We calculate the normalization factor for the one-pion distribution function. The result shows that the multiplicity cannot increase unless the S-channel helicity conservation is violated in the inelastic process.

§ 1. Introduction

It is getting clearer and clearer that the extremely high energy phenomena (phenomena in which the Pomeron plays the dominant role) are more complicated and more fruitful than we thought several years ago. Study of inclusive reactions will certainly give us much information on the high energy interaction. It seems already very likely that the scaling law or equivalently the limiting fragmentation is realized not far above 20–30 GeV/c (incident momentum) where the total cross section becomes constant. Distribution functions of Benecke, Chow, Yang and Yen seem to have a simple exponential form although its theoretical interpretation is not yet definite. We know very little about the multiparticle distribution functions both experimentally and theoretically.

In this article the calculation of the normalization factor of the single distribution function is presented. It can of course be easily generalized to multiparticle distribution functions. Together with the assumption of scaling, this factor gives the multiplicity.

In Refs. 1) and 2) the mechanism of increasing multiplicity is suggested. In particular Ref. 2) speculates on the absence of the so-called pionization. But since the concept of pionization in this reference is rather obscure it is not clear what the issue is. On the other hand we can take the definition à la Weinberg. Here the pionization is the soft-pion production either virtual or real. We think it is still obscure since we do not know where we should cut off the momentum for the soft pion. We can say there is the pionization if we take the semi-soft pion but we can also say there is not when we consider only very soft pion. Whether there is the pionization or not in this sense, it is not inconsistent with the existence of the limiting fragmentation. All we do in this paper is to calculate $\rho_1(k=0)$ where $\rho_1$ is defined in (2). It turns out that the contribution of more than one soft (very soft) pion production is relatively small if the pomeron can flip the S-channel helicity. In this case the calculation is very simple and we need not refer to the beautiful but very complicated formalism by Weinberg.
In the inclusive reaction
\[ A + B \rightarrow C + \text{anything}, \] (1)
the distribution function or the differential cross section is defined as
\[ \frac{d\rho(K, s)}{d^3K} = \frac{1}{(2\pi)^6} \frac{1}{VTv_{AB}} \sum_n \langle n \rangle_{\text{out}} a_{\text{out}}(K) |AB\rangle_{\text{in}}^2, \] (2)
where \( K \) is the C.M. momentum of \( C \), \( VT = (2\pi)^6 \delta^4(0) \), \( v_{AB} \) is the relative velocity (C.M. system) of \( A \) and \( B \) and \( s \) is the square of the C.M. energy. \( a_{\text{out}}(K) \) is the annihilation operator of \( C \). We can easily show that
\[ \int \frac{d\rho}{d^3K} d^3K = \langle n \rangle \sigma_{\text{TOT}}, \] (3)
where \( \langle n \rangle \) is the average multiplicity and \( \sigma_{\text{TOT}} \) is the total cross section for \( A + B \). Generalization of \( \rho \) to processes like
\[ A + B \rightarrow C + D + E + \cdots + \text{anything} \] (4)
is obvious. We want to calculate \( \rho(K=0, s(\text{large})) \) with the following assumptions.
(1) We take \( \pi^+ + p \rightarrow \pi^+ + \text{anything} \) as an example. 'Anything' is assumed to contain \( \pi \)'s and \( K \)'s but no baryon anti-baryon pairs.
(2) \( K=0 \) can be smoothly continued to \( K = 0 \) where we define \( \phi_s = \epsilon \partial_s A_s \), where \( \phi \) is the pion field, \( c = g_s / M \mu g_A \) and \( A \) is the axial current (see Ref. 6).

The first assumption is experimentally justified. The second one is more than the ordinary PCAC. Because the role of C.M. system is crucial here.\(^*\)

\( \rho \) can be rewritten in the following form though we do not use this form in this paper.
\[ \frac{d\rho}{d^3K} = \frac{1}{v_{\text{in}}} \frac{1}{(2\pi)^3} \frac{1}{2K_0} \langle P_1 P_2 |_{\text{in}} \int d^4xe^{-iK_0} J_s(x) J_s^+(0) |P_1 P_2 \rangle_{\text{in}}, \] (5)
where we have specified \( A, B \) and \( C \) as proton \( (P_1) \), pion \( (P_2) \) and pion \( (K) \) respectively, \( J_s(x) = (\Box + \mu^2) \phi_s \). We use the Drell-Bjorken metric with the normalization \( \langle P_1 |P_1' \rangle = (2\pi)^3 \delta^3(P_1 - P_1') \).

§ 3. Summation of Born terms

The first step of applying the Bloch-Nordsieck type calculation\(^7\) in Q.E.D. to pion physics was taken by Lewis, Oppenheimer and Wouthuysen\(^8\) more than

\(^*\) In Lab. frame it is not clear to which point of the distribution the soft pion \( (K=0) \) corresponds.
two decades ago. Recent development in current algebra \(^6\) puts this type of theory on a firm ground, and it also clarifies the limitation of the theory: not all the pions emitted are soft.

Let us start with the expression (2) and rewrite this (see Fig. 1) as

\[
\frac{d\rho}{d^3K} = \frac{1}{(2\pi)^3} \frac{1}{VTv_{\text{in}}} \sum_i \frac{1}{2K_0} |\langle n|_{\text{out}}\rangle^2 \times \int e^{iKz} (\Box + \mu^2) \phi(x) |P, P_2\rangle_{\text{in}}|^2.
\]  

(6)

Here we suppress the isospin indices. We now define soft and hard pion in the following way:

- **soft pions** = pions with the magnitude of their momentum less than certain value.
- **hard pions** = all the other pions.

One-particle phase space for the soft pion is therefore

\[
\frac{1}{(2\pi)^3} \int_{|K| < |\tilde{K}|} \frac{d^3K}{2K_0} = A^2.
\]

We take \(A\) to be \(\simeq \mu/2\pi\) (very soft pion).

We now go back to Eq. (2) and take the soft pions out of the \(\langle n\rangle\) states. In doing this we use the following approximations.

1. We neglect the dependence of the \(\delta\) function of the energy momentum conservation on the soft pion momenta.
2. In \(\sqrt{2K_0} \alpha_{\text{out}}(K_i)\) we put \(K_i = K\) for all \(i\). Because of the second assumption we are forced to get Born terms only. The current algebra terms are proportional to the difference of soft pion momenta. We neglect the \(\sigma\) terms in this section. These terms will be considered in the next section.

Then we can easily show that

\[
\sum_i |\langle n|_{\text{out}} \alpha_{\text{out}}(K) |P, P_2\rangle_{\text{in}}|^2 = \sum_i \frac{1}{\pi^3} \int ds_1 ds_2 ds_3 \exp(-Z_i Z_i) |\langle n'|_{\text{out}} \exp(AZ_i A_{\text{out}}) \alpha_{\text{out}}(K) |P, P_2\rangle_{\text{in}}|^2.
\]  

(7)

Here \(n'\) contains no soft pions. \(A_i = \lim_{K \to 0} \sqrt{2K_0} \alpha_{\text{out}}(K)\), where \(i\) is the isospin suffix, \(ds_i = d \text{Re} Z_i \times d \text{Im} Z_i\). Making use of the PCAC equation

\[
A_{t,\text{out}} - A_{t,\text{in}} = \frac{i g_r}{M g_A} (X_{t,\text{out}} - X_{t,\text{in}})
\]

(see Ref. 6),
we can compute this matrix element:

\[
\frac{1}{2\kappa A} |\langle n'|_{\text{out}} \exp(\mathbf{A}_t \mathbf{A}_{t,\text{out}}) a_{j,\text{out}}(K) |P_1P_2\rangle_{\text{in}}|^2
\]

\[
= \frac{1}{2\kappa A} |\langle n'|_{\text{out}} \frac{\partial}{\partial Z_i} \left[ \exp\left( iA - \frac{A_r}{M\mathbf{g}_A} Z_i X_{t,\text{out}} \right) \exp\left( - iA - \frac{A_r}{M\mathbf{g}_A} Z_i X_{t,\text{in}} \right) \right] |P_1P_2\rangle_{\text{in}}|^2,
\]

(9)

where \( X_i \) is the chirality operator. We now use the assumption (1) of the previous section together with the following:

*The isospin of the nucleon is conserved in the hard pion processes.*

We have

\[
\langle n'|_{\text{out}} \exp\left( iA - \frac{A_r}{M\mathbf{g}_A} Z_i X_{t,\text{out}} \right) \exp\left( - iA - \frac{A_r}{M\mathbf{g}_A} Z_i X_{t,\text{in}} \right) |P_1P_2\rangle_{\text{in}}
\]

\[
= \langle n'|_{\text{out}} \exp\left( i\frac{A_r}{2M} (h_f - h_i) Z_i \tau_{t,N} \right) |P_1P_2\rangle_{\text{in}},
\]

(10)

where \( h_f \) and \( h_i \) are the helicity of the final and initial nucleons and \( \tau_{t,N}/2 \) is the isospin operator of the nucleon. On taking the matrix element for each nucleon state (helicity and isospin) we obtain

\[
\frac{d\sigma^\pm}{d^3K} = \frac{1}{(2\pi)^3} \frac{1}{2\kappa A} \frac{1}{\pi^2} \int d\xi ds d\bar{s} d\bar{\xi} \exp\left( -\frac{1}{2M} Z_i Z_i \right)
\]

\[
\times \left\{ \left| \frac{\partial}{\partial Z_i} \cos \frac{A_r}{M\sqrt{Z_i}} \right|^2 + \left| \frac{\partial}{\partial Z_i} \sin \left( \frac{A_r}{M\sqrt{Z_i}} \right) \right|^2 \right\} \sigma^{h.t.,}
\]

(11)

where \( \partial^z = (1/\sqrt{2}) \left( \partial/\partial Z_1 \mp i \partial/\partial Z_2 \right) \), \( Z^2 = Z_1^2 + Z_2^2 + Z_3^2 \) and \( Z \) is the isospin vector \( (Z_1, Z_2, Z_3) \). \( \sigma^{h.t.} \) corresponds to the emission of \( \pi^\pm \). \( \sigma^{h.t.} \) is the helicity flip part of \( \sigma (\pi + p \rightarrow \text{hard pions + kaons, etc.}) \). Similarly we have

\[
\sigma^{\text{soft}} = \sigma (\pi + p \rightarrow p + \text{soft pions})
\]

\[
= \frac{1}{\pi^3} \int d\xi ds d\bar{s} d\bar{\xi} \exp\left( -\frac{1}{2M} Z_i Z_i \right) \left( \left| \cos \frac{A_r}{M\sqrt{Z_i}} \right|^2 + \left| \sin \left( \frac{A_r}{M\sqrt{Z_i}} \right) \right|^2 \right) (Z^2)^{\gamma} - 1 \]

\[
\times \sigma^{h.t.}.
\]

(12)

Using formulae such as

\[
\frac{1}{\pi^3} \int d\xi ds d\bar{s} d\bar{\xi} \exp\left( -\frac{1}{2M} Z_i Z_i \right) (Z^2)^{\gamma}
\]

\[
= (2n + 1)!,
\]

(13)

we can compute (11) and (12).

The final results are

*) We do not yet know if \( \pi + p \rightarrow n + \text{anything} \) actually shows this is correct.
\[
\frac{d\rho^+}{d^3K_{\pi^0}} = \frac{1}{(2\pi)^3} \frac{1}{2K_0} \frac{1}{3M^4} \left[4 \cosh X + \frac{X^2 + 2}{X} \sinh X\right] \sigma^{h.t.}, \tag{14a}
\]
\[
\frac{d\rho^-}{d^3K_{\pi^0}} = \frac{1}{(2\pi)^3} \frac{1}{2K_0} \frac{1}{3M^4} \left[(X - \frac{1}{X}) \sinh X + \cosh X\right] \sigma^{h.t.}, \tag{14b}
\]
and
\[
\sigma^{soft} = \{(X + 1) \sinh X + (2X + 1) \cosh X - 1\} \sigma^{h.t.}. \tag{14c}
\]
Here
\[
X = \frac{\Lambda^2 g^2}{M^3} = \frac{1}{\pi} \frac{\mu^2}{M^3} \frac{g^2}{4\pi} \ll 0.1. \tag{15}
\]

This result suggests the following things:

1) The smallness of \(X\) shows that we may be able to neglect the multi-soft-pion production
\[
\frac{d\rho^+}{d^3K_{\pi^0}} \sim \frac{1}{(2\pi)^3} \frac{1}{2K_0} \frac{2g^2}{M^4} \sigma^{h.t.}, \tag{16a}
\]
\[
\frac{d\rho^-}{d^3K_{\pi^0}} \sim \frac{1}{(2\pi)^3} \frac{1}{2K_0} \frac{7g^2}{18M^2 \lambda^2} \sigma^{h.t.}. \tag{16b}
\]

2) If we use the expressions (16) we get \(d\rho^-/d\rho^+ \approx 7/36 \cdot X^2 \ll 1\). The conclusion (1) also suggests that we may be able to neglect higher orders in calculating current algebra terms including \(\sigma\) terms since \(X^2\) appears in the same manner. The result (2) is due to the fact that the single \(\pi^-\) production is impossible in the process considered here (Fig. 1) because of the nucleon isospin conservation. Since the factors \(X^2\) may also appear in the current algebra terms (no single pion production) we expect that we can totally neglect current algebra terms. In this case we get from (16)
\[
\sqrt{X^2 + \frac{4(\mu^2 + P_\perp^2)}{S}} \frac{d\sigma^+}{dP_\perp^2dX} \approx \frac{g^2}{4\pi M^4} \frac{2}{4\pi} \sigma^{h.t.}, \tag{17}
\]
and
\[
\frac{d\sigma^-}{dP_\perp^2} \bigg|_{P_\perp=0} \ll 0. \tag{18}
\]

We conclude therefore that the multiplicity cannot increase if the s-channel helicity conservation holds at high energy. We also have \(d\rho^-/d\rho^+ \approx 0,^{**)\} The present experimental data\(^{10}\) do not seem to be consistent with this, but we must wait for a decisive result.

*) If we take \(\sigma^{h.t.} = (1/2)\sigma^{TOT}\) as in the statistical model we have \(\sqrt{s^2 + 4(\rho^2 + P^2)/S}\, d\sigma^+ /dP^2 d\sigma \approx 20\text{mb}/(\text{GeV}/c)^2\). This is consistent with experiment.\(^9\)

**) This should not be taken seriously. Although our calculation shows \(d\rho^- (K=0) \approx 0\) this is when \(s \to \infty\). At finite energy we should take \(d\rho^- (K=0)\), and \(K\) actually may correspond to the semi-soft pion.
§ 4. Current algebra term and concluding remarks

As is stressed at the end of the preceding section each extra soft pion gives the factor $X^3$ because of the phase space integration. Since at least two soft pions are emitted in the current algebra terms we expect that these are small corrections. The same argument holds both for $\sigma$-terms and the virtual pion emissions. To illustrate this circumstance let us calculate the two-soft-pion-emission amplitude. We have

$$\langle n'|a_{n+1}^{(2)}(k_i) a_{n+1}^{(2)}(k)|i\rangle$$

$$= -\frac{g^2}{(Mg_A)^2 2k_0 \cdot 2k_1} \int \langle n'|d^4xd^4ye^{i k_0 (x) \cdot (x')} T(\partial x A_{\mu}^i(x) \partial y A_{\nu}^j(y)) |i\rangle.$$  \hspace{1cm} (19)

The current algebra term is

$$\lim_{\{k_{\mu} \to 0\}} \frac{g^2 g_{i\alpha\beta}}{(Mg_A)^2 2k_0 \cdot 2k_1} \int d^4xe^{i k_0 (x) \cdot (x')} \langle n'| V^f_\nu(x) |i\rangle.$$ \hspace{1cm} (20)

As the low energy theorem of Low\(^{11}\) shows that this cannot be calculated unambiguously. It depends on how we take the limit. However, if the current is emitted from the nucleon it becomes

$$\langle n'|a_{n+1}^{(2)}(k_i) a_{n+1}^{(2)}(k)|i\rangle = \frac{ig^2 g_{i\alpha\beta}}{4(Mg_A)^2 \sqrt{2k_0 \cdot 2k_1}} \langle n'| \tau^f_{\mu\nu} - \tau^f_{\nu\mu} |i\rangle,$$ \hspace{1cm} (21)

where $\tau^f$ is the nucleon isospin. To derive this we first put $k_\mu = 0$, then carry out the $d^4x$ integration. This contributes to the distribution function

$$\frac{d\rho}{d^4K} \sim \frac{X g_{i\mu\nu}}{M^2},$$ \hspace{1cm} (22)

where $X$ is the parameter defined earlier and

$$\sigma^{C.E.} = \sigma(P \to n + \text{pions and kaons}).$$

Since we have the factor $X$ in front and also $\sigma^{C.E.}$ is small,\(^*) this can be neglected.

$\sigma$-term and virtual pion emission will be also small since at least the factor $X$ appears in the amplitudes. But we cannot exclude the possibility that these contributions are proportional to the helicity non-flip cross section. We get therefore as the condition that our approximation is true:

$$\sigma^{h.f.} \sim X^3 \sigma^{h.n.},$$ \hspace{1cm} (23)

where $\sigma^{h.n.}$ is defined as the helicity non-flip part of $\sigma(P \to P + \text{pions and kaons})$.

As concluding remarks we point out the following things. Our result is very peculiar since their is a large discrepancy between the $\pi^\pm$ emissions. If the

\(^*) See the footnote on p. 1481.
experiment tells us that this is not the case we must compute the semi-soft pions making the parameter $X$ larger ($\approx 1$). We cannot neglect pion emissions from resonances, and higher orders will become important. Weinberg\textsuperscript{5} already has the method of formulating the problem.

Finally, although the multiplicity increases logarithmically in our case this does not show that the multiplicity of the soft pions goes up. In our case the soft pion cannot be emitted more than 1. Of course the situation will be different if we include semi-soft pions.

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