Shifted $Z_6$ Orbifold Model and 1·4 Model

Hiroshi Kawabe, Tatsuo Kobayashi* and Noriyasu Ohtsubo**

Department of Physics, Kanazawa University, Kanazawa 920
*Department of Physics, College of Liberal Arts and Sciences
Kyoto University, Kyoto 606-01
**Kanazawa Institute of Technology, Ishikawa 921

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We explicitly construct a two-dimensional 'shifted' $Z_6$ orbifold model equivalent to a two-dimensional twisted $Z_6$ orbifold model at an enhancement point. Then we study equivalence of the shifted $Z_6$ super-orbifold model and the 1·4 model. As applications to the $(2, 2)$ heterotic string theories, we investigate correspondence of the $Z_6 \times Z_6$ ($Z_3 \times Z_6$) orbifold model and $(1 \cdot 4)^3$ ($1^3 (1 \cdot 4)^2$) model.

§ 1. Introduction

A huge number of four-dimensional string vacua are derived from several types of constructions, e.g., orbifold models, tensoring of coset constructions and so on. It is very important to relate models obtained by different constructions and further to understand the whole vacua in a unified framework. For example, some Gepner's models have the same generation numbers as orbifold models. It is meaningful to investigate in more detail the relations of the models besides coincidence of the massless spectra.

In the $E_8 \times E_8$ heterotic string theory, internal spaces of the four-dimensional string vacua are represented in terms of the $N=2$ superconformal field theories (SCFTs) with $c=9$, as well as the $SO(10) \times E_8$ internal space for the left-mover. The huge number of string vacua implies that there are lots of possibilities to construct $c=9$ SCFTs. For example, $c=9$ theories in the orbifold models are represented in terms of six-dimensional superstring satisfying twisted boundary conditions, where we call the superstring super-orbifold models. In the Gepner's models, $c=9$ theories are constructed by tensoring of $N=2$ minimal models.

Recently, it has been shown that two-dimensional $Z_3, Z_4$ and $Z_6$ super-orbifold models are equivalent to $1^9, 2^9$ and $1\cdot 4$ theories, respectively, from the point of view of the SCFT. Those are applied to correspondence between $1^9 (2^6)$ models and $Z_3 \times Z_3 (Z_4 \times Z_3)$ orbifold models at the maximal enhancement point in the $(2, 2)$ heterotic construction. In Ref. 7), it has been verified that the two-dimensional twisted super-orbifold models are equivalent to $c=3$ tensoring of the minimal models. On the other hand, the shifted orbifolds equivalent to the twisted orbifolds were treated in Ref. 5) (6)), where it has been clarified that the two-dimensional shifted $Z_4 (Z_6)$ super-orbifold model is one sort of free-bosonic representations of the $1^3 (2^3)$ SCFT. The shifted formalism represents an enhanced $U(1)$ diagonal basis naturally. In addition,
the formalism is very meaningful in order to understand relations between the twisted orbifolds and the minimal models more deeply and it gives geometrical aspects of the minimal theories. Therefore, one of purposes of this paper is to construct two-dimensional 'shifted' $Z_6$ super-orbifold explicitly, equivalent to the twisted orbifold at the enhancement point and then to show correspondence between the $1\cdot 4$ model and the shifted $Z_6$ super-orbifold model. In addition to the $c=3$ SCFTs, it is important to investigate explicit correspondences of the Gepner's models and the orbifold models in the $(2, 2)$ heterotic construction. For the purpose, it is required that we investigate tensoring conditions deriving $c=9$ theories from $c=3$ theories in order to show equivalence of the Gepner's models and the orbifold models in the $(2, 2)$ heterotic constructions.

This paper is organized as follows. In § 2, we review the two-dimensional twisted $Z_6$ and $Z_3$ super-orbifold models briefly and construct the shifted $Z_6$ orbifold as well as the shifted $Z_3$ orbifold in § 3. The super conformal algebra SCA elements of the $1\cdot 4$ theory are compared with those of the shifted $Z_6$ super-orbifold and then we discuss correspondence of chiral primary states and Yukawa couplings of them in § 4. These results are applied to equivalence between $(1\cdot 4)^3 (1^3(1\cdot 4)^2)$ model and $Z_6 \times Z_6 (Z_6 \times Z_6)$ orbifold model at the enhancement point under the $(2, 2)$ heterotic construction, in § 5. The last section is devoted to conclusion and discussion.

§ 2. 2-dim twisted $Z_6$ super-orbifold model

Two-dimensional $Z_6$ and $Z_3$ orbifolds are quotients of an $SU(3)$ torus by $Z_6$ twist $\Theta$ and $Z_3$ twist $\Theta^{2}$, respectively, while the $SU(3)$ torus is obtained as a division of two-dimensional flat space $R^2$ by an $SU(3)$ lattice $\Lambda_{SU(3)}$, whose simple roots are denoted by $a_1$ and $a_2$. Hereafter, two-dimensional string coordinates and their left (right)-moving zero-mode momenta are denoted by $X^i$ and $P^i_{(L,R)}$ ($i=1, 2$). There is a specific point in the moduli space, i.e., a specific radius of the $SU(3)$ torus so that $(P_L^1)^2+(P_L^2)^2=2$. The point is called an enhancement point and here we consider the orbifolds at the enhancement point.

It is well known that closed strings on the orbifolds are classified into two types. One is the untwisted string and the other is the twisted string. In the heterotic construction, left-moving untwisted states with momenta $(P_L^1)^2+(P_L^2)^2=2$ correspond to massless states, i.e., gauge bosons. The states are denoted by vertex operators $E(a_1, a_2)\equiv \exp[i\sum_i P^i X^i]$, where $a_1 a_1 + a_2 a_2 = (P_1^1, P_2^2)$ and we choose $a_1 = (\sqrt{2}, 0)$ and $a_2 = (-\sqrt{2}/2, \sqrt{6}/2)$. Since physical states should be invariant under the $Z_6$ twist in the $Z_6$ orbifold model, we obtain a $Z_6$-invariant state as

$$\frac{1}{\sqrt{6}}(E(1, 0)+E(1, 1)+E(0, 1)+E(-1, 0)+E(-1, -1)+E(0, -1)), \quad (2\cdot 1)$$

from linear combinations of the states. In this way, we can find that the two-dimensional $Z_6$ orbifold model has an enhanced $U(1)$ symmetry corresponding to the gauge boson described by Eq. (2·1). In the case of the $Z_3$ orbifold, there is another $Z_3$-invariant state in addition to the state of Eq. (2·1), i.e.,
\[
\frac{i}{\sqrt{6}}(E(1, 0) - E(1, 1) + E(0, 1) - E(-1, 0) + E(-1, -1) - E(0, -1)).
\] (2.2)

As a result, two-dimensional \(Z_3\) orbifold model has an enhanced \(U(1)^2\) gauge group. The \(\Theta^m\)-twisted string has a twisted boundary condition as

\[
X(\sigma = 2\pi) = \Theta^m X(\sigma = 0) + e,
\] (2.3)

where \(e\) is a vector on \(A_{su(1)}\). Its zero-mode satisfying the same condition as Eq. (2.3) is called a fixed point and sometimes denoted by the space group element \((\Theta^m, e)\). Independent fixed points are classified in terms of conjugacy classes. In the heterotic construction, physical states are derived from tensoring of \(\Theta\)-invariant states, which are obtained as linear combinations of the twisted states corresponding to the independent fixed points.\(^*)\) In addition, we can often construct other physical states in terms of eigenstates of \(\Theta\) with eigenvalues \(\Theta \pm 1\), as we will see in § 5. The third column of Table I shows only \(Z_3\)-invariant states and we can easily obtain the other eigenstates of \(\Theta\), changing relative phases. The \(\Theta^m\)-twisted bosonic ground states have conformal dimensions \(h_a = m(1 - m/6)/12\). Their superpartner is denoted by a bosonized field \(H\), whose momenta \(\check{p}\) span an \(SO(2)\) weight lattice in the untwisted sector. For the NS states, the untwisted sector has the \(SO(2)\) momenta \(\check{p} = 0, 1\). For instance the states corresponding to gauge bosons have vanishing momentum. The \(\Theta^m\)-twisted sector has the \(SO(2)\) momentum \(\check{p} = m/6\) and the momentum contributes to conformal dimension as \(h = (\check{p})^2/2\). Then the \(\Theta^m\)-twisted ground states of the super-orbifold have total conformal dimensions \(h = m/12\) and the states are chiral primary as we will see.

The \(\bar{\Theta}\)-twisted sector of the \(Z_3\) orbifold has the same three independent fixed points as the \(\Theta^2\)-twisted sector of the \(Z_3\) orbifold. Of course the \(\bar{\Theta}\)-twisted sector has the same conformal dimension and \(SO(2)\) momentum as the \(\Theta^2\)-twisted sector of the

\begin{table}[h]
\centering
\caption{Chiral primary state in \(Z_3\) orbifold and 1·4 models.}
\begin{tabular}{|c|c|c|c|c|}
\hline
Twist & \(SO(2)\) momentum & Twisted orbifold & Shifted orbifold & 1·4 model \\
\hline
0 & 0/6 & \(|0\rangle\) & \(|0, 0\rangle\) & \(|0, 0\rangle\) \\
\hline
1 & 1/6 & \(|0\rangle\) & \(\sqrt{2}/12, 0\rangle\) & \(0, 1\rangle\) \\
\hline
2 & 2/6 & \(|0\rangle\) & \(\sqrt{2}/6, \sqrt{2}/6\rangle + \sqrt{2}/6, -\sqrt{2}/6\rangle\) & \(1, 0\rangle\) \\
\hline
3 & 3/6 & \(\sqrt{2}/6, \sqrt{2}/6\rangle + \sqrt{2}/6, -\sqrt{2}/6\rangle\) & \(\sqrt{3}/4, 0\rangle\) & \(0, 3\rangle\) \\
\hline
4 & 4/6 & \(\sqrt{2}/6, \sqrt{2}/6\rangle + \sqrt{2}/6, -\sqrt{2}/6\rangle\) & \(\sqrt{3}/6, 0\rangle\) & \(0, 4\rangle\) \\
\hline
5 & 5/6 & \(\sqrt{2}/6, \sqrt{2}/6\rangle + \sqrt{2}/6, -\sqrt{2}/6\rangle\) & \(\sqrt{3}/12, 0\rangle\) & \(1, 2\rangle\) \\
\hline
6 & 6/6 & \(\sqrt{2}/6, \sqrt{2}/6\rangle + \sqrt{2}/6, -\sqrt{2}/6\rangle\) & \(\sqrt{3}/12, 0\rangle\) & \(1, 4\rangle\) \\
\hline
\end{tabular}
\end{table}

\(^*)\) See in detail Ref. 9).
§ 3. Shifted $Z_6$ orbifold

The twisted ground states do not have definite enhanced $U(1)$ charges under the current $(2, 1)$ or $(2, 2)$ in the basis corresponding to the independent fixed points, so that we must change the basis in the twisted orbifolds, in order to investigate the definite enhanced $U(1)$ charges. Instead of doing so, we here study 'shifted' orbifolds, where ground states have the enhanced $U(1)$ charges naturally. Further the shifted orbifolds have a closer relation to the minimal models in the free bosonic representation.

Now let us construct 'shifted' $Z_6$ orbifold equivalent to the twisted $Z_6$ orbifold by transforming the $SU(3)$ Kac-Moody algebra basis,\textsuperscript{10,12} similar to the $Z_4$ orbifold,\textsuperscript{6} where we consider an $SO(4)$ Kac-Moody algebra. Cocycle factors play a notable role in the shifted $Z_6$ orbifold, while the $Z_4$ orbifold do not need cocycle factors. The $SU(3)$ Kac-Moody algebra consists of vertex operators $E(\pm 1, 0), E(\pm 1, \pm 1), E(0, \pm 1)$ corresponding to the non-zero roots and Cartan elements $i\partial X^1, i\partial X^2$. The cocycle factors $\epsilon_{\alpha\beta}$ are defined by\textsuperscript{13}

$$E_a(z)E_b(w) \sim \frac{\epsilon_{\alpha\beta}}{z-w} E_{a+b}(w).$$

Here we choose $\epsilon_{(1,0)(0,1)} = -i$.\textsuperscript{*} The cocycle factors are represented by antisymmetric tensor $B^{ij}$ as $\epsilon_{\alpha\beta} = \exp[-\pi i a_i B^{ij} \beta^j/2]$, where $\alpha, B^{ij}, \beta^j = 1$.

We regard the currents $(2\cdot 1)$ and $(2\cdot 2)$ as new Cartan elements, i.e.,

$$i\partial X^a = \frac{1}{\sqrt{6}} (E(1, 0) + E(1, 1) + E(0, 1) + E(-1, 0) + E(-1, -1) + E(0, -1)),
\tag{3·1}$$

$$i\partial X^a = \frac{i}{\sqrt{6}} (E(1, 0) - E(1, 1) + E(0, 1) - E(-1, 0) + E(-1, -1) - E(0, -1)).
\tag{3·2}$$

Then we can obtain new elements corresponding to non-zero roots under the new Cartan elements as

$$E'(\pm 1, 0) = \frac{1}{\sqrt{3}} e^{\pm i\theta} \left( -E^x_{(\pm 1)} + E^y_{(\pm 1)} + \frac{i}{\sqrt{2}} (\partial X^1 \pm i\partial X^2) \right),$$

$$E'(0, \pm 1) = \frac{1}{\sqrt{3}} e^{\pm i\theta} \left( -\omega^{\pm 1} E^x_{(\pm 1)} + E^y_{(\pm 1)} + \frac{i\omega^{\mp 1}}{\sqrt{2}} (\partial X^1 \pm i\partial X^2) \right),$$

$$E'(\pm 1, \pm 1) = \frac{1}{\sqrt{3}} e^{\pm i(\sigma+\beta)} \left( E^x_{(\pm 1)} - \omega^{\pm 1} E^y_{(\pm 1)} + \frac{i\omega^{\mp 1}}{\sqrt{2}} (\partial X^1 \pm i\partial X^2) \right),$$

where $\omega = \exp[2\pi i/3]$ and

\textsuperscript{*} When we choose $\epsilon_{(1,0)(0,1)} = -1$, the vertex operator $E(1, 0)$ transforms into $-E(0, -1)$ under the $Z_4$-twist. The minus sign is not needed under a $Z_4$ outer-automorphism of the $SU(3)$ algebra if we choose $\epsilon_{(1,0)(0,1)} = -1.$
Here the phases $\alpha$ and $\beta$ are ambiguous.

Under the $Z_6$-twist, these elements are transformed as

$$
\begin{pmatrix}
  i\partial X^1 \\
  i\partial X^2
\end{pmatrix}
\rightarrow
U
\begin{pmatrix}
  i\partial X^1 \\
  i\partial X^2
\end{pmatrix}
=\begin{pmatrix}
  1 & 0 \\
  0 & -1
\end{pmatrix}
\begin{pmatrix}
  i\partial X^1 \\
  i\partial X^2
\end{pmatrix},
$$

$$
E'(\pm 1, 0) \rightarrow -\omega^{2\gamma} E'(\pm 1, 0), \quad E'(0, \pm 1) \rightarrow -e^{\pm i\gamma} E'(\mp 1, \mp 1),
$$

where $\gamma=\alpha+2\beta$. It is remarkable that the matrix $U$ does not commute with the antisymmetric tensor $B$. In such a case, Ref. 15) shows that the above transformation of the vertex operator $E'_a$ is represented as

$$
E'_a \rightarrow \exp\left[2\pi i V^i a^i - i\frac{\pi}{2} (Ua)^i C^{ij} (Ua)^j\right] E'_{\alpha a},
$$

so as to preserve the correct cocycle property, where the vector $V^i$ is some shift on the $SU(3)$ torus and the matrix $C$ is a symmetric tensor defined by

$$
a^i \alpha^j C^{ij} a^j = \frac{1}{2} a^i (B - UBU)^i a^j \quad (\text{mod 2})
$$

Thus the matrix $C$ satisfies $a^i C^{ij} a^j = 1$. Therefore, the matrix contributes to the phases of the transformation as $E'_a \rightarrow -e^{\pi i v a} E'_{\alpha a}$ only in the case $Ua=\pm(\alpha + \alpha_2)$ and otherwise it does not. As a result, if we choose the ambiguous phase $\gamma=-\pi/6$, we can easily obtain the shift $V^i = (\sqrt{2}/12, 0)$. Notice that each vertex operator is eigenstate of $\Theta^3$-twist and that $E'(1, 0), E'(0, 1)$ and $E'(-1, -1)$ have an eigenvalue $\omega$ under $\Theta^3$-twist and the others have $\omega^2$. In the new basis, the $\Theta^3$-twist is represented as $E'_a \rightarrow e^{\pi i v a} E'_a$, where $V^i = (\sqrt{2}/6, \sqrt{6}/6)$.

Representing the $Z_6$-twist as the above shift and the reflection of $X^2$, we consider 'shifted' ground states. At first it must be remarked that there is only one independent fixed line under the reflection of $X^2$ on the new coordinate space in the same way as the two-dimensional shifted $Z_4$ orbifold. Therefore, we cannot distinguish ground states in terms of the reflection.

For the $1/6$-shifted sector of the 'shifted' orbifold, the momentum $P^i$ is shifted by $\sqrt{2}/12$ from original $P^i$ values on the $SU(3)$ weight lattice, i.e., $\hat{P}^i = P^i + \sqrt{2}/12$ and the $1/6$-shifted states have conformal dimensions $h_B = (\hat{P}^i)^2/2 + 1/16$, where the second term is a contribution from the $Z_6$ twisted boundary condition of $X^2$. The conformal dimension takes the minimum values at $P^i = 0$.

For the $2/6$-shifted sector, momenta $\hat{P}^1$ and $\hat{P}^2$ are shifted by $(\sqrt{2}/6, \sqrt{6}/6)$, i.e., $(\hat{P}^1, \hat{P}^2) = (P^1 + \sqrt{2}/6, P^2 + \sqrt{6}/6)$ and $2/6$-shifted states have conformal dimensions $h_B = ((P^1)^2 + (P^2)^2)/2$. The conformal dimension takes the minimum value at $(\hat{P}^1, \hat{P}^2) = (-\sqrt{2}/3, 0)$ and $(\sqrt{2}/6, \pm \sqrt{6}/6)$. All of the shifted ground state with the above momenta are not eigenstates of the $Z_6$-twist, i.e., reflection of $X^2$. From those
Table II. Chiral primary state in $Z_3$ orbifold and $1^a$ models.

| Twist $m$ | SO(2) momentum $|p|$ | Twisted orbifold $|e>$ | Shifted orbifold $|\tilde{P}^a, \tilde{P}^{a'}>$ | $1^a$ model $|q, q', q''>$ |
|-----------|-----------------|------------------|-----------------|-----------------|
| 0         | 0/3             | $|0>$             | $|0, 0>$          | $|0, 0, 0>$      |
| 1         | 1/3             | $|\alpha>$       | $|\sqrt{2}/6, -\sqrt{6}/6>$ | $|1, 0, 0>$      |
|           |                 | $|\alpha>$       | $|\sqrt{2}/6, \sqrt{6}/6>$ | $|0, 1, 0>$      |
| 2         | 2/3             | $|\alpha>$       | $|\sqrt{2}/6, -\sqrt{6}/6>$ | $|1, 0, 1>$      |
| 3         | 3/3             | $|0>$             | $|0, 0>$          | $|1, 1, 1>$      |

states we can derive eigenstate of $\Theta$ as

$$
| -\frac{\sqrt{2}}{3}, 0 > , \frac{\sqrt{2}}{6}, \frac{\sqrt{6}}{6} > \pm \frac{\sqrt{2}}{6}, -\frac{\sqrt{6}}{6} >.
$$

Similarly, momentum $\tilde{P}^a$ of the 3/6-shifted sector is shifted by $\sqrt{2}/4$ and the conformal dimension of the 3/6-shifted states is obtained as $h_3=(\tilde{P}^a)^2/2+1/16$. The conformal dimension takes the minimum value at $\tilde{P}^a=\pm \sqrt{2}/4$ and we can derive eigenstates under the $Z_6$-twist from states with $\tilde{P}^a=\pm \sqrt{2}/4$. The fourth column of Table I shows only $Z_6$-invariant states. As a matter of course, a $U(1)$ charge of each state among the above is described as $\tilde{P}^a$ and the $m/6$-shifted ground states have the same conformal dimensions as the $\Theta^m$-twisted ground states of the twisted orbifold. At last, the $m/6$-shifted sector of the $Z_6$ super-orbifold has a conformal dimension $h=m/12$.

The 1/3-shifted sector of the two-dimensional shifted $Z_3$ orbifold has the same states as the 2/6-shifted ground states on the $Z_6$ orbifold, as shown in Table II. A $U(1)^2$ charge of each state is given by $(\tilde{P}^a, \tilde{P}^{a'})$.

§ 4. $N=2$ superconformal algebra

It is known that an $SU(2)$ Kac-Moody algebra with the level $k=4$ is represented by two free bosons $\phi$ and $\phi'$ as

$$
J_{\pm}=2i\partial \phi , \quad J_{\pm}=\sqrt{2}e^{\pm i\frac{1}{2}t}(e^{i\sqrt{2}\phi'}+e^{-i\sqrt{2}\phi'}).
$$

(4.1)

Following Ref. 17), we derive $N=2$ SCA elements of the minimal model with the level $k=4$ from the above Kac-Moody algebra as

$$
T_{(a)}=\frac{1}{2}((\partial \phi)^2+(\partial \phi')^2) , \quad J_{(a)}=i\sqrt{\frac{2}{3}}\partial \phi ,
$$

$$
G_{(a)}=\sqrt{\frac{2}{3}}e^{i\sqrt{2}\phi}(e^{i\sqrt{2}\phi'}+e^{-i\sqrt{2}\phi'}).
$$

(4.2)

Primary states of the $k=4$ minimal model have conformal dimensions $h$ and $U(1)$
charge $Q$ under the current $J_\omega$, which are given by

$$h = \frac{l(l+2) - q^2}{4(k+2)} + \frac{1}{2} s^2, \quad Q = \frac{q}{k+2} + s,$$

where $l$, $q$ = integer, $s = 0, \pm 1/2, 1$ and they satisfy $l \leq k$, $|q-2s| \leq l$ and $l + q + 2s = \text{even}$. NS primary states satisfying with $2h = Q$ are called chiral. In the case $k=4$, the chiral primary states have $h=q(4)/12$ and $Q=q(4)/6$ ($q=0, 1, \cdots, 4$).

As well known, $N=2$ SCA elements of the $k=1$ minimal model are represented by a free boson $\phi''$ as

$$T_\omega = -\frac{1}{2} (\partial \phi'')^2, \quad J_\omega = -i \partial \phi'',$$

$$G_\omega^+ = \sqrt{\frac{2}{3}} e^{i \frac{2}{3} \phi''}.$$

The theory has two chiral primary states, whose conformal dimensions $h$ and $U(1)$ charges $Q$ are labelled by a quantum number $q^{(1)}(=0, 1)$ as $h = q^{(1)}/6$ and $Q = q^{(1)}/3$.

On the other hand, $N=2$ SCA elements of the two-dimensional $Z_6$ twisted superorbifold are represented as

$$T = -\frac{1}{2} ((\partial X^1)^2 + (\partial X^2)^2), \quad J = i \partial H,$$

$$G^\pm = e^{\pm iH}(\partial X^1 \mp i \partial X^2). \quad (4\cdot4)$$

Transforming the $SU(3)$ Kac-Moody basis, we obtain $N=2$ SCA elements of the $Z_6$ shifted superorbifold, i.e.,

$$T = -\frac{1}{2} ((\partial X^1)^2 + (\partial X^2)^2), \quad J = i \partial H,$$

$$G^\pm = i \sqrt{\frac{2}{3}} e^{\pm iH}(e^{i/3}e^{\pm i a E}'(\mp 1, 0) + \omega^{\pm 1}e^{i/3}E'(0, \mp 1) + \omega^{\pm 1}e^{i(a+\beta)/3}E'(\pm 1, \pm 1)), \quad (4\cdot5)$$

where $E'(a, b) = \exp[i(aa_1 + ba_2) \cdot X']$ with $X' = (X^1, X^2)$.

Now let us compare this algebra elements $(4\cdot5)$ with the algebra elements of the $1\cdot4$ model, $(4\cdot2)$ and $(4\cdot3)$. It is easy to show correspondence between $J$ and $J_\omega + J_\omega$ if we identify the fields as follows:

$$H = \phi_\omega = \frac{1}{\sqrt{3}}(\sqrt{2} \phi + \phi''). \quad (4\cdot6)$$

Defining a field $\phi_\omega = (1/\sqrt{3})(\phi - \sqrt{2} \phi'')$, we can represent the supercurrent $G_\omega + G_\omega^+$ as

$$G_\omega^+ + G_\omega = \sqrt{\frac{2}{3}} e^{i \phi''}(e^{i \frac{2}{3} \phi} - e^{i \frac{2}{3} \phi'} + e^{-i \frac{2}{3} \phi'} + e^{-i \frac{2}{3} \phi}). \quad (4\cdot7)$$

It coincides with the supercurrent $G^+$ of Eq.$(4\cdot5)$' up to an overall phase if we set adequate phases to $\alpha$ and $\beta$ and identify the fields as

$$X^1 = \phi_\omega, \quad X^2 = \phi', \quad (4\cdot8)$$
in addition to Eq. (4·6). Similarly, we can obtain coincidence between $G^{(4)} + G^{(1)}$ and $G^{-}$ as well as correspondence between $T^{(4)} + T^{(1)}$ and $T$. Further it is remarkable that the algebra of the $Z_{6}$ shifted super-orbifold has another current $i\delta X^{1}$, i.e., the enhanced $U(1)$ current. Its existence indicates reducibility of the SCA (4·5). Of course the current $i\delta X^{1}$ corresponds to the current $i\delta \phi_{-} = (1/\sqrt{2})(J^{(4)} - 2J^{(1)})$. The identification of the fields (4·6) and (4·8) gives relations between the quantum numbers labelling the chiral primary states. As a result, we obtain relations,

$$\bar{p}^{1} = q^{(4)} + q^{(1)} \over 3, \quad \bar{P}^{1} = \frac{1}{6\sqrt{2}}(q^{(4)} - 4q^{(1)}). \tag{4·9}$$

Under Eq. (4·9), correspondence between chiral primary states and shifted ground states is found in Table I.

Equation (4·5) also represents $N=2$ SCA elements of the two-dimensional $Z_{3}$ shifted super-orbifold. As shown in Ref. 5), these elements correspond to a sum of three copies of the $k=1$ SCA elements (4·3), where chiral primary states of each algebra are labelled by $q^{(i)}(i=1, 2, 3)$. It is remarkable that the $Z_{3}$ orbifold also has other two currents $i\delta X^{1}$ and $i\delta X^{2}$, too. That suggests reducibility of the $N=2$ SCA. We can derive relations between quantum numbers from the identification, i.e.,

$$\bar{p}^{1} = \frac{1}{3}(q^{(1)} + q^{(2)} + q^{(3)}), \quad \bar{P}^{1} = \frac{\sqrt{2}}{6}(q^{(1)} + q^{(2)} - 2q^{(3)}), \quad \bar{P}^{2} = \frac{\sqrt{6}}{6}(q^{(1)} - q^{(2)}). \tag{4·10}$$

Under the relations, correspondence of the states is summarized in Table II for the sake of discussions in the next section.

Next, we discuss Yukawa couplings of three chiral primary states in brief. For the 1·4 model, three chiral primary states denoted by $(q^{(i)}, q^{(j)})(i=1, 2, 3)$ are allowed to couple if they satisfy

$$\sum_{i=1}^{3} q^{(i)} = 1, \quad \sum_{i=1}^{3} q^{(j)} = 4. \tag{4·11}$$

Using the relation (4·9), the condition (4·10) leads to conditions as $\sum \bar{p}^{1} = 1$ and $\sum \bar{P}^{1} = 0$. The conditions are nothing but coupling conditions for the shifted $Z_{6}$ orbifold, that is, they imply the $SO(2)$ invariance and the enhanced $U(1)$ invariance. Similarly, coupling conditions of the 1·3 model are equivalent to those of the shifted $Z_{3}$ orbifold.

§ 5. 6-dim orbifold models and Gepner's models

In this section, we are going to discuss the correspondence of the $Z_{6} \times Z_{6}$ ($Z_{3} \times Z_{3}$) orbifold model at the maximal enhancement point and the 1·4 ($1^{5} 4^{5}$) Gepner's model in the ($2, 2$) heterotic construction. It is known that the models have the same gauge group $E_{6} \times U(1)^{8} \times E_{8}$ ($E_{6} \times U(1)^{8} \times E_{8}$) and the same numbers of 27 and 27 massless matter fields. The $Z_{6} \times Z_{6}$ ($Z_{3} \times Z_{3}$) super-orbifold can be regarded as a tensor
product of three 2-dim $Z_6$ super-orbifolds (one $Z_3$ and two $Z_6$ super-orbifolds) with some tensoring conditions. Similarly, the $1^3 4^3$ ($1^4 4^2$) SCFT can be regarded as a tensor product of three $1\cdot 4$ (one $1^3$ and two $1\cdot 4$) SCFTs. The equivalence of the 2-dim $Z_6$ super-orbifold and the $1\cdot 4$ SCFT (the $Z_3$ super-orbifold and the $1^3$ SCFT) has been proved with respect to the 2-dim $N=2$ SCA and the chiral primary states in the last section. However, physical massless states of the orbifold models and the Gepner’s models are not always derived from tensor products of the chiral primary states. Physical state conditions restricting the tensoring of the states do not explicitly relate to each other. Therefore, we compare the physical massless $27$ and $\bar{27}$ states derived from each physical state conditions and examine the correspondence of both the models.

At first, we study the $27$ massless matter fields of the $Z_6 \times Z_6$ orbifold model, where the whole string coordinates are divided by two transformations $\theta_1$ and $\theta_2$ with order 6. They are general terms for twists $\Theta \otimes 1 \otimes \Theta^{-1}$ and $1 \otimes \Theta \otimes \Theta^{-1}$ of the 3-dim complex coordinates of the internal bosonic strings, shifts $v_1=(0, 1, 0, -1)/6$ and $v_2=(0, 0, 1, -1)/6$ of right-moving $SO(8)$ momenta $P^i(t=0\sim 3)$ and shifts $V_1=(1, 0, -1, 0, \cdots, 0)/6$ and $V_2=(0, 1, -1, 0, \cdots, 0)/6$ of left-moving $E_8 \times E_8$ momenta $P^i(t=11\sim 26)$ in the $(2, 2)$ compactification, respectively.

A $\theta_1^k \theta_2^l$-twisted sector $T_{kl}$ has $k/6$, $l/6$ and $(6-k-l)/6$ twists for the three complex coordinates respectively, and has right-moving shifted momenta $\tilde{p}^i = P^i + kv_1^i + lv_2^i$ and left-moving shifted momenta $\tilde{P}^i = P^i + kv_1^i + lv_2^i$. In the shifted formalism, $1/6$, $2/6$ and $3/6$-twists of the complex coordinates are represented by the shift $(\sqrt{2}/12, 0)$ of the $SU(3)$ weight lattice with the reflection $X'^2 \rightarrow -X'^2$, the shift $(\sqrt{2}/6, \sqrt{6}/6)$ and the shift $(\sqrt{2}/4, 0)$ with the reflection $X'^2 \rightarrow -X'^2$ respectively, as shown in § 3. Making use of the shifted momenta $\tilde{P}^i (i=1\sim 6)$ of the three $SU(3)$ weight lattices, we obtain massless conditions of the $27$ and the $\bar{27}$ representations for the $T_{kl}$ sector as

$$
\frac{1}{2} \sum_{i=1}^{6} (\tilde{P}^i)^2 + \frac{1}{2} \sum_{i=0}^{3} (\tilde{p}^i)^2 + c_{k+l} = \frac{1}{2} (P_0^6)^2 + h_{(6)} + h_{(6-k-l)} = \frac{1}{2},
$$

$$
\frac{1}{2} \sum_{i=1}^{6} (\tilde{P}^i)^2 + \frac{1}{2} \sum_{i=11}^{26} (\tilde{P}^i)^2 + N + c_{k+l} = \frac{1}{2} \sum_{i=15}^{26} (P^i)^2 + h_{(6)} + h_{(6-k-l)} = 1,
$$

where $c_{k+l}$ denote the normal ordering constant caused by the reflections of $X'_i$, $X_i$ and $X'_i$, and $h_{(m)}$ is the conformal dimension of the $m/6$-shifted state of the 2-dim super-orbifold model. The NS chiral primary state has $h_{(m)} = m/12$, as derived in the previous section. Any $m/6$-shifted states have corresponding chiral primary states of the $1\cdot 4$ model as shown in Table I. When the shifted states are replaced by the corresponding states, the massless conditions of the $(1\cdot 4)^3$ model are the same as Eqs. (5.1) as far as chiral primary states are considered.

The physical massless states should be invariant under the transformations $\theta_1$ and $\theta_2$. In the $(2, 2)$ compactification, the $T_{kl}$ twisted state induces phases:

$$
\Phi_1 \exp[2\pi i (\sum v_1^i (p^i + kv_1^i + lv_2^i) - \sum V_1^i (P^i + kV^i_1 + lV_2^i))],
$$

$$
\Phi_2 \exp[2\pi i (\sum v_2^i (p^i + kv_1^i + lv_2^i) - \sum V_2^i (P^i + kV^i_1 + lV_2^i))],
$$

where $\Phi_{1,2}$ denote the normal ordering constant caused by the reflections of $X'_i$, $X_i$ and $X'_i$, and $h_{(m)}$ is the conformal dimension of the $m/6$-shifted state of the 2-dim super-orbifold model. The NS chiral primary state has $h_{(m)} = m/12$, as derived in the previous section. Any $m/6$-shifted states have corresponding chiral primary states of the $1\cdot 4$ model as shown in Table I. When the shifted states are replaced by the corresponding states, the massless conditions of the $(1\cdot 4)^3$ model are the same as Eqs. (5.1) as far as chiral primary states are considered.
under the $\theta_1$ and $\theta_2$ transformations respectively, where $\Phi_1$ and $\Phi_2$ indicate contributions from the complex 3-dim bosonic coordinates\textsuperscript{18,19}. The 27 states have no contribution from the shifts of the $SO(8)$ and the $E_8 \times E_8$ shifted momenta because of cancellations of their contributions. Since the shifted momenta $\tilde{P}'_k$ and $\tilde{P}'_\ell$ should belong to the same conjugacy class and are identical in this case, cancellations also occur on $\Phi_1$ and $\Phi_2$ with respect to the shifts of the $SU(3)^3$ shifted weights. Therefore, the phases of 27 massless states are derived only from the reflections of $X'^2$, $X'^4$ and $X'^6$.

The 2-dim states of the fourth column of Table I are invariant under the reflection, so the physical 27 states are obtained from tensor products of such invariant states. We obtain 83 states in this way. Also physical are the states which are not invariant under each 2-dim reflection but the total $\theta_1$ and $\theta_2$ transformations. Although $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$ orbifold models have no such state, the $\mathbb{Z}_6 \times \mathbb{Z}_6$ orbifold model has it in the $T_{22}$ sector and it is written with the shifted momenta as

$$\left(\begin{array}{c}
\sqrt{2}/6, \sqrt{6}/6 \\
\sqrt{2}/6, -\sqrt{6}/6
\end{array}\right). \quad (5.3)$$

(In the twisted formalism, this 27 massless state is represented by $\otimes_{i=1}^3 (|a_i\gamma_i - \bar{a}_i\gamma_i|).$

Physical states of the Gepner’s models are described by vectors $V = V_R \otimes V_L$ with

$$V_R = (\lambda; q_1^{(4)}, q_2^{(4)}, q_3^{(4)}, q_4^{(4)}, s_1^{(4)}, s_2^{(4)}, s_3^{(4)}, s_4^{(4)}, s_5^{(4)}, s_6^{(4)}), \quad (5.4a)$$

$$V_L = (\Lambda; q_1^{(4)}, q_2^{(4)}, q_3^{(4)}, q_4^{(4)}, s_1^{(4)}, s_2^{(4)}, s_3^{(4)}, s_4^{(4)}, s_5^{(4)}, s_6^{(4)}), \quad (5.4b)$$

where $\lambda$ and $\Lambda$ are $SO(2)$ and $SO(10)$ weight vectors respectively and they are related by the heterotic condition. The physical state condition is given by a relation:

$$V_L = V_R + m_0 \beta_0 + m_i \beta_i \quad (m_0, m_i = \text{integer}) \quad (5.5)$$

with $\beta_0 = (\bar{s}; 1, \ldots, 1; 1, \ldots, 1)$ and $\beta_i = (\nu; 0, \ldots, 0; 0, \ldots, 0, 2, 0, \ldots, 0)$ (2 in the $i$-th position), where $\bar{s}$ and $\nu$ are a conjugate-spinor and a vector of $SO(2)$, respectively. When $m_0 = m_i = 0$, we find 83 massless physical 27 states which correspond to the tensor product states of the 2-dim $\mathbb{Z}_6$ invariant states in the $\mathbb{Z}_6 \times \mathbb{Z}_6$ orbifold model. The correspondences of the states are obtained from the rows of the fourth column and the fifth column in Table I, where only the NS states are manifested. The other physical state of the (1-4)$^3$ model is given by

$$V_R = (0; 0, 0, 2, 2, 2; 0, 0, 0, 0, 0),$$

$$V_L = (\nu; 0, 0, 0, -4, -4, -4; 0, 0, 0, -2, -2), \quad (5.6)$$

as its representative, which corresponds to the state of Eq. (5.3). Here the states $(i, q, s) = (2, -4, -2)$ in the $k=4$ minimal theory are transformed into the state $(2, 2, 0)$ by the $\mathbb{Z}_2$ automorphism $l \rightarrow k-l, \ q \rightarrow q+k+2, \ s \rightarrow s+2$. On the other hand, 2-dim states $|\sqrt{2}/6, \sqrt{6}/6\rangle - |\sqrt{2}/6, -\sqrt{6}/6\rangle$ in Eq. (5.3) and $|\sqrt{2}/6, \sqrt{6}/6\rangle + |\sqrt{2}/6, -\sqrt{6}/6\rangle$ in Table I are two eigenstates of the reflection $X'^2 \rightarrow -X'^2$ in the $\mathbb{Z}_6$ super-orbifold. These facts suggest that the $\mathbb{Z}_2$ automorphism corresponds to the reflection.
In this way we have completed the correspondence of total 84 states of 27 massless fields in the $Z_6 \times Z_6$ orbifold and the $(1 \cdot 4)^3$ models. There are no 27 states, but many singlet states whose correspondence could be proved in a similar manner.

Next, we study the $Z_3 \times Z_6$ orbifold model with the $(2,2)$ compactification, where the twists of the 3-dim complex coordinates are $\Theta^2 \otimes 1 \otimes \Theta^{-2}$ and $1 \otimes \Theta \otimes \Theta^{-1}$ and the shifts are $v_1 = (0,1,0,-1)/3$, $v_2 = (0,0,1,-1)/6$, $V_1 = (1,0,-1,0,\cdots,0)/3$ and $V_2 = (0,1,0,\cdots,0)/6$. The first 2-dim super-coordinate of the $Z_3 \times Z_6$ orbifold model is described by the $Z_3$ super-orbifold, while the second and the third super-coordinates are the $Z_6$ super-orbifolds. From the massless conditions such as Eqs. (5.1) and the invariant states of the $Z_3$ orbifold in Table II and the $Z_6$ orbifold in Table I, we get 68 physical 27 massless states which correlate with the left-right symmetric states ($m_0 = m_i = 0$) of the $1^3(1 \cdot 4)^2$ model. Using the similar way to the $Z_6 \times Z_6$ model, we find five $Z_3 \times Z_6$ invariant states which are not invariant with respect to each 2-dim transformation. Three of them belong to $T_{12}$, one to $T_{02}$ and one to $T_{04}$.

Supposing that the physical states of the $1^3(1 \cdot 4)^2$ model are represented as $V = V_R \otimes V_L$ with

$$V_R = (\lambda; q^{(1)}, q^{(3)}, q^{(4)}, q^{(6)}, q^{(4)}, s^{(1)}, s^{(2)}, s^{(3)}, s^{(4)}, s^{(4)}), \quad (5.7a)$$

$$V_L = (\Lambda; \bar{q}^{(1)}, \bar{q}^{(3)}, \bar{q}^{(4)}, \bar{q}^{(6)}, \bar{q}^{(4)}, \bar{s}^{(1)}, \bar{s}^{(2)}, \bar{s}^{(3)}, \bar{s}^{(4)}, \bar{s}^{(4)}), \quad (5.7b)$$

the five left-right asymmetric NS states are given by

$$V_R = (0; 0, 0, 0, 0, 2, 2; 0, \cdots, 0),$$

$$V_L = (v; 0, 0, 0, 0, -4, -4; 0, 0, 0, 0, -2, -2), \quad (5.8)$$

where the underlines indicate simultaneous permutations of $V_R$ and $V_L$. Examining quantum numbers of these states, we can confirm correspondences of the 73 physical 27 states of the $Z_3 \times Z_6$ and the $1^3(1 \cdot 4)^2$ models.

The $Z_6 \times Z_6$ orbifold model has no 27 physical state, but the $Z_3 \times Z_6$ orbifold model has one 27 state in the $T_{03}$ sector described as

$$|0, 0; 0, 0\rangle_{t=2,3} \otimes (|\sqrt{2}/4, 0; -\sqrt{2}/4, \sqrt{6}/2\rangle, +|\sqrt{2}/4, 0; -\sqrt{2}/4, -\sqrt{6}/2\rangle),$$

where the ket vectors denote $|\vec{P}_R^1, \vec{P}_R^2, \vec{P}_L^1, \vec{P}_L^2\rangle$, etc. (In the twisted formalism, this 27 massless state is represented by $|0\rangle \otimes (|\alpha_3 + \omega|a_3\rangle + |\alpha_3 + \omega^2|a_3\rangle + |\alpha_3 + \omega^3|a_3\rangle + \omega|a_3 + \omega a_3\rangle) \otimes (|\alpha_3 + \omega^3|a_3\rangle + \omega|a_3 + \omega^2 a_3\rangle).$) The corresponding state in the $1^3(1 \cdot 4)^2$ model is

$$V_R = (0; 0, 0, 0, 0, 3, 3; 0, 0, 0, 0, 0, 0),$$

$$V_L = (v; 0, 0, 0, 0, -3, -3; 0, 0, 0, 0, 0, 0). \quad (5.10)$$

In this manner, we have shown the correspondence of 27 and 27 states between the 6-dim $Z_3 \times Z_6$ orbifold model and the $1^3(1 \cdot 4)^2$ model. We could also make sure of the correspondence of the singlet states, similarly to the $Z_3 \times Z_3^0$ and $Z_4 \times Z_4$ orbifolds. Thus we would complete correspondences of all the massless physical states in the $Z_3 \times Z_6$ and $Z_6 \times Z_6$ orbifolds.
§ 6. Conclusion

We have constructed the shifted $Z_6$ orbifold model in two-dimensional space and then shown that the shifted $Z_6$ super-orbifold model is equivalent to the $1\cdot4$ model. The equivalence has been applied to the correspondence between the $Z_6 \times Z_6$ ($Z_3 \times Z_6$) orbifold model and the $(1\cdot4)^3(1^3 \cdot 1\cdot4)^3$ model in the $(2,2)$ heterotic construction. In addition, we have indicated that there is the orbifold counterpart to the $Z_2$ automorphism of the $k=4$ minimal model, i.e., the $Z_2$ automorphism of the $k=4$ minimal model corresponds to the relation between states with different eigenvalues under the $Z_2$ reflection of the $Z_6$ shifted orbifold model.

Similarly we could relate $(2,2) Z_6$ orbifold models to $(1\cdot4)^3$ models modded by some twists. Since Refs. 5), 6) and this paper have completed the proof of the equivalence between all the two-dimensional shifted $Z_N (N=3, 4, 6)$ super-orbifold models and all the $c=3$ tensor products of the minimal models ($1^3$, $2^2$ and $1\cdot4$ models), we could investigate geometrical aspects of models obtained by tensor products of the above $c=3$ theories. Further, it is very important to study relations among tensoring of the other minimal models, Kazama-Suzuki models and the other orbifold models. Thus we could make a chart of some region in the four-dimensional string vacua.

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