves to \(k\)'s such that \(k = 1, 2, \ldots, f < \infty\). As first shown by Takahashi and one of the present authors (S.K.),\(^8\) the trilinear commutation relations to be satisfied by \(a_k\) and \(a_k^\dagger\) can be derived group-theoretically in the following manner. We assume that \(a_k\) and \(a_k^\dagger\) are such operators as to undergo the infinitesimal transformations:

\[
a_k \rightarrow a_k' = a_k - i \sum_{m=1}^f (\xi_{km} a_m + \gamma_{km} a_m^\dagger),
\]

\[
a_k \rightarrow a_k'' = a_k^\dagger + i \sum_{m=1}^f (\xi_{mk} a_m^\dagger - \gamma_{mk} a_m),
\]

(2·1)

where the infinitesimal parameters \(\xi_{km}\), \(\gamma_{km}\) and \(\zeta_{km}\) are taken to satisfy

\[
\xi_{km} = \xi_{mk}, \quad \gamma_{km} = \gamma_{mk}, \quad \gamma_{km} \pm \gamma_{mk} = 0.
\]

(2·2)

It is easy to see that the transformations (2·1) leave invariant the quantities \(A_{\pm}\) defined by

\[
A_{\pm} = \sum_k (a_k^\dagger a_k \pm a_k a_k^\dagger)
\]

(2·3)

and therefore correspond to the groups \(SO(2f)\) and \(Sp(2f, R)\) for the cases of upper and lower signs, respectively. It is further assumed that any of the above transformations can be generated by a unitary operator \(U\):

\[
a_k' = U^{-1} a_k U, \quad a_k'' = U^{-1} a_k^\dagger U,
\]

(2·4)

where \(U\) can be written as

\[
U = 1 - i \sum_{l,m} (\xi_{lm} N_{lm} + \frac{1}{2} \gamma_{lm} L_{lm} + \frac{1}{2} \zeta_{lm} M_{lm}),
\]

(2·5)

with the generators \(N_{lm}, L_{lm}\) and \(M_{lm}\) satisfying the relations:

\[
N_{lm}^\dagger = N_{ml}, \quad L_{lm} = M_{ml}, \quad L_{lm} \pm L_{ml} = M_{lm} \pm M_{ml} = 0.
\]

(2·6)

From (2·4) and (2·1) it then follows that

\[
[a_k, N_{lm}] = \delta_{kl} a_m,
\]

\[
[a_k, L_{lm}] = \delta_{ml} a_m^\dagger \mp \delta_{km} a_l^\dagger,
\]

\[
[a_k, M_{lm}] = 0.
\]

(2·7)

Requiring the integrability condition that the difference between two successive transforms \(\delta_a\delta_b\) and those in the reversed order \(\delta_b\delta_a\) shall equal a third related transformation \(\delta_{[a,b]}\), i.e., \(\delta_a \delta_b - \delta_b \delta_a = \delta_{[a,b]}\), we obtain the following Lie commutation...
relations:

\[
\begin{align*}
[N_{kl}, N_{mn}] &= \delta_{lm}N_{kn} - \delta_{kn}N_{ml}, \\
[L_{kl}, L_{mn}] &= [M_{kl}, M_{mn}] = 0, \\
[L_{kl}, N_{mn}] &= -\delta_{kn}L_{ml} \pm \delta_{lm}L_{mk}, \\
[M_{kl}, N_{mn}] &= \delta_{km}M_{nl} \mp \delta_{ln}M_{nk}, \\
[L_{kl}, M_{mn}] &= -\delta_{km}N_{ln} \pm \delta_{kn}N_{lm} - \delta_{lm}N_{km} \pm \delta_{lm}N_{kn}.
\end{align*}
\]

(2.8)

The quadratic Casimir operators are found to be

\[
C = \sum_{k,l} (\pm 2N_{kl}N_{lk} + L_{kl}M_{lk} + M_{kl}L_{lk}),
\]

(2.9)

which agrees with (A·31) apart from an overall factor. As regards explicit forms for the generators, it is consistent to introduce the following Ansatz:

\[
\begin{align*}
N_{kl} &= \frac{1}{2} [a^\dagger_k, a_l], \\
L_{kl} &= \frac{1}{2} [a^\dagger_k, a_l^\dagger], \\
M_{kl} &= \frac{1}{2} [a_k, a_l^\dagger] .
\end{align*}
\]

(2.10)

Substituting (2.10) in (2.7), we obtain the trilinear commutation relations of the usual form

\[
\begin{align*}
[a_k, [a^\dagger_l, a_m^\dagger]] &= 2\delta_{kl}a_m, \\
[a_k, [a^\dagger_l, a_m]] &= 2\delta_{kl}a_m^\dagger \mp 2\delta_{km}a_l^\dagger, \\
[a_k, [a_l, a_m]] &= 0.
\end{align*}
\]

(2.11)

We thus see that the cases of upper and lower signs correspond to the parafermi and parabose commutation relations, respectively.

In the parafermi case, i.e., taking upper signs throughout the preceding relations we can make a further observation: The relations (2.10) can be reinterpreted as a kind of the Lie commutation relations, and as was firstly shown by Ryan and Sudarshan\(^a\) all the relations in (2.8), (2.7) and (2.10), when taken together, form in fact the Lie commutation relations for the group \(SO(2f+1)\). In view of the similarity between the parafermi and parabose cases which has been exhibited so far we may expect that all the relations for the parabose case could also be related to the Lie commutation relations for a group larger than \(Sp(2f, R)\). A difficulty lies, however, in the fact that the right-hand sides of (2.10), when the lower signs are chosen, become anticommutators.

In order to incorporate anticommutators into a unified framework we invoke the generalized version of the Lie algebras and the Lie groups as mentioned in § 1. In fact the case of \(Z_2\)-grading\(^b\) suffices for our present purpose. In what follows we shall show that when the operators \(a_k\) and \(a^\dagger_k\) are regarded as a kind of 'spinor' operators to play the role of grading, all the relations in (2.8), (2.7)
and (2.10) can indeed be unified in the form of the Lie commutation relations for the graded $Sp(2f, R)$. Since both the parafermi and parabose cases can be dealt with in a similar manner, we shall again describe the two cases together: As before, upper and lower signs correspond to the parafermi and parabose cases, respectively.

Instead of (2·1) let us consider infinitesimal transformations such as

$$a_k \rightarrow a_k' = a_k - i \sum_{m=1}^{f} \left( \tilde{\zeta}_{km} a_m + \eta_{km} a_m^\dagger + \bar{\theta}_m M_{km} + N_{mk} \theta_m \right),$$

$$a_k^\dagger \rightarrow a_k'^\dagger = a_k^\dagger + i \sum_{m=1}^{f} \left( \tilde{\zeta}_{mk} a_m^\dagger + \zeta_{mk} a_m \pm L_{mk} \theta_m + \bar{\theta}_m N_{km} \right),$$

$$N_{kl} \rightarrow N_{kl}' = N_{kl} + i \sum_{m=1}^{f} \left( N_{ml} \tilde{\zeta}_{km} - \tilde{\zeta}_{lm} N_{km} - \eta_{lm} L_{km} + M_{lm} \zeta_{km} \right) + \frac{i}{2} (\bar{\theta}_k a_l - a_k^\dagger \theta_l),$$

$$L_{kl} \rightarrow L_{kl}' = L_{kl} + i \sum_{m=1}^{f} \left( L_{ml} \tilde{\zeta}_{km} + \tilde{\zeta}_{ml} L_{km} + \zeta_{km} N_{lm} + N_{km} \zeta_{ml} \right) + \frac{i}{2} (\bar{\theta}_k a_l^\dagger - a_k^\dagger \theta_l),$$

$$M_{kl} \rightarrow M_{kl}' = M_{kl} - i \sum_{m=1}^{f} \left( M_{ml} \tilde{\zeta}_{km} + \tilde{\zeta}_{lm} M_{km} + N_{ml} \eta_{km} + \eta_{ml} N_{km} \right) - \frac{i}{2} (a_k \theta_l - \theta_k a_l).$$

Here $N_{kl}$, $L_{kl}$ and $M_{kl}$ are subject to (2·6), the ordinary group parameters $\tilde{\zeta}_{km}$, $\eta_{km}$ and $\zeta_{km}$ are subject to (2·2), and the new group parameters $\theta_k$ and $\bar{\theta}_k (= h.c. of \theta_k)$ are assumed to obey commutation relations such as

$$[\theta_k, \theta_l] = [\tilde{\zeta}_{kl}, \tilde{\zeta}_{kl}] = [\eta_{kl}, \tilde{\eta}_{kl}] = [\zeta_{kl}, \tilde{\zeta}_{kl}] = 0,$$

and similarly for $\bar{\theta}_k$'s, where $a_k$ stands for either $a_k$ or $a_k^\dagger$. Under the transformations (2·12) the following quantities remain invariant:

$$C' = C + A_\pm,$$

with $C$ and $A_\pm$ being given by (2·9) and (2·3), respectively: In fact, $C'$ turns out to be the Casimir operator for the group $SO(2f+1)$ or the graded $Sp(2f, R)$ (cf. (A·31)).

We now apply to the above transformations the usual or modified procedure for the Lie groups. The only new feature that arises here is that owing to the anticommutativity of $\theta_l$'s and $\bar{\theta}_l$'s for the case of lower signs some of the commutators are changed into anticommutators when such parameters are taken out of commutator brackets. We note first of all that any transformation (2·12) can
Fermi-Bose Similarity

be generated by a unitary operator $U$ in such a way as

$$a'_k = U^{-1}a_k U, \quad a''_k = U^{-1}a_k^* U, \quad N_k' = U^{-1}N_k U,$$

$$L_k' = U^{-1}L_k U, \quad M_k' = U^{-1}M_k U$$

(2.15)

with $U$ given by

$$U = 1 - i \sum_{l,m} \left( \xi_{lm} N_{lm} + \frac{1}{2} \gamma_{lm} L_{lm} + \frac{1}{2} \eta_{lm} M_{lm} \right) + \frac{i}{2} \sum_l \left( \bar{\theta}_l a_l + a_l^* \theta_l \right),$$

(2.16)

provided that the relations (2.7), (2.8) and (2.10) hold true. Further, a straightforward calculation shows that the Lie commutation relations obtained from the integrability condition $\delta_0 \delta_0 - \delta \delta_3 = \delta_{[2,1]}$ consist precisely of (2.7), (2.8), (2.10) and their hermitian conjugates. We thus come to the conclusion that the set of parafermi (parabose) operators $a_k, a_k^*, N_k, L_k$ and $M_k$ form the Lie algebra of $SO(2f+1)$ (the graded $Sp(2f,R)$).

§ 3. A system of parafields transforming to each other

In the present section we shall show that the interrelations of parafermi and parabose fields which satisfy trilinear commutation relations among themselves can also be exploited by the method of generalized Lie algebras and Lie groups. Let us consider a system of parafields $\alpha, \beta, \cdots$, and distinguish the corresponding operators by attaching superscripts $\alpha, \beta, \cdots$. We also define signature $(\alpha\beta) = (\beta\alpha) = \pm$ that depend on a pair of fields $\alpha$ and $\beta$, and in particular, put $(\alpha\alpha) = -1 (+1)$ for the case $\alpha = a$ parafermi (parabose) field. This time we start with the transformations

$$a_k^\alpha \rightarrow a_k'^{\alpha'} = a_k^\alpha - i \sum_{m,\beta} \left( \xi_{km}^\alpha a_m^\beta + \gamma_{km}^\alpha a_m^{\beta*} \right),$$

$$a_k^{\alpha*} \rightarrow a_k'^{\alpha*'} = a_k^{\alpha*} + i \sum_{m,\beta} \left( \alpha_m^\beta \gamma_{km}^\alpha a_m^{\beta*} + \alpha_m^{\beta*} \gamma_{km}^\alpha a_m^\beta \right),$$

(3.1)

where

$$\xi_{km}^\alpha = \xi_{mk}^\alpha, \quad \gamma_{km}^\alpha = \gamma_{mk}^\alpha.$$

(3.2)

Under the transformations (3.1) the expression $A$ defined by

$$A = \sum_{k,\alpha} (a_k^{\alpha*} a_k^\alpha - (\alpha\alpha) a_k^\alpha a_k^{\alpha*})$$

(3.3)

remains invariant, provided that

$$(\alpha\beta) (\beta\alpha) a_k^\alpha a_m^{\beta*} z_{km}^{\alpha \beta} = \bar{z}_{mk}^{\alpha \beta} a_k^{\alpha*} a_m^\beta,$$

$$a_k^{\alpha*} \gamma_{km}^\alpha a_k^\alpha = (\beta\beta) \gamma_{mk}^\alpha a_k^{\beta*} a_k^\alpha.$$

(3.4)

We now specify the commutation properties of the group parameters $\xi_{km}^{\alpha \beta}, \gamma_{km}^{\alpha \beta}$ and $z_{km}^{\alpha \beta}$ in the following way. Let $A^{\alpha \beta}, B^{\alpha \beta}, \cdots$ be any such parameters with superscript $\alpha \beta$, and let $\tilde{a}^\alpha$ be either $a_k^\alpha$ or $a_k^{\alpha*}$ with $k$ being arbitrary. For these
quantities we assume that

\[ \hat{a}^\alpha A^{\beta r} = (\alpha \beta) (\alpha \gamma) A^{\beta r} \hat{a}^\alpha, \]

\[ A^{\alpha \beta} B^{\gamma} = (\alpha \gamma) (\beta \delta) (\beta \gamma) B^{\gamma} A^{\alpha \beta}, \]

and that the signatures \((\alpha \beta)\) are freely commuted with any other quantities. Under this assumption the first relation in (3.4) is automatically satisfied, whereas the second relation therein requires

\[ \eta_{km}^{\beta \gamma} = (\alpha \alpha) (\beta \beta) \eta_{nm}^{\beta \alpha}, \]

which determines the symmetry property of \(\eta\)'s with respect to suffixes. For the case when the system concerned is made up of a single parafield, all the above relations are reduced to those given by (2.1), (2.2) and (2.3).

As in the foregoing cases we assume that any transformation (3.1) can be generated, in the same way as (2.4), by a unitary operator \(U\) which we now write as

\[ U = 1 - i \sum_{\alpha, \beta} \left( \hat{a}_{\alpha} A^{\beta r} \hat{a}_{\alpha}^\dagger + \frac{1}{2} \eta_{\alpha}^{\beta} \hat{a}_{\alpha} A^{\beta r} \hat{a}_{\alpha}^\dagger + \frac{1}{2} M_{\alpha}^{\beta r} \hat{a}_{\alpha} A^{\beta r} \hat{a}_{\alpha}^\dagger \right), \]

where

\[ N_{\alpha}^{\beta r} = (\alpha \alpha) (\beta \beta) N_{\alpha}^{\beta r}, \quad L_{\alpha}^{\beta r} = M_{\alpha}^{\beta r}, \]

\[ L_{\alpha}^{\beta r} = (\alpha \alpha) (\beta \beta) \hat{a}_{\alpha} \hat{a}_{\alpha} A^{\beta r} \]

As for the commutation properties of the generators \(N_{\alpha}^{\beta r}, L_{\alpha}^{\beta r}\) and \(M_{\alpha}^{\beta r}\) with the group parameters we assume the same as the second relation in (3.5) but with \(B^{\beta r}\) now representing any of the generators.

From (2.4), (3.7) and (3.1) we obtain

\[ [a_{\alpha}^\alpha, N_{\alpha}^{\beta r}] = (\alpha \beta) (\alpha \gamma) \delta_{\alpha} \hat{a}_{\alpha} A^{\beta r}, \]

\[ [a_{\alpha}^\alpha, L_{\alpha}^{\beta r}] = (\alpha \beta) (\alpha \gamma) \delta_{\alpha} \hat{a}_{\alpha} A^{\beta r} + (\beta \beta) \delta_{\alpha} \hat{a}_{\alpha} A^{\beta r}, \]

\[ [a_{\alpha}^\alpha, M_{\alpha}^{\beta r}] = 0, \]

where we have put \(\varepsilon = - (\alpha \beta) (\alpha \gamma)\). The Lie commutation relations derived from the integrability condition \(\delta_{\alpha} \hat{a}_{\alpha} \delta_{\beta} = \delta_{\beta} \hat{a}_{\alpha} \delta_{\alpha} \) then read:

\[ N_{\alpha}^{\beta r}, N_{\alpha}^{\beta r} \] = (\alpha \gamma) (\alpha \lambda) (\gamma \lambda) \delta_{\alpha} \hat{a}_{\alpha} A^{\beta r}, \]

\[ L_{\alpha}^{\beta r}, L_{\alpha}^{\beta r}, M_{\alpha}^{\beta r} \] = 0,

\[ L_{\alpha}^{\beta r}, N_{\alpha}^{\beta r} \] = \delta_{\alpha} \hat{a}_{\alpha} A^{\beta r}, \]

\[ M_{\alpha}^{\beta r}, N_{\alpha}^{\beta r} \] = (\alpha \alpha) (\beta \beta) \delta_{\alpha} \hat{a}_{\alpha} A^{\beta r}, \]

\[ L_{\alpha}^{\beta r}, M_{\alpha}^{\beta r} \] = - (\alpha \alpha) (\beta \beta) (\alpha \lambda) (\gamma \lambda) \delta_{\alpha} \hat{a}_{\alpha} A^{\beta r} - (\alpha \alpha) (\beta \beta) (\beta \gamma) \delta_{\alpha} \hat{a}_{\alpha} A^{\beta r} - (\alpha \alpha) (\beta \beta) (\lambda \lambda) \delta_{\alpha} \hat{a}_{\alpha} A^{\beta r} \]

(3.10)
where we have put \( \varepsilon' = -(\alpha\gamma)(\alpha\lambda)(\beta\gamma)(\beta\lambda) \). It is of interest to note that by combining (3·10) with the expressions (2·10), i.e., \( N_{ki}^{\alpha\beta} = 1/2[a_k, a_i]^{\alpha\beta}_{(\alpha\beta)} \), \( L_{ki}^{\alpha\beta} = 1/2[a_k, a_i]^{\alpha\beta}_{(\alpha\beta)} \) and \( M_{ki}^{\alpha\beta} = 1/2[a_k, a_i]^{\alpha\beta}_{(\alpha\beta)} \) we can uniquely determine the general expressions for \( N_{ki}^{\alpha\beta} \), \( L_{ki}^{\alpha\beta} \) and \( M_{ki}^{\alpha\beta} \) with \( \alpha \neq \beta \). For example, in the case \( \alpha = \beta = \gamma = \lambda = \) the first relation in (3·10) reads \( [N_{ki}^{\alpha\beta}, N_{km}^{\alpha\gamma}] = \delta_{im}N_{kl}^{\alpha\delta} \). When the above expression for \( N_{ki}^{\alpha\beta} \) is substituted in the left-hand side and use is made of the first relation in (3·9), \( N_{ki}^{\alpha\beta} \) is obtained. In this way we are led to the following results:

\[
N_{ki}^{\alpha\beta} = \frac{1}{2}(\alpha\alpha)(\beta\beta)[a_k, a_i]^{\alpha\beta}_{(\alpha\beta)},
L_{ki}^{\alpha\beta} = \frac{1}{2}(\alpha\alpha)(\beta\beta)[a_k, a_i]^{\alpha\beta}_{(\alpha\beta)},
M_{ki}^{\alpha\beta} = \frac{1}{2}(\alpha\beta)(\beta\beta)[a_k, a_i]^{\alpha\beta}_{(\alpha\beta)}.
\]  

(3·11)

The quadratic Casimir operator for the present case is given by

\[
C = \sum_{k, \gamma} (-2(\alpha\alpha)N_{ki}^{\alpha\gamma}N_{kj}^{\gamma\alpha} + L_{ki}^{\alpha\gamma}M_{kj}^{\gamma\alpha} + M_{ki}^{\alpha\gamma}L_{kj}^{\gamma\alpha}),
\]  

(3·12)

which is a generalization of (2·9) (cf. (A·31)). By substituting (3·11) in (3·9) we obtain the trilinear commutation relations in the most general form:

\[
[a_k^\gamma, [a_i^\beta, a_m^\gamma]_{(\beta\gamma)} - (\alpha\beta)(\alpha\gamma)] = 2\delta_{ki}\delta_{\beta\alpha}a_m^\gamma,
\]

\[
[a_k^\gamma, [a_i^\beta, a_m^\gamma]_{(\beta\gamma)} - (\alpha\beta)(\alpha\gamma)] = 2\delta_{ki}\delta_{\beta\alpha}a_m^\gamma + 2(\beta\gamma)\delta_{km}\delta_{\alpha\beta}a_i^\gamma,
\]

\[
[a_k^\gamma, [a_i^\beta, a_m^\gamma]_{(\beta\gamma)} - (\alpha\beta)(\alpha\gamma)] = 0,
\]  

(3·13)

which agree with the results we have obtained elsewhere by a different method.\(^9\)

It is worth noticing that the trilinear commutation relations involving three parafields \( \alpha, \beta \) and \( \gamma \) are determined by the three signature factors \( (\alpha\beta), (\beta\gamma) \) and \( (\gamma\alpha) \), each depending on a pair of parafields.

It has been proved previously\(^9\) that those parafields which obey trilinear commutation relations of the type (3·13) must have the same order. That is to say, those parafield operators that are transformed to one another according to (3·1) must have the same order, irrespective of whether they are parafermi or parabose fields. In a way similar to §2 the group of transformations (3·1) can be enlarged to a still larger group by introducing further parameters \( \theta_{k}^{\alpha} \) such that the entire set of operators \( a_k^\gamma, a_k^{\alpha\gamma}, N_{ki}^{\alpha\beta}, L_{ki}^{\alpha\beta} \) and \( M_{ki}^{\alpha\beta} \) form the Lie algebra of the latter group. The most general case of this kind will be discussed in the following section.

\[\text{§ 4. Families of parafields}\]

In the present section we consider a general system of parafields which is divided into families such that fields belonging to each family are mutually trans-
formed only within the family. Denoting the annihilation or creation operators for the parafield \( \alpha \) of the \( i \)-th family by \( a_{\alpha i} \), we thus write their transformations as

\[
a_{\alpha i} \rightarrow a_{\alpha i}' = a_{\alpha i} - \sum_{m, p \in i} \left( \eta_{km} a_m a_{\beta k} + \eta_{km} a_m a_{\beta k}' - (\beta_{\alpha i}) \bar{\eta}_{m} a_{\beta k} M_{km} \right) + (\beta_{\alpha i}) N_{mk} a_{\beta k},
\]

\[
a_{\alpha i} \rightarrow a_{\alpha i}' = a_{\alpha i} + \sum_{m, p \in i} \left( a_m a_{\beta k} a_{\alpha i} + a_m a_{\beta k}' a_{\alpha i}' \right) + (\beta_{\alpha i}) \bar{\eta}_{m} a_{\beta k} M_{km},
\]

\[
N_{k_{1}l_{1}} \rightarrow N_{k_{1}l_{2}} = N_{k_{1}l_{1}} + \sum_{m, l \in i} \left( (\alpha_{i} a_{\alpha i}) (\beta_{j} a_{\beta j}) N^{j_{m}l_{m}}_{k_{2}k_{1}} - \eta_{i_{m}} l_{m} N_{k_{2}k_{1}} \right) - \eta_{i_{m}} l_{m} q_{i_{m}} N_{k_{2}k_{1}} + (\alpha_{i} a_{\alpha i}) (\beta_{j} a_{\beta j}) L_{k_{2}l_{2}} + \frac{i}{2} (\alpha_{i} a_{\alpha i}) (\beta_{j} a_{\beta j}) (\bar{\eta}_{i} a_{\beta j} - a_{\alpha i} a_{\beta j} \bar{l}_{i}),
\]

\[
M_{k_{1}l_{1}} \rightarrow M_{k_{2}l_{2}} = M_{k_{1}l_{1}} + \sum_{m, l \in i} \left( (\alpha_{i} a_{\alpha i}) (\beta_{j} a_{\beta j}) M^{j_{m}l_{m}}_{k_{2}k_{1}} - \eta_{i_{m}} l_{m} M_{k_{2}k_{1}} \right) + \frac{i}{2} (\alpha_{i} a_{\alpha i}) (\beta_{j} a_{\beta j}) (\bar{\eta}_{i} a_{\beta j} - a_{\alpha i} a_{\beta j} \bar{l}_{i}),
\]

for all \( i \). Here we postulate as before that

\[
\bar{\eta}_{i} a_{\beta j} = \bar{\eta}_{i} a_{\beta j}, \quad \bar{\eta}_{i} a_{\beta j} = \bar{\eta}_{i} a_{\beta j}, \quad (\alpha_{i} a_{\alpha i}) (\beta_{j} a_{\beta j}) = (\alpha_{i} a_{\alpha i}) (\beta_{j} a_{\beta j}), \quad N_{k_{1}l_{1}} = N_{k_{1}l_{1}}, \quad L_{k_{1}l_{1}} = L_{k_{1}l_{1}}.
\]

Further the commutation relations for \( \xi_{k_{1}l_{1}}, \eta_{k_{1}l_{1}} \) and \( \xi_{k_{1}l_{1}} \) are taken to be the same as (3·5) but with signatures \( (\alpha_{i}, \beta_{j}) \), etc., and those for \( \theta_{k_{1}l_{1}} \) are now assumed to be

\[
[\theta_{k_{1}l_{1}}, \theta_{k_{2}l_{2}}]_{(\alpha_{i}, \beta_{j})} = [\theta_{k_{1}l_{1}}, \bar{\theta}_{k_{2}l_{2}}]_{(\alpha_{i}, \beta_{j})} = [\theta_{k_{1}l_{1}}, \bar{\theta}_{k_{2}l_{2}}]_{(\alpha_{i}, \beta_{j})} = [\theta_{k_{1}l_{1}}, \bar{\theta}_{k_{2}l_{2}}]_{(\alpha_{i}, \beta_{j})} = \cdots = 0,
\]

where \( A^{\beta_{j}}_{\alpha_{i}} \) stands for \( N_{k_{1}l_{1}}, L_{k_{1}l_{1}}, M_{k_{1}l_{1}}, \xi_{k_{1}l_{1}}, \eta_{k_{1}l_{1}} \) and \( \zeta_{k_{1}l_{1}} \).

The above transformations (4·1) leave invariant each expression in the square brackets of

\[
C' = \sum_{i} \sum_{\alpha_{i}, \beta_{j}} \left( -2 (\alpha_{i} a_{\alpha i}) N_{i_{k_{1}}l_{1}} a_{\beta k_{1}} M_{i_{k_{2}}l_{2}} + 2 M_{i_{k_{1}}l_{1}} M_{i_{k_{2}}l_{2}} + M_{i_{k_{1}}l_{1}} M_{i_{k_{2}}l_{2}} a_{\beta k_{1}} \right)
\]
Fermi-Bose Similarity

$\sum_{k,a} (a_k^a \bar{a}_k^a - (\alpha, \alpha) a_k^a \bar{a}_k^a)$, \hspace{1cm} (4.4)

and form a group $G$ such that $G = G_1 \otimes G_2 \otimes \cdots \otimes G_i \otimes \cdots$, where $G_i$ consists of the transformations (4.1) for the $i$-th family. Again we assume that any transformation (4.1) can be generated by a unitary operator $U$ in the same way as (2.15) with $U$ now given by

$$U = 1 - i \sum_{a_i, \beta, i} \left( \frac{\gamma_{\alpha \beta i} N_{\alpha \beta i} + \frac{1}{2} \gamma_{\alpha \beta i} L_{\alpha \beta i} + \frac{1}{2} M_{\alpha \beta i} \nu_{\alpha \beta i}}{2} \right) + i \sum_{a_i, \alpha, i} (\bar{\theta}_i^{\bar{a}} a_i^{a} + a_i^{a} \bar{\theta}_i^{\bar{a}}). \hspace{1cm} (4.5)$$

This assumption is justified by the results that follow therefrom.

All necessary commutation relations for the operators $a_k^a$, $a_k^{a_i}$, $N_{\alpha \beta i}$, $L_{\alpha \beta i}$ and $M_{\alpha \beta i}$ with $i=1, 2, \cdots$ are now included in the Lie commutation relations of $G$ and can be derived in the same way as in the preceding sections. Among the relations thus obtained those involving operators only of a single family are precisely the same as (3.9), (3.10) and (3.11). As for the relations involving operators of different families we have, in particular,

$$\frac{1}{2} (\alpha_i \alpha \alpha) \left[ a_k^{a_i}, a_i^{a_j} \right]_{\alpha \beta \gamma} = \delta_{ij} N_{\alpha \beta i},$$

$$\frac{1}{2} (\alpha_i \alpha \beta) \left[ a_k^{a_i}, a_i^{a_j} \right]_{\alpha \beta \gamma} = \delta_{ij} L_{\alpha \beta i},$$

$$\frac{1}{2} (\beta_i \beta \gamma) \left[ a_k^{a_i}, a_i^{a_j} \right]_{\alpha \beta \gamma} = \delta_{ij} M_{\alpha \beta i}. \hspace{1cm} (4.6)$$

This implies that operators of different families must either commute or anticommute, i.e., satisfy the bilinear commutation relations. Any other commutation relations involving operators of different families can be derived from (4.6).

The above result enables us to define families of parafields in an alternative way, i.e., by referring to the types of commutation relations: i) Within each family all parafields satisfy trilinear commutation relations (and hence have the same order $p$ irrespective of whether they are parafermi or parabose fields); ii) Any two parafields belonging to different families satisfy bilinear commutation relations. (For those families of order $p=1$ an exceptional situation occurs, however: Ordinary fermi or bose fields simultaneously satisfy both bilinear and trilinear commutation relations.) In fact this is the definition given in our previous paper. Further it has been shown there that with each family there is an associated gauge group (relating to the so-called colour degrees of freedom) which gives rise to some superselection rules governing the family members.

On the strength of the results obtained above we can thus make the following statement. What matters most concerning the difference between two arbitrary parafields is not whether they are parafermi or parabose fields, but whether they belong to the same family or different ones. Within a family parafermi and parabose fields behave so similarly to each other that no separate treatment
is needed for them. The fermi-bose similarity is complete in this sense.

Appendix

A formal generalization of Lie algebras and Lie groups

The purpose of this appendix is to study group-theoretical properties of a group $G$ whose elements $x, y, \cdots$ are specified as $x = (x^i, x^2, \cdots, x^r), y = (y^i, y^2, \cdots, y^r), \cdots$, where the coordinates or essential parameters $x^i, y^i, \cdots$ are assumed to possess commutation properties such as

$$[x^i, x^j]_{(ij)} = [x^i, y^j]_{(ij)} = [y^i, y^j]_{(ij)} = \cdots = 0,$$  \hspace{1cm} (A·1)

with $[x^i, x^j]_{(ij)} = x^i x^j - (ij) x^j x^i$, and $(ij) = (ji) = +$ or $-$. In particular, the unit element $e$ of $G$ is defined as $e = (0, 0, \cdots, 0)$. For example, $x^\mu s, y^\mu s, \cdots$ may be taken to be $x^i = \sum k \xi_k \sin(kx^i), y^i = \sum k \eta_k \sin(ky^i), \cdots$, where the $\xi_k, \eta_k, \cdots$ are ordinary commuting variables and the 'operators' $\xi_k, \eta_k$ satisfy $[\xi_k, \eta_l] = 0$ for all $k, l$. Any function $f$ of such variables may be assumed to be expressible as a formal series such as $f = \sum f(n_1, n_2, \cdots, n_1', n_2', \cdots) (x^1)^{n_1} (x^2)^{n_2} \cdots (y^1)^{n_1'} (y^2)^{n_2'} \cdots$, where $n_i, n_i' = 0$ or 1 for $(ii) = -$. In differentiating such functions we have, of course, to make distinction between left and right differentiations. In what follows, however, we shall exclusively be concerned with the former for the sake of convenience. Further, integration is defined as the inverse operation of left differentiation. Integration constants, in general, must be so fixed as to be consistent with the commutation property of integrated quantities.

Let us begin by writing the product of two elements $x$ and $y$ as

$$xy = f(x, y).$$  \hspace{1cm} (A·2)

From $xe = x$ and $ey = y$ we obtain

$$f(x, e) = x$$  \hspace{1cm} (A·3)

and

$$f(e, y) = y,$$  \hspace{1cm} (A·4)

respectively.

The coordinates $f^i(x, y)$ of $f(x, y)$ can formally be expanded in power series

$$f^i(x, y) = x^i + y^i + x^i y^k a^j_k + x^j y^k y^l a^i_{jk} + x^i y^k y^l y^m a^j_{kl} + \cdots,$$  \hspace{1cm} (A·5)

where summation over repeated indices is to be understood. It should be noted in (A·5) that terms such as $x^i x^k x^i, y^i y^k y^m, \cdots$ need not be included owing to (A·3) and (A·4).

In order to make $f^i$ satisfy (A·1) we require that

$$[a^j_{ik}, x^i]_{(ii)} = \cdots = 0,$$  \hspace{1cm} (A·6)
and more generally that any quantity $A_{kl}^{ij}$ has the same commutation property as $x^i x^j \cdots x^k x^l \cdots$. Further the signatures $(ij)$ are assumed to commute with any other quantities. We define the structure constants $c_{jk}^l$ by

$$c_{jk}^l = a_{jk}^l - (jk) a_{lj}^i$$

for which the following theorem holds true.

**THEOREM I**

The $c_{jk}^l$'s satisfy the relations

$$c_{jk}^l = -(jk) c_{kj}^l,$$  \hspace{1cm} (I·1)

$$(lk) c_{ij}^m c_{km}^l + (kj) c_{il}^m c_{jm}^l + (jl) c_{ei}^m c_{lm}^i = 0.$$  \hspace{1cm} (I·2)

Here and in what follows the summation convention is applied only when the same indices are repeated in factors other than signature symbols.

**Proof**

The relation (I·1) is obvious from the definition. To prove (I·2) we make use of the associative law $(xy)z = x(yz)$. Equating the coefficients of $x^i y^j z^k$ in the expansions (A·5) for both quantities, we obtain

$$(jk) (lk) (mk) a_{ji}^m a_{km}^l - a_{ki}^m a_{jm}^l = h_{ij}^l (k) h_{kl}^j - g_{ij}^l (j) g_{kl}^i.$$  \hspace{1cm} (A·7)

The combination of the above equations for all permutations of $j$, $k$ and $l$ with coefficients such that the resulting right-hand side vanishes then leads to (I·2).

**q.e.d.**

We have thus found that given a group $G$ the structure constants are uniquely determined and satisfy (I·1) and (I·2). Next let us consider the inverse problem of constructing the group for a given set of structure constants $c_{jk}^l$ which satisfy (I·1) and (I·2). To do this we introduce as usual the auxiliary function $v_j^i(x)$ in the following way. Let $x + \delta x$ be the element whose coordinates are given by $x^i + \delta x^i$. Then $p(x) = (x + \delta x) x^{-1}$, the increment of $x$ corresponding to $\delta x^i$, can be expanded as

$$p^i(x) = \delta x^i v_j^i(x) + \cdots.$$  \hspace{1cm} (A·7)

From the definition it immediately follows that

$$v_j^i(e) = \delta_j^i.$$  \hspace{1cm} (A·8)

For $v_j^i(x)$ there holds the following theorem.

**THEOREM II**

The function $f(x, y)$ with $y$ being fixed is the solution of the differential equation

$$\frac{\partial f^k}{\partial x^j} v_j^i(f) = v_j^i(x),$$  \hspace{1cm} (II·1)

corresponding to the initial condition (A·4). The integrability condition of (II·1),
Further let \( x(t) \)'s form a one-parameter subgroup of \( G \) where \( t \) is an ordinary parameter, and \( a \) be its tangential vector. Then

\[
\frac{d}{dt} a = v_j^t(x(t)).
\]

Here and in what follows we use \( \partial \) to denote left differentiation.

**Proof**

Writing \( f(x+\delta x, y) = f(x, y) \) as \( f(f) = f(x, y) \), we obtain

\[
(f + \delta f) = (x + \delta x) y y^{-1} x^{-1} = (x + \delta x) x^{-1},
\]

which leads to (II·1) when expressed in terms of coordinates via (A·9).

To prove (II·2) we introduce the inverse matrix \( u^j_i \) of \( v_j^t \)

\[
v_j^t u_i^j = \delta_i^j = u_i^j v_i^t,
\]

its existence being guaranteed by the fact that the group has \( r \) essential parameters (cf. (A·9)). It is then easy to show that the integrability condition can be rewritten as

\[
\sum_{k=1}^{r} v_k^t \frac{\partial v_j^t(f)}{\partial f^r} = \sum_{m=1}^{r} v_m^t \frac{\partial v_j^t(f)}{\partial f^r},
\]

thereby implying that either side of (A·13) must be a constant. This constant can then be determined by use of (II·1) and (A·5). Lastly (II·3) is derived by adjusting (A·9) to the case of the one-parameter subgroup. q.e.d.

The converse of the above theorem is stated as follows.

**Theorem III**

Suppose that given a set of \( c_{jk} \)'s there exists a matrix \( v_j^t(x) \) which has its inverse, such that

\[
v_j^t(e) = \delta_j^t,
\]

\[
\frac{\partial v_k^t}{\partial x^j} = (jk) \frac{\partial v_j^t}{\partial x^k} = (kl) v_j^t v_k^m c_{lm}.
\]

Then the differential equation

\[
\frac{\partial f^k}{\partial x^l} v_j^t(f) = v_j^t(x)
\]

is integrable, so that for arbitrary \( x_0 \) and \( f_0 \) there exists the solution \( f(x, f_0, x_0) \)
Fermi-Bose Similarity

with the initial value \( f = f_0 \) for \( x = x_0 \). The function \( f(x, y) \) defined by \( f(x, y, e) = f(x, e) = x \) and \( f(e, y) = y \). The elements \( x, y, \ldots \) thus form a group in the sense that the product of \( x \) and \( y \) is defined by

\[ xy = f(x, y), \quad \text{(III·4)} \]

and the unit element is given by \( e \).

**Proof**

To prove that products defined by (III·4) are consistent with the associative law we put \( u = f(x, y), \ v = f(y, z), \ w = f(u, z) \) and \( w = f(x, v) \). It suffices then to show that when \( y \) and \( z \) are fixed, \( w \) and \( w \) satisfy, as functions of \( x \), the same differential equation and the same initial condition. This can be done by use of (III·3). Further the existence of inverse elements can easily be confirmed by checking that for a given \( y \) there always exists an \( x \) such that \( f'(x, y) = 0 \). q.e.d.

We now turn to the question of how to determine the function \( v_j^i(x) \) for a given set of \( c_{jk} \)’s. In so doing we shall lean exclusively on standard coordinates of the first kind. As usual, the necessary and sufficient condition for \( x^a \)’s to be this kind of coordinates is

\[ x^i v_j^i(x) = x^i. \quad \text{(A·14)} \]

It can be seen below that such \( x^a \)’s satisfying this condition always exist. The following theorem then holds.

**Theorem IV**

Let \( w_j^i(t) \) be the function defined, in a system of standard coordinates, by

\[ w_j^i(t) = w_j^i(t, a) = tv_j^i(at), \quad \text{(IV·1)} \]

where \( a \) is a fixed vector, and \( at \) is the one whose coordinates are given by \( a^1t, a^2t, \ldots, a^nt \). Then the following relations hold true:

\[ v_j^i(x) = w_j^i(1, x), \quad \text{(IV·2)} \]
\[ w_j^i(0, a) = 0, \quad \text{(IV·3)} \]
\[ \frac{dw_j^i(t)}{dt} = \delta_j^i + (jk) a^k w_j^i(t) c_{ki}. \quad \text{(IV·4)} \]

**Proof**

Equations (IV·2) and (IV·3) are evident from the definition (IV·1). From (A·14) and (III·2), on the other hand, there follows the equation

\[ v_j^i + x^k \frac{\partial v_j^i}{\partial x^k} = \delta_j^i + (jk) x^k v_j^i c_{ki}, \quad \text{(A·15)} \]

which immediately leads to (IV·4) when \( x^k = a^k t \) is substituted therein. q.e.d.

We can also prove the converse of the above theorem as follows.
THEOREM V

For a given set of $c_{jk}^i$'s that satisfy (I·1) and (I·2) consider simultaneous differential equations such as

$$
\frac{dw^i_j(t)}{dt} = \delta^i_j + (jk)a^s w^i_j c_{kl}^i,
$$

(V·1)

where $a$ is a fixed vector. Let $w^i_j(t, a)$ be the solution of (IV·1) satisfying the initial condition

$$
w^i_j(0, a) = 0.
$$

(V·2)

Then the function $v^i_j(x)$ defined by

$$
v^i_j(x) = w^i_j(1, x)
$$

satisfies (III·2), (A·10) and (A·14).

Proof

To find $v^i_j(e)$ for the $v^i_j(x)$ defined by (V·3) we put $a^s = 0$ in (V·1), thereby obtaining the solution $w^i_j(t) = \delta^i_j t$. This result, together with (V·3), gives $v^i_j(e) = \delta^i_j$. To prove that $v^i_j(x)$ satisfies (III·2) let us introduce a function $F^i_{jk}(t)$ defined by

$$
F^i_{jk}(t) = \frac{\partial w^i_j(t, a)}{\partial a^j} - (jk) \frac{\partial w^i_j(t, a)}{\partial a^k} - (kl) w^i_j(t, a) w^m_k(t, a) c_{lm}^i.
$$

(A·16)

It is then easy to see that this function is the solution of

$$
\frac{dF^i_{jk}(t)}{dt} = (lm) F^m_{jk} a^l c_{im},
$$

(A·17)

corresponding to the initial condition $F^i_{jk}(0) = 0$. Hence $F^i_{jk}(t) = 0$, which means that (III·2) holds true. It can be shown in a similar manner that $v^i_j(x)$ satisfies (A·14). q.e.d.

Let us now derive the Lie commutation relations for generalized Lie algebras which have frequently been referred to in the text. To do this we consider a function $F(z)$ of $z = f(x, y)$. For the increment $\delta F(z)$ due to that of $x$, $\delta x$, we have

$$
\delta F(f) = \delta f^i \frac{\partial F(f)}{\partial f^i} = \delta x^i \frac{\partial f^j}{\partial x^i} \frac{\partial F}{\partial f^j} = \delta x^i v^i_j(x) u^i_j(f) \frac{\partial}{\partial f^j} F(f),
$$

(A·18)

where we have used (II·1). Introducing the operators

$$
X_i = -u^i_j(f) \frac{\partial}{\partial f^j},
$$

(A·19)

we can write

$$
\delta F(f) = -\delta x^i v^i_j(x) X_j F(f) = -\rho^i(x) X_i F(f),
$$

(A·20)
owing to (A·9). Since \( p^i(x) \) represents the increment of a group element \( x \) corresponding to \( \delta x^i \), the \( X_i \)'s play the role of generators for \( F(f) \). Now, by using the equation
\[
u^i_j(x) \frac{\partial u^i_k(x)}{\partial x^j} - (jk) u^i_k(x) \frac{\partial u^i_j(x)}{\partial x^k} = c^j_{kl} u^i_l(x),
\]
(\(A\cdot21\))
which is obtained from (II·2), we find that the \( X_i \)'s satisfy the Lie commutation relations
\[
[X_i, X_j]_{\alpha \beta} = (ij) c^k_{ij} X_k.
\]
(\(A\cdot22\))
Further, (I·1) and (I·2) provide us with
\[
[X_i, X_j]_{\alpha \beta} = -(ij) [X_i, X_j]_{\alpha \beta},
\]
(\(A\cdot23\))
\[
(i k) [X_i, [X_j, X_k]_{\alpha \beta}]_{\gamma \delta} + (ji) [X_j, [X_k, X_i]_{\alpha \beta}]_{\gamma \delta} + (kj) [X_i, [X_k, X_j]_{\alpha \beta}]_{\gamma \delta} = 0.
\]
(\(A\cdot24\))
Lastly we derive the expression for the quadratic Casimir operator \( C \) in terms of the \( X_i \)'s. To do this we define
\[
g_{kl} = (il) (kl) (ii) c^l_{ik} c^i_{jl},
\]
(\(A\cdot25\))
with the symmetry property
\[
g_{kl} = (lk) g_{lk}.
\]
(\(A\cdot26\))
We also define the inverse \( g^{-1} \) of this quantity by
\[
g_{kl} (g^{-1})^{km} = \delta^m_i = (g^{-1})^{mk} g_{ik}
\]
(\(A\cdot27\))
with the symmetry property
\[
(g^{-1})^{kl} = (ll) (lk) (kk) (g^{-1})^{1k}.
\]
(\(A\cdot28\))
The structure constants defined by
\[
c_{ik, m} = c^l_{ik} g_{ml}
\]
(\(A\cdot29\))
satisfy
\[
c_{km, i} = (mk) (mi) c_{ik, m} = (mi) (ki) c_{mi, k}.
\]
(\(A\cdot30\))
The quadratic Casimir operator \( C \) is then written as
\[
C = X_i (g^{-1})^{ik} X_k,
\]
(\(A\cdot31\))
for which we have the following relations:
\[
[X_i, C] = X_i (g^{-1})^{ik} \{ (ki) qi \} (gk) c_{iq, k} + c_{ik, q} \} (g^{-1})^{km} X_m
\]
\[
= X_i (g^{-1})^{ik} \{ - (ki) (qk) c_{iq, k} + c_{ik, q} \} (g^{-1})^{km} X_m
\]
\[
= X_i (g^{-1})^{ik} \{ - c_{ik, q} + c_{ik, q} \} (g^{-1})^{km} X_m = 0.
\]
(\(A\cdot32\))
References

   Y. Ne'eman and S. Sternberg, Rev. Mod. Phys. 47 (1975), 573.
5) Y. Ohnuki and S. Kamefuchi, to be published.
6) S. Kamefuchi and J. Strathdee, Nucl. Phys. 42 (1963), 166.