Second Quantization and Lorentz Invariance

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The possibility of constructing the relativistic invariant theory using the particle representation is investigated. It is shown that the Hamiltonian formalism is not adequate in this representation. The general structure of the invariant S-matrix is investigated, and some correspondences to the ordinary theory are obtained. An application to the Compton scattering is also made.

§ 1. Introduction

The problem of Lorentz invariance is treated in the conventional quantum field theory as follows:

I) Field variables are transformed as classical quantities.

II) Next, the procedure of the second quantization is performed in the new coordinate system. Then the commutation relations are retained to be invariant.

However, as Wigner\(^1\) has pointed out, the invariance of the commutation relations does not mean the existence of unitary operators which transform the field quantities as \(\varphi' = U\varphi U^{-1}\). In fact these operators have never been constructed in the conventional theory. Moreover, the above procedure produces the invariance of the Feynman amplitudes, but the connection between the Feynman amplitudes and the scattering amplitudes does not seem to be Lorentz invariant.

In spite of these incompletenesses, however, the present field theory has given the correct results in a rather wide range of problems. Therefore, if the present field theory does not contain any inconsistency in itself the above procedure may be considered to be satisfactory, and the problem of Lorentz invariance need not be pursued any more. But, if one considers the local field theory to be unsatisfactory and intends to find out the possibility to construct new theories, this problem must be reinvestigated as a guiding principle, and the following two problems may become important:

I) Whether it is possible to construct any theory which can describe the creation and annihilation of particles and possesses the unitary representation of Lorentz group.

II) In the theory which does not need the unitary representation of Lorentz group, how are the Lorentz transformations treated in a consistent way?

As for the latter problem, Segal\(^2\) has recently proposed an interesting attempt. This paper will be devoted to the investigation of the former problem. Then we
must be aware of the fact that there exist infinitely many inequivalent representations of the creation-annihilation operators.\(^3\) We now intend to deal with the unitary representation of Lorentz group in the ring of operators generated by the creation-annihilation operators, so we must adopt the simplest one, namely the representation possessing the vacuum state. We adopt, therefore, Fock-Cook-Friedrichs’ particle representation. In addition, we use Foldy’s canonical form for the one-particle equation to simplify the procedure of the second quantization.

In § 2 the particle representation and Foldy’s canonical form are explained briefly. In § 3 we examine the possibility of Hamiltonian formalism, and the origin of difficulties is pointed out. In § 4 the S-matrix formalism is investigated and some correspondences to the ordinary theory are shown, and an application to the Compton scattering is given in § 5. Finally in § 6 we briefly mention about the treatment of decay process.

§ 2. Particle representation and Foldy’s canonical form

We first explain the particle representation. One-particle state is represented by \(\varphi(x, s, a)\), where “s” is spin variable and “a” is a variable which discriminates between particle and antiparticle. The latter will be called the particle variable. These variables together with coordinates will be denoted by \(\tilde{\xi}\), and we write \(\int d\tilde{\xi}\) instead of \(\sum_{\tilde{\xi},a} \int dx\). It is assumed that the states of particle and antiparticle do not mix with each other, and \(\varphi(\tilde{\xi}) \in L_2(\mathbb{R})\).

The second quantized state is represented by

\[
\Phi = \{\varphi_0, \varphi_1(\tilde{\xi}_1), \varphi_2(\tilde{\xi}_1, \tilde{\xi}_2), \ldots, \varphi_n(\tilde{\xi}_1, \ldots, \tilde{\xi}_n), \ldots\}\equiv\{\varphi_a(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)\},
\]

\[||\Phi||^2 = \sum_{n=0}^{\infty} ||\varphi_n||^2 < \infty,
\]

where \(\varphi_0\) is a constant and \(\varphi_n\)’s are assumed to be symmetrized or antisymmetrized according to the statistics of particle. For any self-adjoint operator \(T\) on one-particle space, we define a self-adjoint operator \(\Omega(T)\) on the second quantized space by

\[
\Omega(T) \Phi = \{\sum_{n=1}^{\infty} T_n \varphi_n(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)\} = \{\varphi_n(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)\}, \tag{2.1}
\]

where \(T\) is to be understood to operate on \(\tilde{\xi}\), and further we mean \(T\varphi_0 = 0\). Other operators on \(\Phi\) space will be denoted like \(\tilde{T}\). For any \(\varphi(\tilde{\xi}) \in L_2(\mathbb{R})\), the creation-annihilation operators are defined by

\[
\tilde{A}^+(\varphi) \Phi = \{\sqrt{n} S_n A_n^\dagger \varphi(\tilde{\xi}) \varphi_{n-1}(\tilde{\xi}_1, \ldots, \tilde{\xi}_{n-1})\}\tag{2.2.1}
\]

\[
\tilde{A}^-(\varphi) \Phi = \{\sqrt{n+1} \tilde{\varphi}(\tilde{\xi}) \varphi_{n+1}(\tilde{\xi}_1, \ldots, \tilde{\xi}_{n+1}) d\tilde{\xi}\}, \tag{2.2.2}
\]

where \(S_n\) and \(A_n\) mean symmetrization and antisymmetrization respectively, and \(\tilde{\varphi}\) means complex conjugate of \(\varphi\). Furthermore, we define the symbolic product be-
between an operator on $L^{(2)}_f$ and the creation-annihilation operators by:

$$ T\tilde{A}^\dagger(\varphi) = \tilde{A}^\dagger(T\varphi). \quad (2·3) $$

Then the following relations are verified easily:

$$ \tilde{A}^+(\varphi)^* = \tilde{A}^-(\varphi), \quad (2·4·1) $$

$$ \{\tilde{A}^-(\varphi'), \tilde{A}^+(\varphi)\} = (\varphi', \varphi)E \quad \text{for bosons}, \quad (2·4·2) $$

$$ \{\tilde{A}^-(\varphi'), \tilde{A}^+(\varphi)\} = (\varphi', \varphi)E \quad \text{for fermions}, \quad (2·4·3) $$

where $*$ means hermite-conjugate and $E$ is the unit operator on $\Phi$ space. Let $\varphi_a(\xi)$ be any complete orthonormal system in $L^{(2)}_f$, and $T^{(1)}$ and $T^{(2)}$ be any two self-adjoint operators on that space, then we obtain

$$ \sum_a T^{(1)}_a \tilde{A}^+(\varphi_a) T^{(2)}_a \tilde{A}^-(\varphi_a) = \Omega(T^{(1)} T^{(2)}). \quad (2·5) $$

From this equation and (2·4) we obtain

$$ [\Omega(T), \tilde{A}^+(\varphi)] = T\tilde{A}^+(\varphi), \quad (2·6·1) $$

$$ [\Omega(T), \tilde{A}^-(\varphi)] = -\tilde{T}\tilde{A}^-(\varphi), \quad (2·6·2) $$

$$ [\Omega(T^{(1)}), \Omega(T^{(2)})] = \Omega([T^{(1)}, T^{(2)}]). \quad (2·6·3) $$

Next we summarize Foldy's canonical form. We shall use the same notation as used in Foldy's paper. The state of a particle specified by a mass $m$ and a spin $s$ and without antiparticle is represented by a $(2s+1)$-component wave function $\varphi$, and its wave equation is given by

$$ i\hbar \frac{\partial}{\partial t} \varphi = \omega \varphi, \quad \omega = \sqrt{m^2 + p^2}. \quad (2·7) $$

The infinitesimal operators of Lorentz group are represented, on this space, by

$$ P = p = -i\nabla, \quad (2·8·1) $$

$$ H = \omega, \quad (2·8·2) $$

$$ J = x \times p + s, \quad (2·8·3) $$

$$ K = \frac{1}{2}(s o + o x) - (s \times p) / (m + \omega) - t p. \quad (2·8·4) $$

For the particle with antiparticle, the wave function is $2(2s+1)$-component, and the wave equation and the infinitesimal operators become

$$ i\hbar \frac{\partial}{\partial t} \varphi = \beta \omega \varphi, \quad (2·9) $$

$$ P = p, \quad (2·10·1) $$
For the former case, the second quantization is easily performed, using the particle representation. But, for the latter case, some considerations corresponding to the hole theory for the Dirac particle must be given before the second quantization. In consequence, we remove $\beta$'s from (2.9) and (2.10) keeping $\phi$ to be $2(2s+1)$-component, and then perform the procedure of the second quantization. The meaning of this rule is: We learn from the classical theory the existence of the degree of freedom corresponding to the antiparticle, and, converting it to the particle of positive energy, perform the second quantization. According to the above prescription, the second quantized wave equation and infinitesimal operators are given, in both cases, by

\[
\begin{align*}
H &= \beta \omega, \\
J &= x \times p + s, \\
K &= \frac{1}{\hbar} \beta (x \omega + \omega x) - \frac{1}{2} (s \times p) / (m + \omega) - tp. 
\end{align*}
\] (2.10.2)

(2.10.3)

(2.10.4)

From (2.6.3) it is obvious that these operators satisfy the correct commutation relations. Thus the second quantization of free particle is completed. The main defect of this method is that we cannot obtain the relation between spin and statistics.

The improper transformations can also be handled according to Foldy's prescription, but we do not enter into details here. In Foldy's paper the particles with non-zero masses only are treated, but it turns out that the Maxwell equation can also be reduced to Foldy's canonical form with supplementary condition. This fact will be explained in the Appendix.

§ 3. Treatment of interactions. Hamiltonian formalism

In the Schrödinger picture, the interactions can be expressed by adding terms to $\tilde{H}^0$ and $\tilde{K}^0$ only, namely it is enough to put

\[
\begin{align*}
\tilde{P} &= \tilde{P}^0, \\
\tilde{H} &= \tilde{H}^0 + \tilde{\nu}, \\
\tilde{J} &= \tilde{J}^0, \\
\tilde{K} &= \tilde{K}^0 + \tilde{K}'.
\end{align*}
\] (3.1.1)

(3.1.2)

(3.1.3)

(3.1.4)

The commutation relations to be satisfied by $\tilde{H}'$ and $\tilde{K}'$ are
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\[ [\widehat{P}_r, \widehat{P}_r'] = [\widehat{J}_r, \widehat{P}_r'] = 0, \]  
(3.2.1)

\[ [\widehat{P}_r, \widehat{K}_r'] = -i \sigma_3 \widehat{P}_r', \]  
(3.2.2)

\[ [\widehat{J}_r, \widehat{K}_r'] = i \sigma_1 \mu_\beta \widehat{K}_\beta', \]  
(3.2.3)

\[ [\widehat{P}_r, \widehat{K}_r'] + [\widehat{P}_r', \widehat{K}_r'] + [\widehat{P}_r, \widehat{K}_r'] = 0, \]  
(3.2.4)

\[ [\widehat{K}_r', \widehat{K}_r''] + [\widehat{K}_r', \widehat{K}_r''] + [\widehat{K}_r', \widehat{K}_r''] = 0. \]  
(3.2.5)

From (3.2.2) \( \widehat{P}_r' \neq 0 \) implies \( \widehat{K}_r' \neq 0 \). (3.2.4) and (3.2.5) are quadratic in the interaction terms, and this fact makes it difficult to determine \( \widehat{P}_r' \) and \( \widehat{K}_r' \). Here exists the essential difficulty in the relativistic theory.

Hereafter we deal with only the interaction of the identical neutral scalar particles to simplify the argument. We first consider the \( \phi^2 \)-type interaction, and put

\[ \widehat{H}' = \sum a_{\alpha \beta} \tilde{A}_\alpha^+ \tilde{A}_\beta^+ \tilde{A}_\gamma^- + \text{h.c.}, \]  
(3.3)

where \( \tilde{A}_\alpha^+ \) and \( \tilde{A}_\gamma^- \) represent \( \tilde{A}_\alpha^+(\phi_\alpha) \) and \( \tilde{A}_\gamma^-(\phi_\gamma) \) respectively. Using (3.2.1) and (2.6), we find that \( a_{\alpha \beta} \) must satisfy

\[ p_{\alpha \beta} a_{\beta \gamma} + p_{\beta \gamma} a_{\alpha \beta} - a_{\alpha \beta} p_{\beta \gamma} = 0, \]  
(3.4.1)

\[ J_{\alpha \beta} a_{\beta \gamma} + J_{\beta \gamma} a_{\alpha \beta} - a_{\alpha \beta} J_{\beta \gamma} = 0. \]  
(3.4.2)

The general solution of these equations is given by

\[ a_{\alpha \beta} = \int H(p, q) \bar{\phi}_\alpha(p) \bar{\phi}_\beta(q) \phi_\gamma(p + q) \, dp \, dq, \]  
(3.5)

where \( H(p, q) \) is a function of scalars which can be composed of \( p \) and \( q \), and is symmetric with respect to \( p \) and \( q \). (3.2.2) and (3.2.3) is satisfied if we put

\[ \widehat{K}' = \sum b_{\alpha \beta} \tilde{A}_\alpha^+ \tilde{A}_\beta^+ \tilde{A}_\gamma^- + \text{h.c.}, \]  
(3.6)

\[ b_{\alpha \beta} = \int H(p, q) \bar{\phi}_\alpha(p) \bar{\phi}_\beta(q) i \frac{\partial}{\partial p} \phi_\gamma(p + q) \, dp \, dq \]

\[ + \int L(p, q) \bar{\phi}_\alpha(p) \bar{\phi}_\beta(q) \phi_\gamma(p + q) \, dp \, dq, \]  
(3.7)

where \( L \) is a sum of products of momentum vectors and scalar functions, and also symmetric with respect to \( p \) and \( q \). Next we consider (3.2.4). In the present case, it means that the relations

\[ [\widehat{H}'', \widehat{K}'''] + [\widehat{H}'', \widehat{K}'''] = 0, \]  
(3.8.1)

\[ [\widehat{H}', \widehat{K}'''] = 0 \]  
(3.8.2)

must be satisfied respectively. We obtain from (3.8.1)

\[ L_i(p, q) = i \frac{1}{\omega_\mu + \omega_\gamma - \omega_{\mu + \gamma}} \left[ \omega_\mu \frac{\partial}{\partial p_i} + \omega_\gamma \frac{\partial}{\partial q_i} + \frac{p_i + q_i}{2\omega_\mu} + \frac{q_i + p_i}{2\omega_\gamma} \right] H(p, q). \]  
(3.9)
(3·8·2) contains many sorts of terms, and gives as many conditions correspondingly. For example, the requirement that the coefficient of $\tilde{A}_\alpha^+ \tilde{A}_\beta^+ \tilde{A}_\gamma^+ \tilde{A}_\delta^-$ must vanish gives

$$-iH(q, q') - \frac{\partial}{\partial q_i} H(p, q + q') - H(p, q + q') L_i(q, q') + H(q, q') L_i(p, q + q') = 0.$$  

(3·10)

Putting $H(p, q) = (\alpha_p \alpha_g \alpha_{p+q})^{-1/2} h(p, q)$ and using (3·9), (3·10) becomes

$$\frac{q_i + q_i'}{\alpha_p + \alpha_g + \alpha_{p+q}} h(q, q') h(p, q + q') - h(q, q') - \frac{\partial}{\partial q_i} h(p, q + q')$$

$$- h(q, p + q') \frac{1}{\omega_p + \omega_g + \omega_{p+q}} \left( \alpha_p \frac{\partial}{\partial q_i} + \alpha_g \frac{\partial}{\partial q_i} \right) h(q, q')$$

$$+ h(q, q') \frac{1}{\omega_p + \omega_g + \omega_{p+q}} \left( \alpha_p \frac{\partial}{\partial p_i} + \alpha_g \frac{\partial}{\partial p_i} \right) h(p, q + q') = 0.$$  

(3·11)

It is very difficult to solve this equation, and it may be inferred that it has no solution. Consequently we cannot construct the interaction which is represented by (3·3) and (3·5).

The above result seems to imply that if $\tilde{H}'$ contains a term which is of third order or more with respect to $\tilde{A}^\pm$, then it becomes to have infinitely many terms on account of (3·2·4). Therefore we next try to determine $\tilde{H}$ and $\tilde{K}$ expanding them as

$$\tilde{H} = \tilde{H}' + \tilde{H}_1 + \tilde{H}_2 + \cdots,$$

$$\tilde{K} = \tilde{K}' + \tilde{K}_1 + \tilde{K}_2 + \cdots.$$  

(3·12·1)  

(3·12·2)

The result is as follows. If we choose $H_1$ suitably, it can be written as

$$\tilde{H}_1 = i[H^0, \tilde{T}],$$

in case it does not contain a quadratic term with respect to $\tilde{A}^\pm$ and a term corresponding to decay interaction*, and here $\tilde{T}$ is a self-adjoint operator. For example, for the $\tilde{H}_1$ represented by (3·3) and (3·5), $\tilde{T}$ is given by

$$\tilde{T} = \Sigma a_{\alpha\beta} \tilde{A}_\alpha^+ \tilde{A}_\beta^+ \tilde{A}_\gamma^+ \tilde{A}_\delta^- + \text{h.c.},$$

(3·13·1)

$$a_{\alpha\beta} = -i \int \frac{H(p, q)}{\omega_p + \omega_g - \omega_{p+q}} \bar{\psi}_\alpha(p) \bar{\psi}_\beta(q) \varphi_\gamma(p+q) d\rho d\varphi, $$

(3·13·2)

and $\tilde{K}_1$ represented by (3·6), (3·7) and (3·9) is written as $\tilde{K}_1 = i[\tilde{K}_0, \tilde{T}]$. Therefore, giving $\tilde{H}_1$ and then to construct $\tilde{H}_2, \tilde{K}_1$, etc., so as to satisfy the commutation relations is the same as expanding

* Refer to § 6.
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\[ \tilde{H} = \exp(-i\tilde{T}) \tilde{H}_0 \exp(i\tilde{T}) \]  \hspace{2cm} (3\cdot14.1)

\[ \tilde{K} = \exp(-i\tilde{T}) \tilde{K}_0 \exp(i\tilde{T}) \]  \hspace{2cm} (3\cdot14.2)

and if the equation corresponding to (3\cdot10) cannot be solved in any stage of expansion, only (3\cdot14) is essentially a possible form of interaction.

Summarizing above considerations, we obtain the following results:

I) From (3\cdot14) it is obvious that the spectrum of \( \tilde{H} \) is the same as that of \( \tilde{H}_0 \), so we cannot deal with the problem of mass difference by this approach.

II) If we adopt the \( \varphi^2 \)-type interaction as \( \tilde{T} \), no process occurs in perturbation calculation.

III) If \( \tilde{T} \) contains a term corresponding to a real process, for example, a term like \( \tilde{A}_a^+ \tilde{A}_b^+ \tilde{A}_c^- \tilde{A}_d^- \), \( \tilde{H} \) can describe the corresponding process, because in this case both \( \tilde{A}_a^+ \) and \( \exp(-i\tilde{T}) \tilde{A}_a^+ \exp(i\tilde{T}) \) satisfy the condition for the creation operator of a physical particle, that is,

\[ H \tilde{A}_a^+ (\varphi) \Phi_0 = \omega \tilde{A}_a^+ (\varphi) \Phi_0, \]

where \( \Phi_0 \) is the vacuum state; so, considering \( \tilde{A}_a^+ \) to be the creation operator of a physical particle, the real process becomes to occur.

These results are very unsatisfactory, and as to the real process there exist too many arbitrarinesses. In addition, the existence of the relativistic Hamiltonian formalism does not imply the existence of the invariant \( S \)-matrix. Therefore the ordinary expression of the invariance of the \( S \)-matrix, namely \( [\tilde{S}, \tilde{K}^\alpha] = 0 \), is a further requirement for the theory. Accordingly, it is more convenient to deal directly with \( S \)-matrix theory, if we are concerned only with real processes. So, in the next section we investigate the structure of the invariant \( S \)-matrix.

§ 4. Structure of \( S \)-matrix

According to the general theory of \( S \)-matrix, if Yang-Feldman’s \( S \)-matrix exists for free Hamiltonian \( \tilde{H}_0 \) and total Hamiltonian \( \tilde{H} \), the absolutely continuous parts of both operators are unitary equivalent.\(^6\) In this respect, the results of the previous section are consistent. It has sometimes been inferred that Yang-Feldman’s \( S \)-matrix does not exist for the multichannel process on account of the complexity to separate each channel.\(^9\) However, in the case of the relativistic invariant theory, the separation of channels is physically very obvious, and we may suppose that Yang-Feldman’s \( S \)-matrix does exist in our theory. We denote the invariant \( S \)-matrix which can be constructed from \( \tilde{H}_0 \) and \( \tilde{H} \) defined by (3\cdot14), by \( \tilde{S}_n \), and the set of all \( \tilde{S}_n \) by \( \mathcal{S}_n \). \( \tilde{S}_n \) satisfies

\[ [\tilde{S}_n, \tilde{H}_0] = [\tilde{S}_n, \tilde{K}^\alpha] = 0. \]  \hspace{2cm} (4\cdot1)

Of course, it also commutes with \( \tilde{P}^0 \) and \( \tilde{J}^0 \). Let the set of all unitary matrices satisfying (4\cdot1) be \( \mathcal{E} \), then \( \mathcal{E} \supset \mathcal{S}_n \). As it is difficult to deal with \( \mathcal{S}_n \) directly, we
investigate the structure of \( \mathcal{S} \) instead. Still \( \mathcal{S} \) is more restricted than the Hamiltonian of last section because of (4·1).

It can be proved that for any \( \tilde{S} \in \mathcal{S} \) a self-adjoint operator \( \tilde{T} \) which satisfies
\[
[\tilde{T}, \tilde{T}^0] = [\tilde{T}, \tilde{K}^0] = 0
\]
exists and \( \tilde{S} \) is written as \( \tilde{S} = \exp(i\tilde{T}) \). Therefore, to investigate \( \mathcal{S} \) is equivalent to investigating \( \tilde{T} \) defined by (4·2).

To compute the commutation relations with \( \tilde{K}^0 \), the operators on one-particle space defined by
\[
Q^\pm K = \beta x_0 Q^\pm \pm i \beta s Q^\pm - i p Q^\pm
\]
are very useful. For the spinless particle \( Q \) turns out to be
\[
Q_0 = \omega^{-\frac{1}{2}}
\]
except a constant factor.

Now we consider \( \tilde{T} \) corresponding to the scattering of neutral scalar particles. Taking into account that \( \tilde{T} \) must commute with \( \tilde{P}^0 \) and \( \tilde{J}^0 \), we put
\[
\tilde{T} = \sum a_{\alpha \beta \gamma} \tilde{A}_\alpha^+ \tilde{A}_\beta^+ \tilde{A}_\gamma^- \tilde{A}_\gamma^-,
\]

\[
a_{\alpha \beta \gamma} = \int F(p, q, r) \frac{1}{\omega_p} \varphi_\alpha(p) \frac{1}{\omega_q} \varphi_\beta(q) \frac{1}{\omega_r} \varphi_\gamma(r)
\]
\[
\times \frac{1}{\omega_{p+q-r}} \varphi_\gamma(p+q-r) dp dq dr,
\]
(4·5·2)
where \( F(p, q, r) \) is a scalar function.

From (4·2) we obtain
\[
(\omega_p + \omega_q - \omega_r - \omega_{p+q-r}) F(p, q, r) = 0,
\]
(4·6·1)
\[
\left( \omega_p \frac{\partial}{\partial p_i} + \omega_q \frac{\partial}{\partial q_i} + \omega_r \frac{\partial}{\partial r_i} - \frac{p_i + q_i - r_i}{\omega_{p+q-r}} \right) F(p, q, r) = 0.
\]
(4·6·2)
(4·6·1) means
\[
F(p, q, r) = \hat{\theta}(\omega_p + \omega_q - \omega_r - \omega_{p+q-r}) f(p, q, r).
\]
(4·7)

Now,
\[
\left( \omega_p \frac{\partial}{\partial p_i} + \omega_q \frac{\partial}{\partial q_i} + \omega_r \frac{\partial}{\partial r_i} \right) \hat{\theta}(\omega_p + \omega_q - \omega_r - \omega_{p+q-r})
\]
\[
= \left[ (p_i + q_i - r_i) - \frac{p_i + q_i + r_i}{\omega_{p+q-r}} (\omega_p + \omega_q - \omega_r) \right] \hat{\theta}(\omega_p + \omega_q - \omega_r - \omega_{p+q-r})
\]
\[
= \frac{p_i + q_i - r_i}{\omega_{p+q-r}} \frac{\theta'}{\omega_{p+q-r}} = \frac{p_i + q_i - r_i}{\omega_{p+q-r}} \hat{\theta},
\]
(4·7)

* Refer to the reason for finding (3·5).
so the relation to determine \( f \) becomes

\[
\delta (\omega_p + \omega_q - \omega_r - \omega_{p+q-r}) \left( \omega_p \frac{\partial}{\partial p_i} + \omega_q \frac{\partial}{\partial q_i} + \omega_r \frac{\partial}{\partial r_i} \right) f = 0.
\] (4.8)

We examine some special solutions of (4.3) in order. If we adopt \( f = \text{const.} \), then (4.5) gives the same result as the lowest order of \( \psi^4 \) interaction in the ordinary theory. Next, considering

\[
\left( \omega_p \frac{\partial}{\partial p_i} + \omega_q \frac{\partial}{\partial q_i} \right) \left( \omega_p + \omega_q - \omega_{p+q} \right) = 0,
\]

where \( \omega' \) is an energy expression with different mass, a special solution of (4.8) is written as

\[
f \equiv f(\omega_p + \omega_q - \omega'_{p+q}) (\omega_p + \omega_q + \omega'_{p+q}).
\]

Taking \( f(x) = 1/x \), (4.5) gives the same result as the contribution of a Feynman diagram indicated in Fig. 1 with \( \psi^2 \phi \) interaction. Similarly, adopting

\[
f \equiv 1/(\omega_p - \omega_q - \omega'_{p+q}) (\omega_p + \omega_q + \omega'_{p+q}),
\]

the contribution of Fig. 2 is also obtained.

For other processes, the similar correspondences to the lowest order of the ordinary theory can be easily confirmed for the interaction of the scalar particles.

The above results seem to be interesting, for the interaction Hamiltonian of the ordinary theory is not only unsatisfactory in the sense mentioned in last section, but also not a well defined operator on the second quantized space, and in spite of this a part of its contributions has reappeared in the S-matrix theory treated here. This fact seems to show the reason why the ordinary theory, though containing inconsistencies in itself, gives some useful results by perturbation calculation.

Before leaving this section, we briefly mention about the extension of our method to the interaction of particles with spin. For the particles of \( s = \frac{1}{2} \) and \( s = 1 \), the operators characterized by (4.3) are given by

\[
Q_{i\alpha}^\pm = \sqrt{\frac{m + \omega}{2\omega}} \mp \sqrt{\frac{1}{\omega(m + \omega)}} (\sigma \cdot p) \] (4.9.1)

\[
Q_{i}^{\pm} = \sqrt{\omega} \mp \frac{1}{\sqrt{\omega}} (s \cdot p) - \frac{\omega - m}{\sqrt{\omega}} R, \] (4.9.2)

where \( R \) is a projection operator to the longitudinal wave and represented by

\[
R = 1 - (s \cdot p)^2/p^2. \] (4.10)

The scattering of the same particles is described taking, instead of (4.5.2),
\[ a_{\alpha\beta\gamma} = \int F(p, q, r) \left( Q^+ \varphi_\alpha(p), Q^- \varphi_\gamma(r) \right) \times \left( Q^+ \varphi_\beta(q), Q^- \varphi_\beta(p+q-r) \right) dp dq dr \pm (+\leftrightarrow-) \]

or

\[ a_{\alpha\beta\gamma} = \int F(p, q, r) \left( Q^+ \varphi_\alpha(p), sQ^- \varphi_\gamma(r) \right) \times \left( Q^+ \varphi_\beta(q), sQ^- \varphi_\beta(p+q-r) \right) dp dq dr \pm (+\leftrightarrow-) \]

where the inner product is to be taken with respect to the spin variable, and \((+\leftrightarrow-)\) means the term which can be obtained from the first term interchanging + and − signs.

As an application of our method to the practical phenomena, the Compton scattering is treated in the next section.

§ 5. Compton scattering

According to the result of the Appendix, it is possible to handle the electromagnetic field with our method. For the particle specified by \( m=0 \) and \( s=1 \), \ref{eq:4.9.2} becomes

\[ Q^t = \sqrt{p} \left( 1 \mp (s \cdot p) / p \right) (1 - R). \]

\ref{eq:5.1} contains the projection operator to the transverse wave, therefore so long as we use \ref{eq:5.1}, only the transverse photons are treated. It is possible to deal with the longitudinal photons, but we do not mention it here.

Hereafter the spin matrices for \( s=\frac{1}{2} \) and \( s=1 \) will be denoted by \( S \) and \( S \) respectively. We denote the wave function of photon by \( h \) and write \( (s, Q^\pm h) \) as \( h^\pm \). We adopt the ordinary representation for \( S \), then for an arbitrary vector \( e \), we get \( (s \cdot e), S_i = -(s \cdot S_i, e) \). We further use the operators defined by

\[ \pm \{ O^\pm, s_i \} = -\left( \omega_p \frac{\partial}{\partial p_i} + \omega_q \frac{\partial}{\partial q_i} + \omega_r \frac{\partial}{\partial r_i} \right) O^\pm = 0. \]

An example is given by

\[ O_{p+q}^\pm = \omega_p + \omega_q \pm (\sigma \cdot (p + q)). \]

To describe the Compton scattering we take

\[ \vec{T} = \Sigma a_{\alpha\beta\gamma} \vec{A}_\alpha^+ \vec{B}_\beta \cdot \vec{A}_\gamma^+ \vec{B}_\gamma, \]

\[ a_{\alpha\beta\gamma} = q \int \delta(\omega_p + q - \omega_r - |p + q - r|) M_{\alpha\beta\gamma}(p, q, r) dp dq dr, \]

where \( \vec{A}^\pm \) and \( \vec{B}^\pm \) are the creation-annihilation operators of electrons and photons.
respectively, and we use for \( M \) the following form,

\[
M = \{(wp+q-wp+q) (wp+q+(up+q) (up+q))^{-1} \\
\times [(h_{\beta} + Q_{12} \varphi_a, O_{\nu+\sigma}^+ h_{\beta} + Q_{12} \varphi_a) + (h_{\beta} - Q_{12} \varphi_a, O_{\nu+\sigma}^+ h_{\beta} + Q_{12} \varphi_a)] \\
+ m (h_{\beta} + Q_{12} \varphi_a, h_{\beta} - Q_{12} \varphi_a) + m (h_{\beta} - Q_{12} \varphi_a, h_{\beta} + Q_{12} \varphi_a)]
\]

\[ (5·3) \]

Of course, \( M \) cannot be uniquely determined by invariance principle only, but (5·3) is one of its simplest form. In this case the correspondence to the ordinary theory cannot be obtained in the matrix element, but calculating the differential cross section, (5·3) leads to the well-known Klein-Nishina formula if we put \( g = e^2/8\pi \).

Thus it may be said that our method can cover the results of the ordinary theory to some extent, though not completely equivalent to it. Here one may find the possibility to step out from the local field theory, and it may be interesting to investigate whether it is possible to find out the way to determine the S-matrix uniquely according to our approach.

§ 6. Decay interaction

The decay process cannot be treated by the S-matrix formalism. Not only it is meaningless to do so, but also the operator \( \tilde{T} \) of § 4 cannot be a well defined one for the decay interaction. So we have to return to the Hamiltonian formalism. The creation-annihilation operators for particles with mass \( m \) and \( M \) will be denoted by \( \hat{a}_a^+ \) and \( \hat{b}_\beta \) respectively, and we assume \( M > 2m \). If we put

\[
\tilde{H}^1 = \sum a_{a\beta \gamma} \hat{a}_a^+ \hat{a}_\beta^+ \hat{b}_\gamma^+ + \text{h.c.,}
\]

\[ (6·1·1) \]

\[
a_{a\beta \gamma} = \int h(p, q) (\alpha_{a\beta \gamma} \alpha_{a\beta \gamma})^{-1/2} \varphi_a(p) \varphi_\beta(q) \varphi_\gamma(p+q) dp dq,
\]

\[ (6·1·2) \]

then the operator \( \tilde{T} \) defined by \( \tilde{H}^1 = [\tilde{H}^0, \tilde{T}] \) is represented formally by

\[
\tilde{T} = \sum a'_{a\gamma \beta} \hat{a}_a^+ \hat{a}_\beta^+ \hat{b}_\gamma^+ + \text{h.c.,}
\]

\[ (6·2·1) \]

\[
a'_{a\gamma \beta} = -i \int \frac{\tilde{\rho}}{\alpha_{a\beta \gamma} + \alpha_{a\beta \gamma}'} h(p, q) (\alpha_{a\beta \gamma} \alpha_{a\beta \gamma}')^{-1/2} \\
\times \varphi_a(p) \varphi_\beta(q) \varphi_\gamma(p+q) dp dq,
\]

\[ (6·2·2) \]

where \( \tilde{\rho} \) means principal value. But in this case \( \sum_{a,\beta,\gamma} |a'_{a\gamma \beta}|^2 \) becomes to be divergent, and the operator \( \tilde{T} \) cannot be a well defined one as mentioned before. Consequently, the transformation (3·14) loses its meaning for the decay process. However, if we expand (3·14) formally, its first order term, namely \( \tilde{H}^1 \), is well defined for suitable \( h(p, q) \), and any process except decay does not occur by perturbation calculation, and the decay too occurs only in the lowest order. So it may be pos-
sible that the Hamiltonian defined by this formal expansion is a well defined operator. For $K$, such reasoning is not possible, but if we demand its first order term to be well defined, then $h(p,q)$ is restricted to, for example,

$$h(p,q) = h(\omega_p + \omega_q - \omega_{p+q}),$$

(6.3)

where $h(0)$ is finite. Using (6.3), (6.1) gives the same result as the lowest order contribution with $\varphi^2\varphi$ interaction in the ordinary theory. The form of $h$ affects only the line width.

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Appendix. Reduction of the Maxwell equation to Foldy's canonical form

The Maxwell equation for the free field is

$$\epsilon_{jkl} \frac{\partial E_l}{\partial x_k} + \frac{\partial B_j}{\partial x_l} = 0, \quad \epsilon_{jkl} \frac{\partial B_l}{\partial x_k} - \frac{\partial E_j}{\partial x_l} = 0,$$

(A.1)

$$\frac{\partial E_j}{\partial x_j} = \frac{\partial B_j}{\partial x_j} = 0.$$  

(A.2)

If we put $-i\epsilon_{jkl} = (s_j)_{kl}$, $\psi_1 = E + iB$, $\psi_2 = E - iB$ and $\psi = (\psi_1 \psi_2)$, then (A.1) becomes

$$i \frac{\partial}{\partial t} \varphi = \beta(\mathbf{s} \cdot \mathbf{p}) \varphi.$$  

(A.3)

Here $s_j's$ are the spin matrices for $s=1$. (A.2) is used as a supplementary condition. Using a matrix given by

$$U = \begin{pmatrix}
\frac{ip_2 - p_1 p_3}{\sqrt{2p^2(p_1^2 + p_2^2)}} & \frac{p_1}{p} & \frac{-ip_2 - p_1 p_3}{\sqrt{2p^2(p_1^2 + p_2^2)}} \\
\frac{-ip_1 - p_2 p_3}{\sqrt{2p^2(p_1^2 + p_2^2)}} & \frac{p_2}{p} & \frac{ip_1 - p_2 p_3}{\sqrt{2p^2(p_1^2 + p_2^2)}} \\
\frac{p_1^2 + p_2^2}{\sqrt{2p^2(p_1^2 + p_2^2)}} & \frac{p_2^2 + p_3^2}{\sqrt{2p^2(p_1^2 + p_2^2)}} & \frac{p_3^2}{p} \\
\frac{p_1^2 + p_3^2}{\sqrt{2p^2(p_1^2 + p_2^2)}} & \frac{p_2^2 + p_3^2}{\sqrt{2p^2(p_1^2 + p_2^2)}} & \frac{p_3^2}{p}
\end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\
0 & 1
\end{pmatrix},$$

and putting $\varphi = U\varphi'$, (A.3) becomes

$$i \frac{\partial}{\partial t} \varphi' = \begin{pmatrix} p & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -p
\end{pmatrix} \otimes \beta \varphi'.$$

(A.4)

Further, putting $\varphi' = V\varphi'$, where $V$ is a matrix given by
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\[
V = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix},
\]

we get

\[
i \frac{\partial}{\partial t'} \chi' = \begin{pmatrix}
\dot{p} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \dot{p} \\
\end{pmatrix} \otimes \beta \chi'.
\]  
(A·5)

Using (A·2), we get

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & \dot{p} & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \otimes \beta \chi' = pR \otimes \beta \psi = 0,
\]

so, adding this to the right-hand side of (A·5), we obtain

\[
i \frac{\partial}{\partial t'} \chi' = \beta \dot{p} \chi'.
\]  
(A·6)

From the transformation properties of \( E \) and \( B \) we get for \( \psi \)

\[
P = \dot{p},
\]
\[
H = \beta (s \cdot p) + pR \otimes \beta,
\]
\[
J = \mathbf{x} \times \mathbf{p} + s.
\]

But for \( \chi' \), \( J \) becomes to have a complex expression. So we further put \( \chi = U \chi' \), and we obtain for \( \chi \)

\[
P = \dot{p},
\]
\[
H = \beta \dot{p},
\]
\[
J = \mathbf{x} \times \mathbf{p} + s.
\]

Here \( \chi \) turns out to be

\[
\chi = \begin{pmatrix}
E + \frac{i}{\dot{p}} \text{rot} \mathbf{B} \\
E - \frac{i}{\dot{p}} \text{rot} \mathbf{B}
\end{pmatrix}.
\]

Next we consider about the Lorentz transformations, the infinitesimal operators are expressed by

\[
K = \mathbf{x}H - i\beta s - i\mathbf{p}, \quad H = \beta (s \cdot p) + p \beta R,
\]
for $\phi$. Performing the above transformations we get for $X$

$$K = \beta x p - \beta (s \times p) / p - i \beta p / p - t p.$$  

Yet this is not a desirable expression. This is rather natural, judging from the dimension of $X$. Therefore we again put $\phi = C / \sqrt{p} X$, where $C$ is a constant to be determined from the fact that the total energy is given by $(1/8\pi) \int (E^2 + B^2) \, dx$ in the Maxwell theory, and consequently we get $C = 1 / \sqrt{8\pi}$. For $\phi$, $K$ becomes to have the desired expression.

Now, it was necessary to use six component $\phi$ in the beginning, but after the canonical form has been thus obtained, the half components of $\phi$ become redundant. Taking this fact into account, the results are summarized as follows:

$$\phi = \frac{1}{\sqrt{8\pi p}} \left( E + \frac{i}{p} \text{rot} B \right), \quad (A \cdot 7)$$

$$i \frac{\partial}{\partial t} \phi = p \phi, \quad (A \cdot 8)$$

$$P = p, \quad (A \cdot 9 \cdot 1)$$

$$H = p, \quad (A \cdot 9 \cdot 2)$$

$$J = x \times p + s, \quad (A \cdot 9 \cdot 3)$$

$$K = \frac{1}{2} (x p + p x) - (s \times p) / p - t p, \quad (A \cdot 9 \cdot 4)$$

$$R \phi = 0. \quad (A \cdot 10)$$

When the interaction source is present, the Maxwell equation is reduced to

$$i \frac{\partial}{\partial t} \phi = p \phi - \frac{2\pi}{p} \frac{i}{p} \left( j - \frac{p}{p^2} \rho \right),$$

$$R \phi = -i \frac{2\pi}{p} \frac{p}{p^2} \rho.$$  

References

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8) K. O. Friedrichs, loc. cit.
10) P. A. M. Dirac, Rev. Mod. Phys. 21 (1949), 392.