On the Adiabatic Nuclear Potential, I

Kazuhiko Nishijima

Department of Physics, Osaka City University

(Received August 30, 1951)

The "nuclear force", inspite of its longest career, is one of the most complicated unsolved problems in the meson theory. We will investigate this problem on a non-relativistic basis. First, we analyze the special natures of the phenomena, "nuclear forces". Then making use of the foregoing analysis, we discuss the problem qualitatively, for instance, what kind of approximation we should employ. As the results of these discussions, we can conclude that no single approximation such as weak coupling or strong coupling will be capable of explaining the phenomena, and that the higher order calculations would not be able to help the situation out of discrepancies, if the adiabatic nuclear potential computed up to 4-th order could not fit the experimental data in the low energy region where the weak coupling theory is expected to hold. Second, based on the above considerations, we calculate the 4-th order adiabatic nuclear potential by the method of canonical transformations. In order to obtain a meaningful 4-th order adiabatic potential, it is necessary that the 2nd order one can be derived in the Schroedinger approximation without referring to the Pauli approximation. Interesting is the result that thus derived potential agrees with the one derived by the S-matrix method.

§ 1. Introduction

It was not later than three years after the birth of Yukawa theory when the serious difficulty concerning the singularity of the meson potential became emphasized, and so disencouraging was this difficulty, as compared with the Coulomb potential, for both theoretical and experimental reasons, that the efforts of early meson physicists were concentrated on this problem. Recently, however, the laboratory studies of mesons were rapidly advanced, and most meson physicists are interested in the mechanism of the production and capture of π-mesons leaving the problem of "nuclear forces" untouched. Still we have reasons to believe that this phenomenon, though implicitly related to the properties of mesons, will give us some informations about the correct method to be employed in the meson problem as will be shown later. Accordingly, we will study the problem in this paper.

Historically various methods were proposed in order to get rid of the singularity which will be stated simply in the following.

First, we must stress the mixed meson theory due to Möller and Rosenfeld. It is interesting to remember the fact that later on this theory was
abstracted and developed to the mixed field theories. The cancellation of singularities in the mixed meson theory, however, is restricted only to the lowest order in the coupling constant, and moreover it is a general truth that such a theory breaks down in other phenomena.

Second, there appeared the method of "cut off" due to Bethe. This method, contrary to other philosophies, leaves the difficulty as unknown, and treats the problem rather phenomenologically, which though imperfect theoretically, seems to be the orthodox in such a kind of problems.

Third, recently the following fact was emphasized by several authors that the relativistic effect, or synonymously the retardation effect, diminishes the order of singularity. But at the present stage, we know the strong interaction between nucleon and meson fields, and unless the higher singularities originated in the higher order calculations can be removed at the same time, we can no more regard this method as the final one than in the mixed meson theory.

Besides, a non-linear meson theory was proposed quite recently by Schiff. This theory seems promising, but we know nothing about detailed results yet.

Thus we will confine ourselves to such extent that we can discuss with the present stage meson theory. As for the problem of singularity, it will not be solved unless the nature of the self field around a nucleon can fully be clarified, and we shall follow Bethe's standpoint for the time being.

We first discuss the method of approximation, i.e., which one of the weak, intermediate, and strong couplings will be the best.

As for the problem of nuclear forces, no single approximation will do, i.e., we must prepare every variety of approximation for every variety of condition. First we will examine if the weak coupling approximation is correct in the low energy region. Were it correct the main behaviour will be determined by the second plus fourth order potential. (§ 2)

The next problem is the computation of the 4-th order potential in the weak coupling approximation. As the method of computation, we make use of the canonical transformations. And in order to clarify what transformations we must use, we review the derivation of the 2nd order potential. (§ 3)

Based on the method of derivation of the 2nd order potential, further transformations for the 4-th order potential are studied. (§ 4)

By the above studied transformations, 4-th order potential is discussed. (§ 5)

And then the physical interpretation of the potential is given. (§ 6)

Finally we compute the 4-th order potential in individual cases. (§ 7)

§ 2 Qualitative discussions

(1) We first consider the reason why the concept of "potential" is necessary. It is almost certain that the mutual interaction between nucleons can be explained only on the basis of the meson theory, so that if we want to solve
the problem of a nucleon system, it is necessary to take account of the meson field. It turns out to be a quite complicated many body problem which, of course, we cannot solve at the present stage.

Therefore some convenient method to eliminate the meson field is desired. It will perhaps be impossible to perform such a procedure rigorously, but to some extent of approximation it will be possible.

From the standpoint of field theory, it corresponds to an approximate separation of nucleon and meson fields by some suitable method such as the canonical transformation.

In this connection, the nuclear potential plays an important role as the implicit representative of the eliminated meson field, and the mathematical treatment becomes much simpler than in the original form. Moreover it is established experimentally that the concept of nuclear potential is useful in describing the nucleon-nucleon interaction.

Next it is known that the Born approximation is meaningless in the low energy nucleon-nucleon scattering. Theoretically, calculations of the covariant S-matrix is desired which, however, inevitably reduces to lower order Born approximations. Thus the non-relativistic potential is more useful in the analysis of the low energy nuclear forces. This is the second reason.

(2) Second, we will consider the special natures of nuclear forces. Although the character of nuclear forces so seriously depends on the nature of the meson field, only virtual mesons are concerned in these phenomena. While in other mesonic phenomena such as production or capture of π-mesons, we see π-mesons appear not virtually but really at least in one of the initial and final states. Thus these phenomena must involve high energy states in their processes in contrast to the nuclear force. What is essentially important is the fact that the nuclear force is the only low energy mesonic phenomenon.

(3) Third, we are concerned with the force range. Now, we define the essential 2n-th order nuclear force as the phenomenon in which μ mesons are interchanged simultaneously between two nucleons.

Other types of nuclear forces are regarded as radiative corrections to the essential nuclear force. Then the radiative corrections in low energies give only the renormalizations. (See Appendix.)

For this reason, we may investigate only the essential nuclear forces provided that we confine ourselves to the low energy phenomena. Our following discussions on the force range are due to Wick's idea. Suppose a two nucleon system, then the uncertainty of energy of the system due to 2n-th order essential nuclear force is given by

\[ \Delta E \gtrsim n \mu c^2 \]  

(1)

where \( \mu \) is the meson rest mass, and \( c \) the light velocity.

So the time of flight of mesons, \( \Delta t \) is limited by the following uncertainty
relation:  

\[ \Delta E \Delta t \sim \hbar. \]  

(2)

Combining (1) and (2), we see  

\[ \Delta t \sim \hbar / \mu \varepsilon. \]

Thus the distance of flight of mesons, \( R \) is seen to be  

\[ R \leq c \Delta t \sim (1/n) (\hbar / \mu \varepsilon) = (1/n) \mu^{-1}. \]  

(3)

The range of the 2\( n \)-th order essential nuclear force is \( 1/n \) times of that of the 2\( n \)nd order. The above consideration is made by expansion in numbers of mesons but not referring to the perturbation method.

The point nucleon model employed in the above discussion, however, is not correct since a nucleon has its spread with a radius of about \( \hbar \), the nucleon Compton wave length, due to its Zitterbewegung.

For \( n \sim 6 \), we see that \( R \sim \hbar \), and the above consideration does not hold. The validity of the discussion will be at most to \( n \sim 3 \). The more mesons two nucleons interchange, the more uncertain the positions of nucleons will be due to their recoil.

In this way, the point nucleon model cannot be applied within about one third of the force range, and we need to use the velocity dependent potential in place of the ordinary static potential. We have estimated the validity region of the static potential to be outside of about one third of the force range (Cf. Bethe), however, it might be half the force range if other effects are taken into account.

Indeed, we must consider the dynamical effects due to meson clouds around nucleons besides the kinematical effects discussed above. This problem will be discussed later.

What we must notice next is the type of coupling between nucleon and meson fields. For instance, the discussions cannot be applied to the queer interaction \( \gamma_\nu \), i.e. Ps (ps).

Here the capital letter Ps denotes the type of the meson field, and the small letter ps in parentheses the type of coupling. The existence of the coupling \( \gamma_\nu \) makes the transitions from positive (or negative) energy states to negative (or positive) energy states easier than those from positive (or negative) to positive (or negative), and the former discussion breaks down. Thus the inequality (1) turns out to be  

\[ \Delta E \geq 2Mc^2, \quad (M: \text{nucleon rest mass}) \]  

(1')

and (3) to be  

\[ R \sim \hbar / 2Mc. \]  

(3')
As easily be seen from the above result, the meson clouds shrink together and the nucleon anomalous magnetic moment cannot be fit to the experiment.

To conclude, the former discussion is valid only when the 2nd order nuclear potential can be derived in the Schroedinger approximation without referring to the Pauli approximation.

Therefore we confine ourselves only to such couplings from now on.

(4) Fourth, we discuss what kind of approximation we should employ. This is a rather general problem, and we pick up only the low energy problem. The characteristic feature of the low energy phenomena is the applicability of the non-relativistic treatment together with the static (or adiabatic) approximation.

Now consider which approximation will be better, the weak coupling or the strong coupling. For this purpose, we first employ the weak coupling. Then the higher the order of approximation proceeds, the higher the order of singularity and shorter the force range will be.

Therefore, for a large separation of nucleons lower orders will be dominant because of shrinkage of the force range in higher orders. For an intermediary separation, lower orders turn out to be inferior since they are of low singularities and much higher orders will also be inferior because of their short force range. Finally for a suitably small separation higher orders will be dominant by their high singularities and the spread of nucleons, i.e. the criterion in terms of force range breaks down in this region.

Thus it is clear that the perturbation does never converge within some small separation, which cannot be determined in the present stage meson theory but is of great importance. We shall call this separation "the critical range" and denote it by $r_o$. The critical range is similar to the cut off radius in the phenomenological theory, and corresponds to the convergence radius of the perturbation calculation.

This is the dynamical limitation for the validity of static potential, strongly depending on the type of coupling contrary to the kinematical limitation discussed before.

As a whole, both limitations together determine the critical range.

From the above discussion, we can conclude that the approximation to be employed depends on the separation of nucleons, and that the weak coupling method cannot be applied within the critical range.

Although we do not know how to determine the critical range, we shall illustrate it by a simple model.

Suppose that the $2n$-th order nuclear potential is given by

$$V_n = (-g^2)^n \frac{F}{x_r} \left( \frac{x_r}{x_r^2} \right)^n,$$

where $g$ is the coupling constant, $F$ a constant with the dimension of energy. Then the whole potential is given by
In this case, the condition of convergence is readily seen to be
\[ g^2 e^{-x r} / x r < 1. \]

This inequality can be transformed into the condition for \( r \):
\[ r > r_c. \]

Thus we get the value of the critical range. In general, the larger the coupling constant \( g \), the larger the critical range \( r_c \), i.e. the narrower the domain of perturbation method. Inside the critical range, it is clear that other methods than the weak coupling should be employed. But in this simple example, we will use the method of analytic continuation. From the outside solution by the perturbation method, we get the following potential inside the critical range:
\[
V = \frac{F}{x r} \left( -\frac{g^2 e^{-x r}}{x r^2} \right) + \frac{g^2 e^{-x r}}{x r^2 + g^2 e^{-x r}}.
\]

In this case, the singularity at the origin is only \( r^{-1} \) in contrast to the prediction due to perturbation method.

In actual problems, we cannot use the method of analytic continuation since higher orders are unknown, but this situation will be similar.

We know, in this way, that the weak coupling theory cannot explain the phenomena singly, so we shall examine the strong coupling theory next. Well, let us study the behaviour of the nuclear forces at comparatively large separation near the force range by the conventional strong coupling theory. Since large momentum transfer won't occur in this large separation, the adiabatic approximation employed in this theory will give fairly good informations. According to Serber and Dancoff, we know that with
\[ x a \lesssim 0.1, \quad (a: \text{the radius of a nucleon}) \quad (4) \]
no value of the coupling constant gives spin dependent forces large enough to agree with experience.

In general, the strong coupling theory gives, for large enough separation, forces between two nucleons of the same type as those obtained from perturbation theory, but at closer approach the forces become ordinary.

Thus the behaviour of the potential predicted from the strong coupling theory does not give the correct information near the force range, while the nature at closer distance cannot be trusted since the adiabatic approximation is not valid in
this region.

In order to investigate the property at small distances, we must study the relativistic strong coupling theory which to our regret is not known yet.

We conclude from the above discussions that inside the critical range the weak coupling theory is not valid and outside the strong coupling theory is not valid. The knowledge concerning the interior region is furnished only by the relativistic strong coupling theory, for which we must abandon the hope at the present stage and our subject is limited to the outer region. For these reasons we will test the validity of the weak coupling theory in the outer region. The treatment in the intermediate coupling theory is left open in this paper, though it is quite promising. We shall investigate how the adiabatic nuclear potential will be, provided that the weak coupling theory is valid.

If the perturbation treatment is allowed, we may suppose that

\[ \frac{1}{2} \leq x r_o \leq 0.3. \]

Of course, the smaller the value of \( x r_o \), the better the perturbation method. We assume for the moment \( x r_o \sim 0.3 \).

We employ the potential on the weak coupling theory in the outer region, and assume suitable cut off in the interior region. Notice here that the solution does not seriously depend on the mode of "cut off" as has been shown by Bethe. Otherwise the cut off procedure will lose its meaning.

The most difficult question in the perturbation treatment is to which order we must perform the calculation. We compute up to 4-th order in this paper. If it does not show good agreement with experiments, then 6-th order calculation will be required. On the other hand, if it shows good agreement with experiments, then we fear the 6-th order calculation would destroy the agreement.

Fortunately we have reasons to believe that the 4-th order computation will be the decisive one for right or wrong in the low energy nuclear forces which we discuss next.

(5) We compute the nuclear potential up to 4-th order which is different from the 4-th order computation of the S-matrix, since the S-matrix calculated by this potential involves the repetition of the 2nd plus 4-th order potential. The calculation of the 4-th order S-matrix is meaningless, for we are interested in the low energy nuclear force, and the Born approximation is no more valid in this region.

From the relation between the force range and the order of nuclear force stated before, the range of 4-th order nuclear force is half that of 2nd order, i. e. \( (2x)^{-1} \), and 6-th order \( (3x)^{-1} \).

Let us call the shell bounded by two spheres with the radii \( x^{-1} \) and \( (2x)^{-1} \), the region A, the shell with the radii \( (2x)^{-1} \) and \( (3x)^{-1} \sim r_o \), the region B, and inside the sphere with the radius \( (3x)^{-1} \sim r_o \), the region C.
Then the forces acting in each region are:
in A: 2nd order nuclear forces,
in B: 2nd+4-th order nuclear forces,
in C: higher orders, velocity dependent forces and forces due to the strong coupling theory, and possibly of heavy mesons.

Our standpoint is to regard the region C as the cut off region. The experimental evidences show that the nuclear force in the region C is not so singular as has been predicted from the weak coupling theory. Moreover the volume of the region C is small compared with the whole volume of nuclear force, i.e.

$$\frac{\text{volume}(C)}{\text{volume}(A+B+C)} = \frac{1}{27}.$$  \hspace{1cm} (6)

Thus we see that the behaviour of the nuclear force in the region C or the mode of cut off will not seriously affect the results of the analysis, and the adiabatic nuclear potential computed up to 4-th order will give us a good criterion about the validity of the weak coupling meson theory of nuclear forces.

As will be shown later, the 4-th order potential in the case of Ps (pv) is large compared with the 2nd order one, and is comparable even near the force range.

If this largeness is due to some inevitable reason, such as differentiations, the higher orders will diverge even in the neighbourhood of the force range, i.e., \(x_F \sim 1\), but if it is due to some accidental reason, then higher orders will be small and the perturbation method converges. At any rate, it is the question whether the 4-th order potential will fit to the experiments or not. For some reason the intermediate coupling theory seems to be best, and both the weak and strong coupling theories will be ruled out. And it seems to us that the validity of the weak coupling theory depends mainly on the type of coupling and only slightly on the value of the coupling constant.

\section*{§ 3. Second order potentials}

As for the 2nd order potentials, we need not repeat calculations since they are well known, but we review them to get some instructive informations for the derivation of the 4-th order potentials.

The typical method to compute the 2nd order potentials are (1) the perturbation method which has developed into Feynman's \(S\)-matrix theory, and (2) the method of canonical transformations originated by Möller and Rosenfeld and developed into Tomonaga-Schwinger theory.

The former is simpler and convenient to analyze scattering problems, and Nambu computed the 4-th order potential on this method. But the concept of
potential is too non-relativistic by nature to compute by the relativistic S-matrix theory, and the potential has the same transformation property with energy as a part of the Hamiltonian, quite different from the invariant S-matrix.

Moreover, the essential problem is how to separate the 4-th order potential from the repetition of the 2nd order potential.

For these reasons, we are inclined to choose the latter method and in fact we do so. In the method of canonical transformations, the original Hamiltonian is transformed into a more convenient form of other Hamiltonian without changing its transformation property and the separation of the 4-th order potential is automatically done.

We start from the Tomonaga-Schwinger equation for meson-nucleon system:

\[
\frac{i \partial \Psi[\sigma]}{\partial \sigma(X)} = (H_1(X) + H_2(X)) \Psi[\sigma]. \tag{7}
\]

where \(H_1\) and \(H_2\) are Hamiltonian densities of the 1st and 2nd orders in the coupling constant, especially \(H_2\) is added by the requirement of the integrability condition. We employ the unit \(\hbar = c = 1\) from now on.

In order to derive the 2nd order potential, we make use of the following customary transformation:

\[
\Psi[\sigma] = \exp \left( -i \int_0^\sigma H_1(X) dX \right) \Psi_0[\sigma]. \tag{8}
\]

For convenience in the non-relativistic approximation, we employed the following notations:

\[dX = dx \, dy \, dz \, dt, \quad dx = dx \, dy \, dz.\]

By the transformation (8), the equation (7) is transformed into

\[
\frac{i \partial \Psi_\sigma}{\partial \sigma(X)} = \left( H_1(X) - \frac{i}{2} [H_1(X), \int_0^\sigma H_1(X') dX'] \right) \Psi_\sigma[\sigma]. \tag{9}
\]

The nuclear force is obtained by taking the two nucleon, no meson part from the above Hamiltonian, i.e.

\[V_1(X) = \langle H_2(X) - \frac{i}{2} [H_1(X), \int_0^\sigma H_1(X') dX'] \rangle_{\sigma, 0}. \tag{10}\]

For instance, in the case of neutral spinless meson theory

\[H_1 = f \, W \phi + \frac{\phi}{x} M_\mu \frac{\partial \phi}{\partial x_\mu}, \quad H_2 = \frac{1}{2} \left( \frac{\phi}{x} \right)^2 (M_\mu n_\mu)^2, \tag{11}\]

where \(\phi\) is the wave function of the meson field, and \(W, M_\mu\) are the bilinear forms of nucleon wave functions. \(n_\mu\) is the unit normal of the space-like surface.
σ at a point X.

Inserting (11) into (10), we face the following type of integrals:

$$\int \sigma W(X) \sigma(X-X') W(X')dX',$$

where $\sigma$ is the $\sigma$-function of the meson field defined by

$$\sigma(X) = \frac{1}{(2\pi)^3} \int \frac{dk}{k_0} e^{ikx} \sin k_0 t,$$

and

$$\sigma^{(3)}(X) = \frac{1}{(2\pi)^3} \int \frac{dk}{k_0} e^{ikx} \cos k_0 t,$$

with $k_0 = \sqrt{k^2 + x^2}$. Gothic letters refer to three dimensional vectors throughout this paper.

The integrations are readily performed by means of the following formulae:

$$\int \sigma(X-X') \cdot F(X')dX' = \frac{F(X)}{x^2 - \Box},$$

$$\int \sigma(X-X') \cdot \frac{\partial}{\partial x_\mu} F(X')dX' = \frac{\partial \mu F(X)}{x^2 - \Box},$$

$$\int \sigma(X-X') \cdot \frac{\partial^2}{\partial x_\mu \partial x_\nu} F(X')dX' = \frac{\partial_\mu \partial_\nu F(X)}{x^2 - \Box} - F(X) \cdot \eta_\mu \eta_\nu.$$ (13c)

Applying these formulae to the case of (11), we have

$$V_2 = -\frac{f_\sigma^2}{4} \left\{ W_\sigma \frac{W_\sigma}{x^2 - \Box} + \frac{f_\sigma}{4x} \left( \left\{ W_\sigma \frac{\partial_\mu W_\sigma}{x^2 - \Box} - \left\{ \frac{\partial_\mu W_\sigma}{x^2 - \Box}, M_\mu \right\} \right\} \right\}$$

$$+ \frac{1}{4} \left( \frac{\sigma}{x} \right)^2 \left\{ M_\mu \frac{\partial_\nu \partial_\mu M_\nu}{x^2 - \Box} \right\}.$$ (14)

It must be noticed that the normal dependent term appearing in the right hand side of (13c) just cancels $H_\sigma$. Terms like $\partial_\mu M_\mu$ can be simplified by means of Dirac equation.

In order to change $V_2$ into non-relativistic form, we have only to employ the Schroedinger or Pauli approximation, and to substitute like

$$x^2 - \Box \to x^2 - \Delta,$$

which is the definition of the adiabatic approximation.

What we must notice here is the situation that the nucleons take only positive energy states all over the process in the 2nd order nuclear force, which no
more holds in the 4-th order. For instance, cases in which nucleons take negative energy states arise in the 4-th order as has been suggested in §2, and indicated in fig. 2. The coupling \( \gamma \) is an example. Thus the substitution (14) must carefully be performed, and we confine ourselves only to the cases in which the contributions due to the change of sign of energy in the virtual states can be neglected, i.e. the cases in which the 2nd order potential can be derived in the Schroedinger approximation. The adiabatic approximation holds only when the nucleons move so slowly compared with meson velocities that nucleons can be regarded as rest. Of course, this approximation can be applied only in the low energy regions. And the direct interaction \( H_2 \) required by the mathematical condition has no physical meaning as has been shown in this section, so that we omit this term in the following calculations.

**§4. Method of canonical transformations**

Based on the qualitative discussions developed above, we shall compute the 4-th order nuclear potential. We first take up Bloch-Nordsieck transformation.

We start from the Tomonaga-Schwinger equation also in this section:

\[
i \frac{\partial \Psi[\sigma]}{\partial \sigma(X)} = H(X)\Psi[\sigma].
\]  
(15)

We understand that the direct interaction \( H_2 \) is dropped already, and apply the first Bloch-Nordsieck transformation (8) to the equation (15), i.e.

\[
\Psi[\sigma] = U_1[\sigma] \Psi_1[\sigma], \quad U_1[\sigma] = \exp\left(-i\int_0^\sigma H(X) dX\right).
\]  
(16)

then the transformed Hamiltonian is given up to 4-th order by

\[
H'(X) = -\frac{i}{2} \left[ H(X), \int_0^\sigma H(X') dX' \right] - \frac{1}{3} \left[ \left[ H(X), \int_0^\sigma H(X') dX' \right], \int_0^\sigma H(X'') dX'' \right] \\
+ \frac{i}{8} \left[ \left[ \left[ H(X), \int_0^\sigma H(X') dX' \right], \int_0^\sigma H(X'') dX'' \right], \int_0^\sigma H(X''') dX''' \right]
\]
(17)

\( H_1 \) contains the following processes:

1. nuclear forces,  
2. self-energy,  
3. Compton scattering and  
4. double
emission or absorption of mesons. (Cf fig. 3)

\[ \text{Nuclear force.} \quad \text{Self energy.} \quad \text{Compton scattering.} \quad \text{Double emission or absorption of mesons.} \]

\begin{center}
\begin{tabular}{cccc}
(V) & (S) & (T) & (R) \\
\text{nuclear force.} & \text{self energy.} & \text{Compton scattering.} & \text{double emission or absorption of mesons.} \\
\end{tabular}
\end{center}

\textbf{Fig. 3}

Now we drop all the divergent terms, for our calculations are non-relativistic and they give only renormalizations in low energies.

So we drop the term (S) in \( H'_2 \), moreover the diagram (T) does not contribute to the 4-th order nuclear force since we take into account only no meson states in both initial and final states. Important is the diagram (R), for its (2,0) part vanishes in this original form, whereas the iteration survives and contributes to the 4-th order potential, i.e. (R) is virtual in the 2nd order and turns out to be real in the 4-th order. Therefore we must eliminate (R) which gives meson clouds by the second Bloch-Nordsieck transformation. Notice that (R) contains the following real process if relativistically treated:

\[ N + N' \rightarrow \pi + \pi, \]

\( (N: \text{nucleon}, N': \text{anti-nucleon}, \pi: \text{meson}) \)

and this process must be left from the second transformation, but since our consideration is restricted to non-relativistic approximation, we can ignore this process. As for Bloch-Nordsieck transformations in a relativistic treatment, they are discussed in detail by Takeda.\(^{21}\)

Now we apply the second Bloch-Nordsieck transformation:

\[ \mathcal{U}[\sigma] = U_2[\sigma] \mathcal{U}_2[\sigma], \quad U_2[\sigma] = \exp(-i \int R(X) dX), \]

i.e.

\[ \mathcal{U}[\sigma] = U_1[\sigma] U_2[\sigma] \mathcal{U}_3[\sigma]. \]

Thus the transformed Hamiltonian becomes up to 4-th order as

\[ H^{II}(X) = V_2(X) + T(X) - i [V_2(X), \int R(X') dX'] - i [T(X), \int R(X') dX'] \]

\[ - \frac{i}{2} [R(X), \int R(X') dX'] + H'_2(X) + H'_4(X), \]

where \( R, T \) and \( V \) are terms in \( H'_2 \) corresponding to diagrams (R), (T) and
Taking up (2,0) part from (20), we obtain

\[ \langle H^{11}(X) \rangle_{20} = V_2(X) - \frac{i}{2} \left[ R(X), \int^\sigma R(X')dX' \right]_{20} + \langle H_4(X) \rangle_{20}. \tag{21} \]

The first term gives the 2nd order nuclear force, and the second term together with the third term give the 4-th order potential.

The advantage of the method of canonical transformations consists in the automatic separation of the 4-th order potential. And also the non-relativistic approximation in the calculation of the potential which is a purely non-relativistic concept has much benefit. The first is concerned with ambiguity of the potential,\(^{30,39}\) i.e., although the 2nd order static potential has its definite meaning, the non-static part can be dropped by a suitable contact transformation which modifies the form of the static part of the 4-th order nuclear force.

This ambiguity does not occur in our calculation, for the indefinite terms vanish in the adiabatic approximation. Moreover we can regard the problem as a pure two-body problem.

Next we will write down the diagrams of the 4-th order nuclear forces:

![Diagrams](Fig. 4)

We calculate only terms corresponding to the irreducible diagrams (A) and (B), for other diagrams give only renormalizations in low energies. For instance, the contribution from (V) vanishes in the adiabatic approximation in which nucleons are regarded as infinitely heavy, and (L), (S) give renormalizations of the coupling constant and the nucleon rest mass. (Cf. Appendix.) To pick up terms corresponding to the diagrams (A) and (B), we use the suffix 2N. Then the 4-th order adiabatic potential is found to be

\[ V_4(X) = V_a(X) + V_b(X), \tag{22} \]

where

\[ V_a(X) = -\frac{i}{2} \left[ R(X), \int^\sigma R(X')dX' \right]_{2N}, \tag{23a} \]

\[ V_b(X) = \langle H_4(X) \rangle_{2N}. \tag{23b} \]

(to be continued)
References

3) L. L. Foldy, Phys. Rev. 72 (1947), 125.
5) H. A. Bethe, Phys Rev. 57 (1940), 260, 390.
7) Y. Nambu, Prog. Theor. Phys. 3 (1948), 444.
9) L. Van Hove, Phys. Rev. 75 (1949), 1619.
20) See ref. (15).
22) F. Bloch and A. Nordsieck, Phys. Rev. 52 (1937), 54.
26) See ref. (1).
27) G. Takeda, Soryūshiron-Kenkyū 3 (1951), No. 1, 53.
28) See ref. (21).
29) S. T. Ma, Phys. Rev. 82 (1951), 275.