On the Relaxation Process of Homogeneous Condensate of Weakly Interacting Bose Systems

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In this short note, irreversible processes of the order parameter and the occupation number of the homogeneous condensate of degenerate Bose gas are investigated above $T_1$ up to second order of the pair interaction, by projecting the Liouville equation for the density matrix written in terms of coherent representation on to the zero momentum state. Equilibrium fluctuations and stability of the condensate have already been investigated by Glassgold et al. In order to obtain a stochastic equation for the probability distribution of order parameter, we define projection operator $\mathcal{P}$ as $\mathcal{P} f(x, y; t) = f(x, y; t)\langle \psi_{eq}(y; \alpha_0) | f(x, y; t) \rangle = \psi_{eq}(y; \alpha_0) f(x, y; t)$ with symbolical notations $\{\alpha_0, \alpha_0^*\}$ and $\{\alpha_k, \alpha_k^*\}_{k \neq 0} = y$. Because of shorter relaxation time of the $k=0$ states, we can regard them as a reservoir and choose for $\psi_{eq}(y; \alpha_0)$ the normalized equilibrium distribution for $k \neq 0$ states with $\{\alpha_0, \alpha_0^*\}$ given by the equilibrium values $\{\bar{\alpha}_0, \bar{\alpha}_0^*\}$; $\psi_{eq}(y; \bar{\alpha}_0) = Z^{-1} \langle \alpha_{k=0} \rangle \exp[-\beta H(\bar{\alpha}_0, \bar{\alpha}_0^*)] \times |\alpha_{k=0}\rangle$, where, $H(\bar{\alpha}_0, \bar{\alpha}_0^*)$ is obtained from initial Hamiltonian by making Bogoliubov's replacement of the operator $\{a_0, a_0^*\}$ with their equilibrium average values $\{\bar{\alpha}_0, \bar{\alpha}_0^*\}$. The formal equation for $f_1$ can be written as:

$$
L = -\sum_k \frac{\partial}{\partial \alpha_k} \left( \xi_k \alpha_k \right) + \sum_{k_q} \frac{v(q)}{V} \alpha_{k_q}^* \alpha_{k_q} \alpha_{k_q'-q} \\
- \frac{1}{2} \sum_{k,k'} \frac{\partial^2}{\partial \alpha_k \partial \alpha_{k'}} \left( \sum_q \frac{v(q)}{V} \alpha_{k_q}^* \alpha_{k_q'} \alpha_{k_q'-q} \right)
$$

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Here $f(\{\alpha^*\}; \alpha; t)$ are diagonal matrix elements of the density matrix in the basis of the coherent states $|\alpha\rangle$, and $\xi_k = k^2/2m - \mu$. Hereafter we take $V=1$ and use abbreviate notations $\xi \equiv \partial / \partial \alpha_0$ and $\xi^* \equiv \partial / \partial \alpha_0^*$. In terms of fluctuation operators for the zero momentum state. We start with the Liouville equation derived in Ref. 2) (Eq. (2.12) of Ref. 2));

$$
i \frac{\partial f}{\partial t} = L f,
$$

(1)
by choosing the initial condition \( f(x, y; 0) = \psi_{eq}(y; \bar{\alpha}_0) h(x) \) with \( h(x) \) arbitrary and \( \bar{\alpha}_0 = \frac{1}{\beta} \). Let us separate the operator \( L \) into three parts as \( L = L_x + L_y + L_{xy} \), where \( L_x \) and \( L_y \) contain only the variables of \( x \) and \( y \), respectively and \( L_{xy} \) is an inseparable part of \( L \). Above \( T_\pi \), \( \beta = 0 \) and \( L_y \) satisfies the equation \( L_y \psi_{eq}(y; 0) = 0 \). In terms of the important relations that \( L_x = 0 \) and \( L_y = 0 \) together with the identity \( \partial^2 = \partial \), we can obtain perturbationally the equation for the prob. dist. of the order parameter \( P(\alpha_0, \alpha_0^*; t) \) from (2) up to order \( \nu \):

\[
i \frac{\partial P}{\partial t} = (\mathcal{L} + \mathcal{L}_1) P(t)
\]

(3)

where \( \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_x \), \( \mathcal{L}_0 = -\xi (\partial \alpha_0 \alpha_0 - \text{c.c.}) \) and \( \mathcal{L}_x = -\nu (0) (\partial_0 \alpha_0|\alpha_0^2 + \partial_0 \alpha_0^* \alpha_0^*/2| - \text{c.c.}) \), \( \mathcal{L}_1 = \nu \partial_1 (0) \partial_0 \alpha_0 + \nu \partial_1 (\partial_0 \alpha_0^*/2 + \partial_0 \alpha_0^* \alpha_0^*/2) + \nu \partial_1 (\partial_0 \alpha_0^* + \partial_0 \alpha_0^*/2 + \text{c.c.}) \), with the coefficients \( \gamma_1(0) = \sum \nu (0) (0 + \nu (\mathcal{K}_0) + \nu (\mathcal{K}_0) + \nu (\mathcal{K}_0) + \nu \partial_1 (\partial_0 \alpha_0^*/2 + \partial_0 \alpha_0^* \alpha_0^*/2 + \text{c.c.}) \), where \( f_1 \equiv e^{\xi t}, \xi_j \equiv \xi \), and \( \mathcal{K}_0 \equiv \mathcal{K}_0 + \mathcal{K}_0 \). The notation \( \sum \nu \) denotes the sum of \( \mathcal{K}_0 \) and \( \mathcal{K}_0 \) under the restriction that \( \mathcal{K}_0, \mathcal{K}_0, \mathcal{K}_0 \equiv 0 \).

On our next step, we translate (3) in terms of the action-angle variables \( J \) and \( \phi \) (\( \alpha_0 = \sqrt{J} e^{\text{i} \phi} \)). Making Markovian approximation, we get the Fokker-Planck equation for the prob. dist. of the occupation number of the condensate \( p_0(J, t) = f(J) \phi P(\alpha_0, \alpha_0^*; t) \);

\[
i \frac{\partial p_0}{\partial t} = \left[ -\frac{\partial}{\partial J} c_1(J) \right] p_0(J, t)
\]

(4)

with \( c_1(J) = \gamma - \gamma J \) and \( c_2(J) = 2 \gamma J \), where \( \gamma = f_0 \partial \gamma_1(s) \) and \( \gamma = 2 f_0 \partial \gamma(s) \).

From (4), using the general relation \( \langle J \rangle_\pm = \langle N_0 \rangle_\pm + 1 \) and \( \langle N_0 \rangle_\pm = a_0^* a_0 \), we arrive finally at the rate equation:

\[
i \frac{\partial}{\partial t} \langle N_0 \rangle_t = -\nu w_{\text{out}} \langle N_0 \rangle_t + \nu w_{\text{in}} \langle N_0 \rangle_{t+1}.
\]

(5)

The coefficients \( \nu w_{\text{in}} \) and \( \nu w_{\text{out}} \) express transition probabilities of collision processes representing creation and annihilation of one condensate particle; \( \nu w_{\text{in}} = \gamma - 1 = \sum \nu (0) + \nu (\mathcal{K}_0) + \mathcal{K}_0) + \nu (\mathcal{K}_0) + \nu \partial_1 (\partial_0 \alpha_0^*/2 + \partial_0 \alpha_0^* \alpha_0^*/2 + \text{c.c.}) \) and \( \nu w_{\text{out}} = \gamma = \sum \nu (0) (0 + \nu (\mathcal{K}_0) + \mathcal{K}_0) + \nu \partial_1 (\partial_0 \alpha_0^*/2 + \partial_0 \alpha_0^* \alpha_0^*/2 + \text{c.c.}) \).

The elementary solution of (4), which satisfies the initial condition \( p_0(z, t=0) = \delta(z - z_0) \) with a normalized variable \( z = J/\langle \alpha_0^2 \rangle_\pm \) is given by

\[
p_0(z, t|z_0, 0) = \frac{1}{1 - e^{-rt}} \exp \left[ \frac{z + z_0}{1 - e^{rt}} \right] \times I_0 \left( \frac{2 \sqrt{zz_0} e^{-rt}}{1 - e^{rt}} \right) e^{-z},
\]

(6)

\( I_0 \) being the modified Bessel function of order 0.

The detailed derivation of the present results will be reported elsewhere together with discussion on phenomena below \( T_\pi \).

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