Appearance of Essential Singularity
in the Analytically Regularized $1/N$ Expansion

Tsuneo SUZUKI and Hisashi YAMAMOTO

Department of Physics, Kanazawa University, Kanazawa 920
*Research Institute for Theoretical Physics
Hiroshima University, Takehara, Hiroshima 725

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It is shown that analytic regularization gives rise to an essential singularity of the same kind as that of dimensional regularization when used in the $1/N$ expansion of $O(N)(\phi^3)_{d=4}$. Such a singularity appears in models having linear or quadratic divergences in the next-to-leading order of the $1/N$ expansion.

Recently Rim and Weisberger\(^1\) have made interesting analyses. Using dimensional regularization, they have calculated ultraviolet divergences appearing up to the second order of the $O(N)$ non-linear $\sigma$ and the Gross-Neveu models near two dimensions and the $O(N)(\phi^3)^2$ model near four dimensions. The divergences in the first two models are expressed in the form of simple $\epsilon(=2-d)$ poles or a polynomial of $\ln e$, whereas the last model shows an essential singularity at $\epsilon(=4-d)=0$ as the accumulation point of an infinite set of poles. Conventional dimensional renormalization does not work well in the last model. They then have suggested that the difference might arise from a dynamical property of the models, i.e., the former have asymptotic freedom but the latter does not.

However the last suggestion seems to be incorrect. Consider, for example, the $O(N)$ non-linear $\sigma$ model near three dimensions. The model, which is perturbatively unrenormalizable, is renormalizable in the $1/N$ expansion and is non-asymptotically-free.\(^3\) No singularities occur in the self-energy up to the second order. As will be shown in the subsequent papers,\(^3\) the appearance of the essential singularity is not directly related to whether a theory has asymptotic freedom or not. It occurs in models having linear or quadratic divergences in the next-to-leading order of the $1/N$ expansion. Among them are three-dimensional $O(N)$ model with scalar mesons and fermions and the Yukawa theory in four dimensions in addition to $O(N)(\phi^3)_{d=4}$. See Ref. 3 for details.

It is the aim of this note to study whether the pathological behavior observed in the $1/N$ expansion of the $(\phi^3)_{d=4}$ model is peculiar to dimensional regularization or not. It is not so difficult to see that the conventional momentum cutoff or the Pauli-Villars regulator method does not encounter any problem in the $1/N$ expansion of the model.\(^3\) It seems that such a pathological behavior is a general feature of all regularizations utilizing analytic continuation. In fact we have found that analytic regularization specified in the following shows the same pathology in the $1/N$ expansion of $O(N)(\phi^3)_{d=4}$, but not in the $O(N)$ non-linear $\sigma$ model in two dimensions.

We introduce a regularization method used here. A propagator can be expressed as

$$i\Delta_r(k) = -\frac{1}{M^2 - k^2 - i\epsilon}$$

with a small positive constant $\epsilon$. Since large $k^2$ momentum corresponds to small $\alpha$ in the integral, any suppression factor at $\alpha=0$ improves the high-momentum behavior of the propagator. Define a regularized propagator as

$$i\Delta_r(k) = i \int_0^{\infty} d\alpha f(\alpha, \gamma) e^{-i(M^2 - k^2 - i\epsilon)},$$

(2)

where $f(0, \gamma) = 0$ and $f(\alpha, 0) = 1$. Here $\gamma$ is a complex parameter with a positive real part. For simplicity, we assume

$$f(\alpha, \gamma) = C_\gamma e^{i\pi\gamma/2} \alpha^\gamma,$$

(3)

where $C_\gamma = \mu^{i\pi/2}/\Gamma(1 + \gamma)$ and $\mu$ is a dimensional constant. Then $\Delta_r(k)$ is reduced to

$$i\Delta_r(k) = \frac{\mu^{i\pi}}{(M^2 - k^2 - i\epsilon)^{1+i\gamma}},$$

(4)

which is just equal to that previously proposed by Speer.\(^4\) The action of the $O(N)(\phi^3)_{d=4}$ model is written as follows:}
\[ S = S_0 + S_1, \]
\[ S_0 = \int d^4x \left[ \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} M^2 \phi^2 + \frac{N}{2\lambda} \phi^2 \right] + \frac{N}{2} \int d^4x d^4y \chi(x) E^{-1}(x, y) \chi(y), \]
\[ S_1 = \int d^4x \left[ \frac{N}{\lambda} (M^2 - m^2) \chi - \frac{\phi^2}{\lambda} \right] - \frac{N}{2} \int d^4x d^4y \chi(x) E^{-1}(x, y) \chi(y), \]
where \( E^{-1} \) corresponds to a bubble diagram and is given here by

\[ E^{-1}(p^2) = -\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \]
\[ \times \left[ M^2 - (k + p)^2 - i\epsilon \right]^{1+\gamma/2} \left[ M^2 - k^2 - i\epsilon \right]^{1+\gamma}. \]

It is possible to renormalize the model with the following counter terms:

\[ L_c = \frac{\alpha}{2} (\partial \phi)^2 + \frac{N}{2\lambda} \]
\[ \times b(\chi^2 + 2M^2 \chi) - \frac{N}{\lambda} \alpha m^2 \chi. \]

Using (2) and (3) and performing the momentum integral first, we get

\[ E^{-1}(p^2) = C_\gamma \Gamma(2\gamma) \int_0^1 \frac{da}{a} \]
\[ \times \left[ M^2 - a(1-a) p^2 - i\epsilon \right]^{1+\gamma}. \]

This has a pole at \( \gamma = 0 \) which is cancelled by the counter term

\[ b^{(0)} = -\frac{\alpha}{6\pi^2} \frac{1}{\gamma}. \]

The renormalized inverse \( \chi \) propagator is given by

\[ ND^{-1}(p^2) = NE^{-1}(p^2) + \frac{N}{\lambda} \left( 1 - \frac{\alpha}{6\pi^2} \right). \]

The leading-order self-energy corrections for the \( \phi \) field are shown in Fig. 1 and are evaluated to be

\[ \Pi_\phi^{(0)} = D(0) \left\{ \frac{\mu^{2\gamma}}{2(4\pi)^2} \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma/2) \Gamma(1+3\gamma/2) (M^2)^{1+\gamma/2}} \right. \]
\[ \left. - \frac{1}{\lambda} \left[ (1 + b^{(0)}) M^2 - (1 + c^{(0)}) m^2 \right] \right\}. \]

We are able to explicitly evaluate \( I(q^2) \) with the aid of the parameter integral (2):
Likewise we have

\[ E^{-1}(p^2) = \frac{4^r \Gamma(2\gamma) (\mu^2 / 4M^2)^{2r}}{64\pi^2 \Gamma(1 + \gamma) \Gamma(3/2 - \gamma)} B \left( \frac{1}{2}, 1 + \gamma \right) F \left( 2\gamma, 1 + \gamma, 3/2 + \gamma, -\frac{q^2}{4M^2} \right). \]

(17)

Making use of the property of the hypergeometric function, we easily get the asymptotic forms of \( E^{-1}(p^2) \) and \( I(p^2) \) for large Euclidean momentum:

\[ E^{-1}(x \equiv \frac{D_2}{4M^2}) \to -A_1 x^{1-\gamma} + A_2 x^{-2+\gamma} + A_3 x^{-1-\gamma} \]

(19)

and

\[ I(x) \to B_1 x^{1-\gamma} + B_2 x^{-1+\gamma}, \]

(20)

where

\[ A_1 = \frac{4^r \sqrt{\pi} \Gamma(2\gamma) \Gamma(1 - \gamma) \Gamma(3/2 - \gamma)}{64\pi^2 \Gamma(1 + \gamma) \Gamma(3/2 - \gamma)} \left( \frac{\mu^2}{4M^2} \right)^{2r}, \]

(21)

\[ B_1 = \frac{i4^r \Gamma(2\gamma) B \left( \frac{1}{2}, 1 + \gamma \right) \left( \frac{\mu^2}{4M^2} \right)^{2r} - \frac{1}{4M^2}}{32\pi^2 \Gamma(1 + \gamma)} \]

(22)

Here the explicit forms of \( A_2, A_3, B_1 \) and \( B_2 \) are not written down.31 From (18) and (22), we find

\[ B_1 = \frac{2i\mu^{2r}}{(4M^2)^{1+2r}} \left[ D^{-1}(0) - \frac{1}{\lambda} (1 - \frac{\lambda}{64\pi^2 \gamma}) \right]. \]

(23)

When (20) with (23) is inserted into (15), the first term of the right-hand side of (23) cancels the most divergent term of (14). Since

\[ D^{-1}(x) \to -\frac{1}{\gamma} (\gamma A_1 x^{-1+\gamma} - D_1) \]

(24)

with \( D_1 = -\lambda^{-1} \left[ \gamma (64\pi^2)^{-1} \right] \), the most divergent term \( J \) in \( \Pi_a \) and \( \Pi_b \) is given by

\[ J = \frac{i(4M^2)^3 D(0)}{32\pi^2} \times \int_{x_0}^\infty dx \frac{\gamma D_2 x^{-\gamma} + \gamma B_2 x^{-3\gamma}}{(\gamma A_1 x^{-1+\gamma} - D_1)}, \]

(25)

where \( D_2 = 2i\mu^{2r} (4M^2)^{-1-\gamma} x^{-1-\gamma} \), and \( x_0 \) is an infrared cutoff. Note that \( \gamma D_2, \gamma B_2, \gamma A_1 \) are all finite at \( \gamma = 0 \). Changing the variable as \( x^{-\gamma} = u \) and taking a principal part prescription to avoid spurious singularity at the denominator, we can set the lower limit \( x_0^{1-\gamma} \to 0 \) for \( \gamma > 0 \). Then the integral \( J \) is evaluated explicitly with the aid of the Mellin transformation, so that we get

\[ J = \frac{-i(4M^2)^3}{64\pi D_1} \left[ D_2 \left( \frac{\gamma A_1}{D_1} \right)^{-1-2r} \right] + B_2 \left( \frac{\gamma A_1}{D_1} \right)^{-2r+1/2} \tan \frac{\pi}{2-2r}. \]

(26)

We note that \( \lim_{\gamma \to 0} (\gamma A_1 / D_1)^{-1-2r+2\gamma} = \lim_{\gamma \to 0} (\gamma A_1 / D_1)^{-3-2r+2\gamma} = \mu^2 K (2M^2)^{-1} \), where \( K = \exp(1/2 + 2 + 32\pi^2) \) with a polynomial function \( \phi(x) \) and Euler's constant \( C \). Also we see that \( D_2 + B_2 \) is finite at \( \gamma = 0 \). The point \( \gamma = 0 \) is an essential singularity as the accumulation point of the infinite sequence of pole. We encounter the same pathological behavior as that met in dimensional regularization.

Next how about a case of the two-dimensional \( O(N) \) non-linear \( \sigma \) model? Then the corresponding asymptotic forms are given by \( E^{-1}(x) \to A_1 x^{-1-\gamma} + A_2 x^{-2+\gamma} + A_3 x^{-1+\gamma} \) and \( I(x) \to B_1 x^{-1+\gamma} + B_2 x^{-2+\gamma} + B_3 x^{-3+\gamma} \) with \( B_1 = 2i\mu^{2r} (4M^2)^{-1-\gamma} x^{-1+\gamma} \times E^{-1}(0) \) and \( A_1 + A_2 \) finite. Since the
renormalized $\chi$ propagator is $E(p^2)$ itself now, we see that the most divergent terms of $\Pi_{a}^{(0)} + \Pi_{b}^{(1)}$ exactly cancel each other. We are left with the following integral $J$:

$$
J = \frac{iM^2 E(0)}{2\pi} \int_{x_0}^{\infty} dx \frac{B_2 x^{-1} + B_3 x^{-1}}{A_2 \left(1 + \frac{A_1}{A_2} x^\gamma\right)}
$$

$$
= \frac{iM^2 E(0)}{2\pi} \left[-\frac{B_2}{\gamma A_1} \ln \left(1 + \frac{A_1}{A_2} x_0^\gamma\right) - \frac{A_1 B_3}{\gamma A_2} \ln \left(1 + \frac{A_2}{A_1} x_0^{-\gamma}\right) + \frac{B_3}{\gamma A_2} x_0^{-\gamma}\right],
$$

where we have discarded contributions from infinity using an appropriate assumption for $\gamma$. Noting that $B_2$ and $B_3$ have a simple pole at $\gamma = 0$, and from (35), we see that $J$ has singularities like $1/\gamma$ and $\ln \gamma$. No essential singularity appears.

Finally we make a comment on how our results depend on the explicit form (3) of $f(a, \gamma)$. Various forms are possible even without introduction of a new parameter. Since the ultraviolet behaviors are determined essentially near $a = 0$ in (2), we may get a similar form to (3) near $a = 0$ by expanding $f(a, \gamma)$ at $a = 0$. Hence we expect that our results obtained with (3) are qualitatively correct, if $\gamma$ is a unique complex parameter to regularize integrals. We may conclude that simple regularization methods like dimensional or analytic regularizations utilizing analytic continuation of one complex parameter give rise to the pathological behavior in the nonperturbative $1/N$ expansion. Detailed discussions and extension to other models will be published elsewhere.\(^{39}\)

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2) I. Y. Arefëva, Ann. of Phys. 117 (1979), 393.