Note on Astronomical Refraction. By Alan Fletcher, M.A.

(Communicated by W. M. Smart.)

1. The usual equation giving refraction $R$ at apparent zenith distance $\zeta$, on the assumption that the surfaces of equal refractivity are concentric spheres, is

$$R = -\sin \zeta \int \frac{d\mu/\mu}{\sqrt{r^2\mu^2 - \sin^2 \zeta}},$$

where $\mu$ is the refractive index at distance $r$ from the centre, and, as elsewhere in this note, (a) the suffix $0$ indicates values at the place of observation, (b) the integration is to be performed outwards (so that $r$ increases from $r_0$ to $\infty$ and $\mu$ decreases from $\mu_0$ to $1$).

This equation may be written

$$R = -\sin \zeta \left\{ \frac{\sinh \nu}{\sqrt{2u + \cos^2 \zeta}} \right\}$$

in which

$$2u = \frac{r^2\mu^2}{r_0^2\mu_0^2 - 1}, \quad \nu = \log_\mu \mu$$

$u$ is zero when $r = r_0$, and increases with $r$ (except where temperature is increasing with height at more than about $1^\circ$ C. per 9 metres). If we write $\nu$ for the refractivity $\mu - 1$, $\nu'$ equals $\nu - \frac{1}{2}\nu^2 \ldots$ and differs little from $\nu$.

2. If we make the nearly Laplacian hypothesis that

$$\nu' = C \left( 1 + \frac{fu}{l} \right) e^{-u/\mu},$$

where $C, l, f$ are independent of $u$, we find by integration of (2) that

$$R = \frac{C}{\sqrt{2l}} \sin \zeta \left[ fT + 2(1 - \frac{1}{2}f - fT^3)\psi(T) \right]$$

where

$$T = \frac{\cos \zeta}{\sqrt{2l}}, \quad \psi(T) = e^{T^3} \int_T^\infty e^{-t}dt$$

Extensive tables of the function $\psi$ were constructed by Radon.*

3. It is well known that the refraction at a given moderate zenith distance depends for all practical purposes on the two independent variables $\nu_0$ and $H_0$, the latter being the ratio $l_0/r_0$ of the height $l_0$ of the homogeneous atmosphere to $r_0$. For example, Rayleigh † found

$$R = \nu_0(1 - H_0) \tan \zeta - \nu_0(H_0 - \frac{1}{2}\nu_0) \tan^3 \zeta,$$

which is the common element of all expressions for refraction and the fundamental formula of refraction for practical purposes; equivalent though frequently less direct expressions are a commonplace of

* Annales de l'Observatoire de Paris, Mémoires, 18, 1885.
† Phil. Mag., 36, 141, 1893.
refraction theory, inasmuch as (5) must be obtained from any expression for the refraction containing two or more adjustable constants when expanded in powers of \(v_0, H_0\) as far as \(v_0^2, v_0 H_0\) (to which order (5) is correct, the term in \(v_0^2 \tan \zeta\) vanishing); no reasonable theory differs from (5) by more than a few thousandths, hundredths, tenths of a second at \(\zeta = 60^\circ, 70^\circ, 75^\circ\) respectively. At rather larger zenith distances the refraction still depends to a considerable degree of accuracy simply on \(v_0\) and \(H_0\), though (5) ceases to sum up this dependence as accurately as it exists. At large zenith distances the refraction depends to an appreciable extent, and beyond \(\zeta = \approx 85^\circ\) to an undue extent for practical purposes, on the particular function \(v'(u)\) used in integrating (2), and no introduction of a third independent variable suffices to the same extent; nevertheless, the refraction depends mainly on \(v_0, H_0\) and the value of \(d\mu/dr\) in the first few kilometres above the ground, and we shall write \(r_0(-d\mu/dr)_0/v_0 = G_0\).

It remains to relate the quantities \(C, I, f\) occurring in the formula (4) with the quantities \(v_0, H_0, G_0\), which at a given place of observation are determined mainly by the atmospheric density, temperature, and vertical temperature gradient respectively; alternatively, with \(v_0, H_0\), and the horizontal refraction.

4. The identification

\[
C = v_0'
\]  

is implicit in (3), while putting \(\zeta = 90^\circ\) in (4) gives

\[
\frac{\sqrt{\pi(1 - \frac{1}{2}f)C}}{\sqrt{2l}} = \text{horizontal refraction}
\]  

From (3) and the definition of \(v'\) in (2) we have

\[
\frac{1 - f}{l} = \frac{1}{C} \left( -\frac{dv'}{du} \right)_0 = \frac{1}{C v_0} \left( -\frac{d\mu}{du} \right)_0,
\]

while from the definition of \(u\) in (2)

\[
(du)_0 = \frac{(dr)_0 + (d\mu)_0}{\rho_0},
\]

so that, using (6),

\[
\frac{1 - f}{l} = \frac{v_0 G_0}{\nu(v_0 - v_0 G_0)}
\]

which is approximately

\[
\frac{G_0}{1 - v_0 G_0}.
\]

5. The usual equation

\[
H_0 = -\int\frac{\rho}{\rho_0} d\left(\frac{r_0}{r}\right)
\]

where \(\rho, \rho_0\) are the densities at distances \(r, r_0\) from the centre, becomes, from the definition of \(u\) in (2),

\[
H_0 = -\int\frac{\rho}{\rho_0} d\left(\frac{\mu}{\mu_0 \sqrt{1 + 2u}}\right)
\]
Let us for the moment consider only terms of order \( \nu_0, l, \) and \( l^2 \) on the right. We may to this order put \( \rho/\rho_0 = \nu/\nu_0 = \nu'/\nu_0', \mu_0 = 1, \) so that

\[
H_0 = -\int \frac{\rho}{\rho_0} d(1 + \nu)(1 - u + \frac{3}{2}u^2)
\]

\[
= \int \frac{\rho}{\rho_0} (du - d\nu - 3udu)
\]

\[
= \int_{u=0}^{\nu_0} \frac{\nu'}{\nu_0} du' - \int_{\nu=\nu_0}^{0} \frac{\nu'}{\nu_0} d\nu' - 3\int_{u=0}^{\nu_0} \frac{\nu'}{\nu_0} udu',
\]

which, using (3) and (6), gives

\[
H_0 = (1 + f)l + \frac{1}{2}\nu_0 - 3(1 + 2f)l^2,
\]

or

\[
(1 + f)l = H_0 - \frac{1}{2}\nu_0 + 3(1 + 2f)l^2 \quad . \quad . \quad (10)
\]

To carry the approximation to a higher power in \( \nu_0 \) it is necessary to assume \( \rho \) proportional not to \( \nu \) but to \( \nu + kv^2 \), where \( k \) is a constant of which, for example, the value would be \( \frac{1}{2} \) or \( 0 \) if \( \rho \) were proportional to \( \mu^2 - 1 \) or \( \mu - 1 \) respectively. By expanding (9) in powers of \( \nu' \) and \( u \), the value of \( H_0 \) has been found as far as terms of order \( \nu_0^2, \nu_0l, \nu_0l^2, l^3 \); but the only correction to the right-hand side of (10) which might conceivably be sensible is found to be

\[
+ \{ (1 + \frac{1}{2}k)(1 + f) - \frac{1}{2}(1 + k)f^2 \} \nu_0 \quad . \quad . \quad (10a)
\]

In the presence of water-vapour the value of \( H_0 \) determined by (9) with \( \rho \) a function of \( \nu \) is not quite that determined by consideration of surface temperature; the slight difference has been investigated by Radau.*

6. In *Mécanique Céleste*, vol. iv., bk. x., ch. i., Laplace showed that the formula (4) corresponds, subject to certain approximations, to the relation between \( \mu \) and \( r \):

\[
\frac{\mu^2 - 1}{\mu_0^2 - 1} = \left( 1 + \frac{f\mu'}{l} \right) e^{-u'/r}
\]

\[
u' = 1 - \frac{\nu_0}{r} - \frac{\mu_0^2 - \mu^2}{2\mu_0^2}
\]

and with the following identifications of \( C, l, f \):

\[
C = \frac{\mu_0^2 - 1}{\mu_0^2 + 1} \quad . \quad . \quad . \quad (12)
\]

\[
(1 + f)l = H_0 - \frac{\mu_0^2 - 1}{4\mu_0^2} \quad . \quad . \quad (13)
\]

together with (7).

(12) gives \( C = \nu_0 - \frac{1}{2}\nu_0^2 \ldots \), which agrees with the exact form (6) except for terms of order \( \nu_0^3 \), which are quite insensible for all practical purposes; while (7) is an exact deduction from (4).

* *Annales de l'Observatoire de Paris, Mémoires, 19, 17, 1889.*
(13) gives \((1 + f)l = H_0 - \frac{1}{2}v_0 \ldots\). The correction \((10a)\) is minute, but the term \(3(1 + 2f)f^2\) in \((10)\), though small, is not always negligible.

7. If \((4)\) is a possible refraction formula, it must correspond with a definite hypothetical relation between \(\mu\) and \(r\), and therefore with definite expressions for \(\mu_0, H_0, G_0\) in terms of \(C, l, f\), provided in the case of \(H_0\) that we know the connection between refraction and density. The procedure here employed determines them exactly except in the case of \(H_0\), which is found to an improved degree of approximation. Laplace's formula thus becomes available for the purpose of integrating \((2)\), which is linear in \(\nu'\), so that the refractions corresponding with different relations \(\nu'(u)\) can be superposed. At the same time, for \(\zeta < 85^\circ\) the small differences (fractions of a second) which exist between the results of the formula and the tables based on Radau's more elaborate theory are reduced by about 40 per cent. by using \((10)\) in place of \((13)\), the comparison being made for the same values of \(v_0, H_0\), and either \(G_0\) or horizontal refraction. For \(\zeta > 85^\circ\) it becomes of decreasing importance that our fictitious atmosphere shall have the right \(H_0\), and of increasing importance that it shall have the right \(\mu\) in the first few kilometres; the formula agrees reasonably with the tables whichever equation is used for \((1 + f)l\), and there is little point in using \((10)\) rather than \((13)\); beyond \(\zeta = 85^\circ\) or even less the ability of any theory to predict the amount of the refraction rapidly dwindles.

8. When \(f = 0\) the formulae \((4)\) may be written

\[
\begin{align*}
R &= C \tan \zeta \cdot L(T) \\
T &= \frac{\cos \frac{\zeta}{\sqrt{2l}}}{L(T) = 2Te^{T^2} \int_T^\infty e^{-p} \, dp}.
\end{align*}
\]

\((14)\)

\(L(T)\) is Laplace's continued fraction, of which the asymptotic expansion for large \(T\) is

\[
I = \frac{1}{2T^2} + \frac{1}{(2T^2)^2} - \frac{1}{(2T^2)^3} + \ldots;
\]

\(\log_{10} L(T)\) varies only from 0 to 1.08 for the range (approximately \(T > 6\)) normally corresponding with \(\zeta < 80^\circ\). (6) still holds, and from (10) we see that \(l\) is now to be determined by

\[
H_0 = \frac{3}{2}v_0 + 3l^2.
\]

\((15)\)

At \(\zeta = 80^\circ\) these formulae give results differing by about \(\sigma''\cdot05\) from Radau's tables for mean temperature gradient at all temperatures and pressures; by the omission of \(3l^2\) from \((15)\) the difference is approximately doubled. Probably at this point the deviations of the refraction from the mean, due to fluctuations of temperature gradient in the lower atmosphere, are usually less than \(\sigma''\cdot2\), so that the formula express fairly accurately the practical dependence of the refraction on \(v_0\), \(H_0\) which exists up to about \(\zeta = 80^\circ\) (always on the assumption that the surfaces of equal refractivity are concentric spheres).