Stability of Envelope Soliton

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In this decade, many works have been made to show that the stable pulse-like waves which are called solitons play an important role in the development of one-dimensional nonlinear wave phenomena. The success of these attempts is dependent upon whether the soliton concerned is fully stable. It is known that the soliton is stable against a one-dimensional disturbance along the wave propagation. The two-dimensional stability has been studied by Kadomtsev and Petviashvili for a system which the Korteweg-de Vries equation applies. Zakharov has treated the two-dimensional stability of envelope solitons which display a remarkable property of permanence in a self-modulation process of nonlinear plane wave and has pointed out, by using a variational method, the possibility that envelope solitons can be unstable. In this paper we apply the reductive perturbation method to the two-dimensional stability problem and present a result more detailed than Zakharov's.

The complex envelope, $\phi$, of a self-modulated plane wave propagating along the $x$-axis in a nonlinear dispersive medium is described by a nonlinear Schrödinger equation,

$$i\partial_t + (\alpha' / 2) \phi_{tt} + (\lambda / 2k_0) \phi_{xx} + \gamma |\phi|^2 \phi = 0,$$

(1)

where the subscripts denote partial differentiations, $k_0$ the wavenumber of carrier wave, $\lambda$ the group velocity, $\lambda'$ its derivative with respect to $k$, $\gamma$ a constant giving the measure of nonlinearity, $\xi = x - \lambda t$ and $r_\perp$ the two-dimensional coordinate vector, $r_\perp = (0, y, z)$. By making use of the complex envelope $\phi$, the modulated plane wave is expressed as $\phi(\xi, r_\perp, t) \exp\{i(k_0 x - \omega t)\}$.

When $\gamma \lambda' > 0$, Eq. (1) has an envelope soliton solution vanishing at infinity:

$$\phi = S_\gamma = A \sech\{ (\gamma / \lambda')^{1/2} A (\xi - \lambda t) \} \times \exp\{i(\alpha' A^2 / 2 - i(\lambda'/2\lambda') \lambda t + iV \xi / \lambda') \}.\tag{2}$$

Consider a two-dimensional perturbation such that $A$ and $V$ are slowly varying functions of $r_\perp$ and $t$. If the wavenumber and frequency of the perturbation are sufficiently small compared with those of the unperturbed soliton, $(\gamma / \lambda')^{1/2} A$ and $\gamma A^2 / 2$, the perturbed solution is expected to be
little different from the solution (2). It is then anticipated that
\[ \phi(\xi, r, t) = A_\ast \left( A(\xi - \xi_0, r, t) \exp(i\theta + iv\xi/\lambda') \right) \]
(3)
\[ \theta_t = \gamma A^2/2 - \varepsilon^2 v^2 / (2k') \quad \xi_{0,t} = v, \]
(4)
\[ A = A_0 + \varepsilon a(\tau, \rho), \quad v = v(\tau, \rho), \]
(5)
where \( \varepsilon \) is a smallness parameter, \( \varepsilon \ll 1, \) \( \tau \) and \( \rho \) are the stretched variables to denote the slowness of \( t \)- and \( r \)-dependence of \( a \) and \( v, \) \( \tau = \varepsilon^{1/2} t \) and \( \rho = \varepsilon^{1/2} r. \) We solve Eq. (1) by substituting Eqs. (3) (5), setting
\[ g = g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \cdots \]
(6)
and solving the sequence of equations corresponding to the successive powers of \( \varepsilon. \)

In the lowest order, we have
\[ (\lambda' / \gamma) g_{0,tt} - g_0 + 2g_0^2 = 0, \]
(7)
where \( \gamma = A(\xi - \xi_0). \) Provided \( \gamma \lambda' > 0, \) this equation has a soliton solution vanishing at infinity, i.e.,
\[ g_0 = \text{sech} \left( (\gamma / \lambda')^{1/2} \xi \right), \]
(8)
which is equivalent to \( S_\gamma \) with \( V = 0 \) (compare with Eq. (2)).

Dividing \( g_1 \) into the real and imaginary parts, \( g_1 = R + iI, \) we get, in the order \( \varepsilon^{1/2}, \)
\[ (\lambda' / \gamma) R_{tt} - R + 6g_0^2 R = (2/\gamma A^3) \{ \lambda^{1/2} v g_0 + \varepsilon^{1/2} (\lambda A^2/2k') P_\rho^2 \delta g_0 \}, \]
(9)
\[ (\lambda' / \gamma) I_{tt} - I + 2g_0^2 I = - (2/\gamma A^3) \{ a_\bullet (g_0 + \gamma g_0) \}
+ \varepsilon^{-1/2} (\lambda A/2k') P_\rho^2 \delta g_0. \]
(10)
We now impose that \( R \) and \( I \) are bounded at \( \xi = \pm \infty. \) Equation (9) is multiplied by \( g_{0,\gamma} \) and integrated by parts with \( \gamma \) over \( (-\infty, \infty). \) Taking into account Eq. (10) and differentiating the equation so obtained with respect to \( t, \) we have
\[ v_{tt} - (\gamma A^3/3k') P_\rho^2 v = 0. \]
(11)
Applying the same procedure to Eq. (10) but replacing \( g_{0,\gamma} \) by \( g_0, \) we then obtain
\[ a_{tt} + (\lambda A^3/k') P_\rho^2 a = 0. \]
(12)
It is noted that Eqs. (11) and (12) are nothing but the condition for the boundedness of \( R \) and \( I \) at \( \xi = \pm \infty. \)

Equations (11) and (12) indicate that for \( \gamma > 0, \) the soliton is stable with regard to a perturbation with \( v \neq 0 \) and \( a = 0 \) (this mode is called "flutter") and unstable against a perturbation with \( v = 0 \) and \( a \neq 0 \) ("granulation"). This case \( (\gamma > 0) \) corresponds to the case of positive dispersion, \( \lambda' > 0, \) since the envelope soliton solution is possible only for \( \gamma \lambda' > 0. \) Whilst, in a medium with negative dispersion \( (\gamma < 0), \) the soliton exhibits a flutter-type of instability and is stable for granulations. It is concluded that the envelope soliton is, in general, not stable.