Application of the Non-Perturbative Renormalization Group to the Nambu-Jona-Lasinio/Gross-Neveu Model at Finite Temperature and Chemical Potential

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The chiral phase structure of the Nambu-Jona-Lasinio/Gross-Neveu model at finite temperature \( T \) and finite chemical potential \( \mu \) is investigated using the (Wilsonian) non-perturbative renormalization group (NPRG). In the large \( N_c \) limit, the solutions of NPRG with various cutoff schemes are given. For a sufficiently large ultraviolet cutoff, the NPRG results coincide with those of the Schwinger-Dyson equation and have little cutoff scheme dependence. Next, to improve the approximation, we incorporate the mesonic fluctuations. We introduce the auxiliary fields for mesons and then derive NPRG equation for finite \( N_c \).

The chiral phase structure on the \((T, \mu)\) plane beyond the leading order of the \(1/N_c\) expansion is investigated in the sharp cutoff limit. The \(N_c\) dependence of the chiral phase diagram is obtained.

§1. Introduction

Explorations of the phase structure of hot/dense QCD and its toy models are of theoretical interest, and they are relevant to the heavy ion experiments planned at RHIC and LHC. At finite temperature and density, the vacuum is expected to move to the quark gluon plasma (QGP) phase via the phase transition, where the chiral symmetry is recovered and quarks and gluons are deconfined. Since the phase transition to the QGP phase is a non-perturbative phenomenon, we need a non-perturbative analysis to understand it. Unfortunately, there are not many methods to deal with such problems: lattice Monte Carlo simulations, the Schwinger-Dyson equation (SDE), the \(1/N\) expansion, \(\epsilon\) expansion. The non-perturbative renormalization group (NPRG) is also one of the methods for such a purpose. The effectiveness of the NPRG method in the non-perturbative phenomena has been investigated by many authors. 1), 2)

Non-perturbative renormalization group equations describe the response due to the change of the infrared momentum cutoff \( \Lambda \) and can be written down exactly. They are the functional differential equations for the Wilsonian effective action in which the quantum correction from the high energy modes \((p > \Lambda)\) are already incorporated. In practical analysis, we approximate the theory space, the functional space of the effective action, and project the renormalization group equation (RGE) onto this subspace. By enlarging this subspace, we can improve the approximations sys-
tematically. In some cases, the evaluated physical quantities converge rapidly under such a process. This is an advantageous feature of the NPRG method compared to asymptotic series, e.g. perturbation theory, $\epsilon$ expansions and $1/N$ expansions.

In this article we investigate the chiral phase structure of the Nambu-Jona-Lasinio (NJL)/Gross-Neveu (GN) model at finite temperature and finite chemical potential. The exploration of the phase diagram is of fundamental interest. If the analysis is extended to QCD, it is of use for the early universe, the astrophysics of neutron stars and the physics of heavy ion collisions. We employ the NPRG method for the analyses. It is worth while to examine the applicability of the NPRG method, because there do not exist many other tools for non-perturbative analyses, and they do not necessarily work well in any situation.

In lattice Monte Carlo calculations, much exploration has been made of the system at finite temperature, while the behavior at finite density is much less understood. The non-vanishing chemical potential $\mu$ makes the fermion determinant a complex number, and therefore straightforward Monte Carlo methods cannot be applied. At present, two known candidates avoiding this difficulty, the Glasgow algorithm and the imaginary chemical potential $\mu = i\nu$ method, require much larger computer resources, but unfortunately do not bring any definite results. Most effort has been made using the SDE, but it is difficult to improve the approximation systematically.

This paper is organized as follows. In §2 we derive the evolution equation which is one of the non-perturbative renormalization group equations and explain the local potential approximation (LPA). In §3 we discuss the chiral phase structure in the large $N_c$ limit and compare with the results from the SDE for the fermion mass function $\Sigma(q)$. Due to the formal equivalence of the two methods in the large $N_c$ limit our large $N_c$ result should coincide with that of the SDE. We show that consistent results can be obtained in the framework of the NPRG method. In §4, we investigate the phase structure beyond the large $N_c$ approximation. The non-perturbative renormalization group method can approximately incorporate the higher order diagrams in the $1/N_c$ expansion within the local potential approximation. The $N_c$ dependence of the chiral phase diagram will be presented there. Section 5 is devoted to summary and discussion.

§2. Evolution equation and local potential approximation

There are three formulations of the non-perturbative renormalization group, the Wegner-Houghton equation, the Polchinski equation and the evolution equation. They are the continuous versions of the block spin transformation written in the momentum space and describe the response of lowering the infrared momentum cutoff $\Lambda$. We can find the exact form of the renormalization group equation (RGE) for the Wilsonian effective action and/or the effective average action. The latter one is the particle irreducible part of the Wilsonian effective action. By lowering the cutoff we have the effective action at large distance and incorporate the radiative corrections from the high energy modes. In this article, we employ the evolution equation in Ref. 12) and apply it to the Nambu-Jona-Lasinio (NJL)/Gross-Neveu
(GN) model\(^4,5\) at finite temperature \(T\) and chemical potential \(\mu\).

The generating functional of connected Green functions is

\[
W_{\Lambda}[\eta, \bar{\eta}] = \ln \int D\bar{\psi} D\psi \exp \left\{ -S_{\text{cut}}^{f}[\bar{\psi}, \psi] - S_{\text{bare}}[\bar{\psi}, \psi] + \bar{\eta} \cdot \psi - \bar{\psi} \cdot \eta \right\}, \tag{2.1}
\]

where \(S_{\text{cut}}^{f}[\bar{\psi}, \psi]\) is given by

\[
S_{\text{cut}}^{f}\Lambda[\bar{\psi}, \psi] = \int_{0}^{1/kT} d\tau \int d^{d-1}x \bar{\psi} \Delta_{f}^{-1}(-i\partial, \Lambda)\psi. \tag{2.2}
\]

Here \(\Delta_{f}^{-1}\) is a cutoff operator and has the property

\[
\Delta_{f}^{-1}(p, \Lambda) = \begin{cases} 0 & \text{for } p \gg \Lambda, \\ \infty & \text{for } p \ll \Lambda. \end{cases} \tag{2.3}
\]

At finite temperature, \(p_0\) is quantized to the Matsubara frequency,

\[
\omega_{f,n} = (2n + 1)\pi T \quad \text{or} \quad \omega_{b,n} = 2n\pi T. \tag{2.4}
\]

These forms are for fermions and bosons, respectively. Note that, since \(S_{\text{cut}}^{f}\Lambda\) preserves the chiral symmetry, the effective action also respects it. We choose the cutoff function \(C\) as

\[
C^{-1}(p/\Lambda) = \frac{f^{2}(p/\Lambda)}{1 - f^{2}(p/\Lambda)}, \quad \text{with} \quad f(p/\Lambda) = \exp[-a(p/\Lambda)^{2b}]. \tag{2.5}
\]

Since the parameter \(a\) can be absorbed in a redefinition of the cutoff \(\Lambda\), we fix it as \(a = 0.3\) in this paper. The parameter \(b\) is the cutoff scheme parameter.\(^\star\star\) Taking the derivative of Eq. (2.1) with respect to \(\Lambda\) and performing the Legendre transformation \(\tilde{\Gamma}_{\Lambda}[\bar{\psi}, \psi] + \bar{\psi} \cdot \Delta_{f}^{-1} \psi = \bar{\eta} \cdot \psi - \bar{\psi} \cdot \eta - W_{\Lambda}[\bar{\eta}, \eta],\) we get the evolution equation for the effective average action,

\[
\Lambda \frac{d}{d\Lambda} \tilde{\Gamma}_{\Lambda}[\bar{\psi}, \psi] = -\frac{1}{2} \text{str} \left[ \Lambda \frac{d}{d\Lambda} \Delta_{f}^{-1} \left( \Delta_{f}^{-1} + \tilde{\Gamma}_{\Lambda}^{(2)} \right)^{-1} \right], \tag{2.6}
\]

where \(\text{str}\) is the super-trace, which involves momentum (or coordinate) integration, Matsubara summation, spinor summation and color summation. \(\tilde{\Gamma}_{\Lambda}^{(2)}\) is a second (functional) derivative with respect to the fields \(\Phi = (\psi^T, \bar{\psi})\):

\[
\left( \tilde{\Gamma}_{\Lambda}^{(2)} \right)_{xy} \equiv \frac{\delta}{\delta \Phi_{x}} \tilde{\Gamma}_{\Lambda}[\Phi] \frac{\delta}{\delta \Phi_{y}}. \tag{2.7}
\]

This RGE possesses exact information about the response of the effective average action to the coarse graining. However, it is a functional differential equation and we cannot solve it without approximation. As a first step of our approximation, we

\(^\star\) In the following, we choose units such that Boltzmann constant, \(k\), is 1.

\(^\star\star\) In §3, we employ this smooth cutoff regularization to see the cutoff scheme \(b\) (in)dependence.
neglect higher derivative terms and keep the $Z$ factor to be unity. This is called the local potential approximation (LPA).\textsuperscript{(9)} This is the leading order of the derivative expansion.\textsuperscript{(13)} Since the NPRG preserves the homogeneous global symmetries, the effective average action $\tilde{\Gamma}_\Lambda[\bar{\psi}, \psi]$ also respects it. Hence the operator space is restricted to that of the chiral invariants. For example, the independent chiral invariant four-fermi operators are given as $O_1 = (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2$, $O_2 = (\bar{\psi}\gamma_i\psi)^2 + (\bar{\psi}\gamma_i\gamma^5\psi)^2$ and $O_3 = (\bar{\psi}\gamma_0\psi)^2 + (\bar{\psi}\gamma_0\gamma_5\psi)^2$\textsuperscript{*}. Here, color indices are omitted and $i$ is the spatial index running over $i=1,2,3$. In the LPA, the effective average action is

$$\tilde{\Gamma}_\Lambda[\bar{\psi}, \psi] = \int_0^{1/T} d\tau \int d^{d-1}x \left\{ \bar{\psi}(i\partial - i\mu\gamma_0)\psi - \sum_i \frac{G_i}{2N_c} O_i + \cdots \right\},$$

where $N_c$ is the number of colors. In the LPA, the four-fermi operators do not receive any corrections from the multi-fermi operators other than the four-fermi operators. The Feynman diagrams corresponding to the $\beta$ functions of the four-fermi coupling constants are displayed in Fig. 1. In the large $N_c$ limit, the Feynman diagrams (b) and (c) in Fig. 1 do not contribute to our $\beta$ functions.

§3. Phase structure in the large $N_C$ limit

In this section, we explore the phase structure of the Gross-Neveu model\textsuperscript{5) at finite temperature and chemical potential in the large $N_c$ limit. We attempt to apply the NPRG method to the GN model for $T \neq 0$ and $\mu \neq 0$, and we show that NPRG gives results consistent with the SDE.\textsuperscript{14)} In the large $N_c$ limit, only the first diagram (a) in Fig. 1 contributes to the $\beta$ function of the four-fermi couplings. The $\beta$ function of the scalar four-fermi coupling $G_s \equiv G_1$ is a function of $G_s$ alone:

$$\frac{d}{dt} \tilde{G}_s = -(d-2)\tilde{G}_s + 2\tilde{G}_s^2 I(a, b; \tilde{T}, \tilde{\mu}).$$

(3.1)

Here $t$ is the cutoff scale parameter, i.e. $\Lambda = \Lambda_0 \exp(-t)$. A profile of the threshold function $I(a, b; \tilde{T}, \tilde{\mu})$ is given in Appendix A. The characters with hats are the dimensionless coupling constants, e.g. $\tilde{G}_s = G_s / \Lambda^{d-2}$. The first term on the right-hand side (RHS) of Eq. (3.1) corresponds to the canonical scaling, and the second one to the radiative correction. At zero temperature and zero chemical potential, we find the two-phase structure by solving the RGE numerically. The RG flow diagram is shown in Fig. 2. There are two phases here divided by critical coupling, the strong

\textsuperscript{*} We consider only the case with a single flavor.
coupling phase and the weak coupling phase. In the strong coupling phase, the four-fermi coupling constant blows up to infinity at a finite scale $t$. By evaluating the effective potential of meson fields, we can recognize that the chiral symmetry is spontaneously broken in this phase. In the weak-coupling phase, the four-fermi coupling goes to zero, and chiral symmetry is not broken.

Let us now discuss hot and dense matter. The broken chiral symmetry at zero temperature and zero chemical potential is restored at some critical temperature $T_c$ and/or critical chemical potential $\mu_c$. We display the RG flow of the four-fermi coupling constant at finite temperature in Fig. 3. There is a critical temperature, below which the four-fermi coupling constant blows up to infinity. On the other hand, above the critical temperature, the four-fermi coupling decreases exponentially.

The temperature/chemical potential dependence of various quantities can be found by solving the RGE with the same initial condition as that for $T = \mu = 0$. There exists a critical temperature/chemical potential, above which the four-fermi coupling tends to zero, so that chiral symmetry is restored. We can estimate the critical temperature and the critical chemical potential by solving the RG flow equation for the scalar four-fermi coupling constant with some fixed initial condition, or, equivalently, a bare coupling constant. The initial condition should be given in the limit $\Lambda_0 \to \infty$. However, then we must calculate contributions of an infinite number of Matsubara modes, since $T/\Lambda = \tilde{T}$ goes to zero.\footnote{As seen from the explicit expression of threshold function in Appendix A, the contributions of high Matsubara modes are suppressed exponentially for a finite $\tilde{T}$. Hence for some high Matsubara modes, they make no contribution within the accuracy of numerical computation. In the $\tilde{T} \to 0$ limit, however, an infinite number of Matsubara modes contribute.} In the practical analysis, we solve the RG flow equations with common initial conditions at the sufficiently large but finite ultraviolet cutoff $\Lambda_0$. If $\Lambda_0$ is sufficiently large compared with $T$ and $\mu$, the solutions will reach the scaling region, where the renormalized information, i.e. the $\Lambda_0 \to \infty$ limit is obtained. This corresponds to tuning the bare four-fermi coupling
constant to the critical one: $\tilde{G}_s|_{\Lambda=\Lambda_0} = \tilde{G}_s^* + \delta \tilde{G}_s, \delta \tilde{G}_s \to 0$, where $\tilde{G}_s^*$ is the critical coupling constant at $T = \mu = 0$. For sufficiently large $\Lambda_0$ (or sufficiently small $\delta \tilde{G}_s$), the critical temperature and critical chemical potential depend proportionally on the common factor $\delta \tilde{G}_s^{-1/(2-d)}$, i.e. $T_c(\delta \tilde{G}_s) \propto \delta \tilde{G}_s^{-1/(2-d)}$, $\mu_c(\delta \tilde{G}_s) \propto \delta \tilde{G}_s^{-1/(2-d)}$. Thus the ratio of these quantities is independent of the initial value of the four-fermi coupling for the scaling region. The practical problem in the calculations is how large $\Lambda_0$ is needed to reach the scaling region. In Figs. 4 and 5, the temperature and chemical potential dependence of the threshold function $I$ is shown. At small dimensionless temperature $T/\Lambda < 0.03$, the threshold functions seem almost constant and have almost no temperature dependence. Therefore we expect to reach the scaling region if we set the initial condition as $\Lambda_0 > 33$ $T_c$. However, in practice, we must check whether the physical quantities scale correctly or not, since this may depend on the quantities to calculate. We perform the scaling check by calculating a ratio of critical temperature (or chemical potential) and $m_0$. Here $m_0$ is the dynamical fermion mass at $T = \mu = 0$.

We give the results in Fig. 6. If we employ the ultraviolet cutoff $\Lambda_0 = 333$ $T_c$, the critical points $(T_c/m_0, \mu_c/m_0)$ almost coincide with those of the SDE. Here $T_c$ is the critical temperature at $\mu = 0$. There is little cutoff-scheme dependence using this ultraviolet cutoff $\Lambda_0$. With the ultraviolet cutoff $\Lambda_0 = 333$ $T_c$ these quantities reach the scaling region. On the other hand, if we choose the ultraviolet cutoff $\Lambda_0 = 33$ $T_c$, the critical points $(T_c/m_0, \mu_c/m_0)$ slightly deviate from those of the SDE. $\Lambda_0 = 33$ $T_c$ is not sufficient to obtain scaled quantities. We can calculate the scaled critical temperature/chemical potential by setting the initial condition at $\Lambda_0 \geq 333$ $T_c$. We also plot the results in the sharp cutoff case (i.e. $b = \infty$) in Fig. 6.$^{**}$

$^*$ Other physical quantities, such as the fermion effective mass $m_0$ at $T = \mu = 0$ also depend proportionally on the factor $\delta \tilde{G}_s^{-1/(2-d)}$. For a detailed explanation, see Appendix B.

$^{**}$ They are the values at $T = \mu = 0$.

$^{***}$ A detailed explanation for the sharp cutoff case is given in § 4.
It should be noted that the above analyses are based on the flow equation for $1/\hat{G}_s$ rather than Eq. (3.1) for a technical reason. In the large $N_c$ limit, $1/\hat{G}_s$ coincides with the mass squared of the meson fields. Multiplying the RGE (3.1) by $-1/\hat{G}_s^2$, it can be rewritten as

$$
\frac{d}{dt} \left( \frac{1}{\hat{G}_s} \right) = (d-2) \cdot \frac{1}{\hat{G}_s} - 2I(a, b; \hat{T}, \hat{\mu}).
$$

(3.2)

Two phases, the chiral broken phase and the symmetric phase on the $(T, \mu)$ plane can be distinguished by the behavior of the solution of Eq. (3.2). If $1/\hat{G}_s$ goes to a negative value at sufficiently large $t$, it is the strong coupling (broken) phase. If $1/\hat{G}_s$ tends to positive infinity, it is the weak coupling (symmetric) phase. Although Eqs. (3.2) and (3.1) are essentially identical, there emerge differences in the analyses. If the chemical potential $\hat{\mu}$ is sufficiently large, the threshold function $I$ can take negative values. Thus even if $1/\hat{G}_s$ becomes negative, $1/\hat{G}_s$ may turn back to a positive value due to the negative $I$. Thus it should be regarded as a weak coupling phase. With the RGE Eq. (3.1), this behavior cannot be detected because once $G_s$ blows up to infinity, then by mistake we may identify it as the strong coupling phase. Indeed, for a large value of $\mu$, it gives an inaccurate (incorrect by about 10%) result. The analysis using $1/\hat{G}_s$ criticality is a more favorable method, and we may say it is a sort of "environmentally friendly renormalization group."\textsuperscript{15)

In Ref. 14), the chiral phase structure of the GN model in $2 \leq d < 4$ dimensions was calculated using the SDE. The effective potentials are calculated to leading order in the $1/N_c$ expansion. It is known that for $2 \leq d < 3$ the chiral phase transition is of first order at low temperature and large chemical potential, and of second order in other regions of the critical line. For $3 \leq d < 4$, the phase transition is of second order all along the critical line.

Using the NPRG we can also derive consistent phase boundaries in any space-time dimensions. For example, the chiral phase structure of the two-dimensional GN model\textsuperscript{15}) is shown in Fig. 7. In two dimensions, there is the first order critical line as well as the second order one. The phase boundary can be found only by exploring the effective potential $V$ of the collective coordinate $\sigma$, which is introduced in the next section. The second order critical point is where $\partial^2 V/\partial \sigma^2 |_{\sigma=0} = 0$ is satisfied, while the first order critical point is where the value of the effective potential at

\textsuperscript{15}) In two dimensions continuous symmetry cannot be spontaneously broken.\textsuperscript{16}) Thus we consider the phase transition with a discrete chiral symmetry $\psi \rightarrow \gamma_5 \psi$ in two dimensions.
the second minimum is equal to that at the first minimum at the origin. The stars in Fig. 7 correspond to the first order chiral phase boundary, and it terminates at the tricritical point. The filled circles and the open circles are where the signs of the meson masses at the origin $\langle \sigma \rangle = 0$, are changed, and they correspond to the $1/\hat{G}_s$ critical line. The filled circles are identical to the second order chiral phase boundary, but the open circles have nothing to do with the phase transition. The effective potential of the two-dimensional GN model has a gap structure on the first order phase boundary. An example of the RG evolution of the effective potential at a first order critical point is shown in Fig. 8.

§4. Phase structure beyond the $1/N_c$ leading order

In the previous section, we explored the chiral phase structure on the $(T, \mu)$ plane in the large $N_c$ limit. In this section, we improve the approximation beyond the $1/N_c$ leading order in the four-dimensional NJL model. Since we are interested not only in the phase boundary but also in the order of the phase transition, the CJT effective potential\(^{17}\) should be investigated. Let us start from the definition of
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the partition function,

\[ Z_A[J] = \int D\bar{\psi}D\psi \exp \left\{ -S_{\text{cut}}^t A - S_{\text{bare}} + \bar{\eta} \cdot \psi - \bar{\psi} \cdot \eta + \Sigma_{\sigma} \cdot \bar{\psi} \psi + \Sigma_{\pi} \cdot \bar{\psi} i\gamma_5 \psi \right\}, \]

where \( S_{\text{cut}}^t A, S_{\text{bare}} \) and \( J \) are the cutoff action defined in Eq. (2.2), the bare action, and the sources \( J = \{ \eta, \bar{\eta}, \Sigma_{\sigma}, \Sigma_{\pi} \} \), respectively. Here the bare action is taken to be

\[ S_{\text{bare}} = \int d^4x \left\{ \bar{\psi}(i\bar{\partial} - i\mu \gamma_0)\psi - \frac{G_s}{2N_c} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5 \psi)^2 \right] \right\}. \]

We introduce the auxiliary fields \( \sigma \) and \( \pi \) corresponding to the composite operators \( \bar{\psi} \psi \) and \( \bar{\psi} i\gamma_5 \psi \) by the Gaussian integral

\[ Z_A = \int D\sigma D\pi e^{-S_{\text{cut}}^b A - \int d^4x \left\{ \frac{1}{2} P^{-1}(\sigma - \Sigma_{\sigma} - P\bar{\psi}\psi)^2 + \frac{1}{2} P^{-1}(\pi - \Sigma_{\pi} - P\bar{\psi} i\gamma_5 \psi)^2 \right\}}, \]

where \( S_{\text{cut}}^b A \) is the cutoff action of the meson fields and is given by

\[ S_{\text{cut}}^b A = \int d^4x \frac{1}{2} \left( \partial_\mu \sigma C^{-1}(-i\partial/\Lambda) \partial_\mu \sigma + \partial_\mu \pi C^{-1}(-i\partial/\Lambda) \partial_\mu \pi \right). \]

Here \( C^{-1} \) is the cutoff function defined in Eq. (2.5). Note that since \( Z_A \) depends on the fields \( \bar{\psi} \) and \( \psi \) as well as on the sources \( \Sigma_{\sigma} \) and \( \Sigma_{\pi} \), the partition function is deformed by the insertion of \( \bar{Z}_A \), except at \( \Lambda = 0 \). Inserting \( \bar{Z}_A \) into the path-integral of Eq. (4.1), the partition function becomes

\[ Z'_A[J] = \int D\bar{\psi}D\psi D\sigma D\pi e^{-S_{\text{cut}}^t A - S_{\text{cut}}^b A - S_{\text{bare}} + \bar{\eta} \cdot \psi - \bar{\psi} \cdot \eta + P^{-1} \Sigma_{\sigma} - C^{-1}(\partial_\mu \sigma) \partial_\mu \sigma + P^{-1} \Sigma_{\pi} - C^{-1}(\partial_\mu \pi) \partial_\mu \pi} \]

\[ \times e^{-\int d^4x \left\{ \frac{1}{2} P^{-1}(\sigma - \Sigma_{\sigma} - P\bar{\psi}\psi)^2 + \frac{1}{2} P^{-1}(\pi - \Sigma_{\pi} - P\bar{\psi} i\gamma_5 \psi)^2 \right\}} \]

\[ = \int D\bar{\psi}D\psi e^{-S_{\text{bare}}^t A - S_{\text{cut}}^b A - S_{\text{bare}} + \bar{\eta} \cdot \psi - \bar{\psi} \cdot \eta + \Sigma_{\sigma} \cdot \bar{\psi} \psi + \Sigma_{\pi} \cdot \bar{\psi} i\gamma_5 \psi}, \]

where \( S_{\text{bare}}^t \) is given by

\[ S_{\text{bare}}^t = S_{\text{bare}} - \frac{1}{2} \int d^4x \left\{ \left( \bar{\psi} \psi + P^{-1} \Sigma_{\sigma} \right) \left[ (P^{-1} + C^{-1}\Box)^{-1} - P \right] \left( \bar{\psi} \psi + P^{-1} \Sigma_{\sigma} \right) + \left( \bar{\psi} i\gamma_5 \psi + P^{-1} \Sigma_{\pi} \right) \left[ (P^{-1} + C^{-1}\Box)^{-1} - P \right] \left( \bar{\psi} i\gamma_5 \psi + P^{-1} \Sigma_{\pi} \right) \right\}. \]

In the limit \( \Lambda \to 0 \), \( Z'_A = Z_A \) except for the field-independent constant. We modify the generating functional \( W_A[J] = \ln Z'_A[J] \) as \( W'_A = W_A + 1/(2P) \cdot (\Sigma_{\sigma}^2 + \Sigma_{\pi}^2) \). Physically meaningful quantities are not affected by this modification. Indeed, no vacuum expectation value is changed by adding an arbitrary polynomial of the sources to the generating functional, e.g. in our case,

\[ \langle \bar{\psi} \psi \rangle = \frac{dW_A}{d\Sigma_{\sigma}} \bigg|_{\Sigma_{\sigma}=0} = \frac{dW'_A}{d\Sigma_{\sigma}} \bigg|_{\Sigma_{\sigma}=0} - P^{-1} \Sigma_{\sigma} \bigg|_{\Sigma_{\sigma}=0} = \frac{dW'_A}{d\Sigma_{\sigma}} \bigg|_{\Sigma_{\sigma}=0}. \]
We rescale the sources as $J' = \{ \eta, \bar{\eta}, \Sigma'_{\sigma}, \Sigma'_{\tau} \} = \{ \eta, \bar{\eta}, P^{-1} \Sigma_{\sigma}, P^{-1} \Sigma_{\tau} \}$. The initial condition of the RGE at the ultraviolet cutoff $\Lambda = \Lambda_0$ can be given after the Legendre transformation, $\tilde{\Gamma}_{\Lambda_0}^{\text{CJT}}[\tilde{\psi}, \psi, \sigma, \pi] = \bar{\eta} \cdot \psi - \bar{\psi} \cdot \eta + \Sigma'_{\sigma} \cdot \sigma + \Sigma'_{\tau} \cdot \pi - W'[J']$. We have

$$
\tilde{\Gamma}_{\Lambda_0}^{\text{CJT}} = S_{\text{bare}} + S_{\text{cut}}^{\text{f}} + S_{\text{cut}}^{\text{b}}\Lambda_0 + \frac{1}{2} \int d^4x P^{-1} \left\{ (\sigma - P\bar{\psi}\psi)^2 + (\pi - P\bar{\psi}\gamma_5\psi)^2 \right\}
$$

$$
= S_{\text{cut}}^{\text{f+b}}\Lambda_0 + \int d^4x \left\{ \bar{\psi}(i\partial - i\mu\gamma_0 - \sigma - \pi\gamma_5)\psi + \frac{N_c}{2G_s}(\sigma^2 + \pi^2) \right\}, \quad (4.8)
$$

where $S_{\text{cut}}^{\text{f+b}} = S_{\text{cut}}^{\text{f}} + S_{\text{cut}}^{\text{b}}$. On the last line of Eq. (4.8), we set $P = G_s/N_c$. By subtracting the cutoff action, we define the effective averaging action as

$$
\tilde{\Gamma}_{\Lambda}[\tilde{\psi}, \psi, \sigma, \pi] = \int d^4x [\tilde{\psi}^{(2)}(\bar{\psi}\psi - \bar{\psi}\gamma_5\psi)] - S_{\text{cut}}^{\text{f+b}}\Lambda. \quad (4.9)
$$

In this expression there are a Yukawa interaction and an effective potential $V$, which is a function of $\sigma^2 + \pi^2$. Here, due to a lack of Lorentz symmetry, the wave-function renormalization factors of the temporal derivative and those of the spatial one differ.

The RG flow equation for the effective average action $\tilde{\Gamma}_{\Lambda}$ can be derived as done in §2. In our approximation, the RGE becomes a partial differential equation for the effective potential $V(\rho)$, where $\rho = 1/2 \cdot (\sigma^2 + \pi^2)$. If we employ the smooth cutoff function $C(q/\Lambda)$, as done in the previous section, we have to perform the momentum integration with respect to the spatial momenta numerically for each Matsubara mode. This is very time-consuming and not practical. As investigated in the previous section, the results do not depend on the cutoff scheme for a sufficiently large ultraviolet cutoff $\Lambda_0$, at least in the large $N_c$ limit. Therefore it is more promising to take the sharp cutoff limit of the RG flow equation.\(^{19}\) After doing this, we do not need the momentum integration, and therefore the computational time is reduced drastically. We cannot do the Taylor expansion in momentum components $p_{\mu}$, because the sharp cutoff induces non-analyticity at the origin of the momentum space.\(^{19,10}\) Instead, we expand the one-particle irreducible generating functional $\tilde{\Gamma}_{\Lambda}$ in the momentum scale $|p| = \sqrt{p_0^2 + \mu^2}$.

In general we may express the cutoff function $C(q/\Lambda)$ as $C^{-1}(q/\Lambda) = 1/\theta_c(|q|/\Lambda) - 1$ in terms of $\theta_c(|q|/\Lambda)$, which is a smooth regularization of the Heaviside $\theta$ function of width $\varepsilon$. In the sharp cutoff limit ($\varepsilon \to 0$), $\theta_c(|q|/\Lambda)$ reduces to $\theta(|q| - \Lambda)$, and the evolution equation for the effective average action becomes

$$
\frac{d}{d\Lambda} \tilde{\Gamma}_{\Lambda}[\bar{\psi}, \psi, \sigma, \pi] = - \frac{1}{2} \text{str} \left[ \frac{\delta(|q| - \Lambda)}{\gamma(q/\Lambda)} \tilde{\Gamma}_{\Lambda}^{(2)} \left( 1 + G\tilde{\Gamma}_{\Lambda}^{(2)} \right)^{-1} \right], \quad (4.10)
$$

\(^{19}\) The $M$th order approximation is to drop all terms beyond $O(|p|^M)$, and the lowest order approximation coincides with the local potential approximation of the Wegner-Houghton equation.\(^{19}\)
where we separate the field-independent full inverse propagator $\gamma(p, A)$ from the two-point function

$$
\left( \hat{T}^{(2)}_A [\bar{\psi}, \psi, \sigma, \pi] \right)_{pp'} = \gamma(p, A) (2\pi)^4 \delta(p + p') + \left( \hat{T}^{(2)}_A [\bar{\psi}, \psi, \sigma, \pi] \right)_{pp'},
$$

so that $\hat{T}^{(2)}_A [0, 0, 0, 0] = 0$. In Eq. (4.10), $G(p, A)$ is the infrared cutoff propagator,

$$
G(p, A) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2 C^{-1}(p/A) + \gamma(p, A)} = \frac{\theta(|p| - A)}{\gamma(p, A)}.
$$

As mentioned above, in the sharp cutoff limit, the origin of momentum space is not analytic, and the Taylor expansion in $p_\mu$ breaks down. Alternatively, one should expand in the momentum scale. The infrared cutoff function is expanded in terms of the absolute value of the external momenta $|p|$ as

$$
\theta(|p - q| - A) = \theta(q \cdot \hat{p} + |p|/2) = \theta(q \cdot \hat{p}) + \sum_{m=1}^{\infty} \frac{1}{m!} \left( \frac{|p|}{2} \right)^m \delta^{(m-1)}(q \cdot \hat{p}).
$$

Here $\hat{p}_\mu$ is a unit vector parallel to $p_\mu$. Integrating the RHS of Eq. (4.10) with respect to the internal momenta $q$, one can expand it in terms of the momentum scale $|p|$. In the finite temperature case, the momentum integral in Eq. (4.10) becomes

$$
\int d^3q \, \delta \left( |q| - \sqrt{\Lambda^2 - \omega^2_{(b \ or \ f), n}} \right) f(q \cdot \hat{p}) = 2\pi(\Lambda^2 - \omega^2_{(b \ or \ f), n}) \int_{-1}^{1} dz f(z),
$$

we can integrate with respect to the internal momenta $q$ and expand the RHS of the RGE (4.10) in terms of the external momenta $|p|$. The flow equation for the effective potential $V(\rho)$ can be derived. We have

$$
\frac{d}{dt} V = \beta_f + \beta_b,
$$

where $\beta_f$ is the contribution from fermion loop graphs, and $\beta_b$ is that from boson loop graphs. They are expressed as

$$
\beta_f = -\frac{T}{\pi^2} \sum_n \zeta_{f,n} \ln \left[ 1 + \frac{2y^2 \rho}{Z_f^2 A^2 + (Z^2_{f0} - Z_f^2) \omega^2_{f,n} - \mu^2 + 2\rho Z_{f0} \omega_{f,n} \mu} \right],
$$

$$
\beta_b = \frac{T}{4\pi^2 N_c} \sum_n \zeta_{b,n} \ln \left[ Z_b A^2 + (Z_{b0} - Z_b) \omega^2_{b,n} + V' \right] + \frac{T}{4\pi^2 N_c} \sum_n \zeta_{b,n} \ln \left[ Z_b A^2 + (Z_{b0} - Z_b) \omega^2_{b,n} + V' + 2\rho V'' \right] - \frac{T}{4\pi^2 N_c} \sum_n \zeta_{b,n} \ln \left[ Z_b A^2 + (Z_{b0} - Z_b) \omega^2_{b,n} + V' \right]_{\rho=0},
$$

where

$$
\zeta_{f,n} = \frac{1}{2} \sum_{\sigma, \pi} \int d^3q \, \delta \left( |q| - \sqrt{\Lambda^2 - \omega^2_{f,n}} \right) f(q \cdot \hat{p}),
$$

$$
\zeta_{b,n} = \frac{1}{2} \sum_{\sigma, \pi} \int d^3q \, \delta \left( |q| - \sqrt{\Lambda^2 - \omega^2_{b,n}} \right) f(q \cdot \hat{p}).
$$
where $\zeta(b \text{ or } f)_{n} = \sqrt{A^2 - \omega_{(b \text{ or } f)}^2} \cdot A^2$, and the prime $'$ operating on the effective potential $V$ denotes the derivative with respect to $\rho$. $\sum'_{n}$ is the summation over $n$ with the condition $\omega_{(b \text{ or } f)}^2, n \leq A^2$. For the wave function renormalization factor $Z_b$, we have $^*)$

$$\frac{d}{dt}Z_b = \frac{2T}{\pi^2} \sum'_{n} \left( \frac{\zeta_{f,n} Z_f^2 y^2}{Z_f^2 (A^2 - \omega_{f,n}^2)} \cdot \left( \frac{1}{2} - \frac{3}{2} Z_f^2 (A^2 - \omega_{f,n}^2) \omega_{f,n}^2 - \mu^2 - 2i Z_f \mu \omega_{f,n} \right) \right).$$

(4.17)

In this paper, we also approximate $Z(0)$ as $Z(0) = Z_0$. In the large $N_c$ limit, the boson fluctuations disappear, i.e. $\beta_b = 0$ due to the factor of $1/N_c$. However, for finite $N_c$, we must incorporate the boson loop contributions, $\beta_b$. This is a significant difference from the large $N_c$ limit. In general, the fermion contribution makes the effective potential evolve downward as the cutoff is lowered, while the boson contribution lifts the effective potential upward.$^{**) \text{ The latter is the main additional effect of a finite } N_c \text{ on the RG evolution of the effective potential, and hence also on the chiral phase structure on } (T, \mu) \text{ plane. We shall ignore the corrections to fermion’s } Z \text{ factors, } Z_f \text{ and } Z_{f0} \text{ and to the Yukawa coupling, } y,^{***) \text{ as the first step toward the exploration of the phase structure at finite temperature and finite chemical potential beyond the } 1/N_c \text{ leading order.}^{1)}$

The RGE for the effective potential is a non-linear partial differential equation. Here we do not attempt to expand it in powers of the fields, as done in Ref. 3), since if the cubic term of the field, i.e. $\rho^{3/2}$, appears in the effective potential $V(\rho)$,$^{21)}$ then the naive polynomial approximation of it cannot work. We discretized the RG flow equation in the $t$ and $\rho$ directions and solved it numerically. We applied the extended Crank-Nicholson method,$^{22), 23)}$ which is effective for a non-linear partial differential equation.

By evaluating the minimum of the effective potential, we can conclude whether it is a broken phase or a symmetric phase. If there is a non-trivial minimum at a nonzero value of $\sigma$, it is a broken phase. If the temperature (or the chemical potential) is greater than the critical value, the effective potential has a minimum at the origin, i.e. the chiral symmetry is recovered. If the temperature (or the chemical potential) is below the critical one, the effective potential has a nontrivial minimum. More precisely, in the broken phase, the effective potential evolves toward the so-called

$^*)$ In the momentum scale expansion of the sharp cutoff effective average action, there emerges a kinetic term proportional to $|p|$. The non-analytic dependence on the momenta at $p = 0$ leads to a non-locality in position space. Since the resulting physics in the limit $A \to 0$ should not suffer from this non-locality, it should be absorbed by a certain counter term. Here we simply neglect it in our approximation.

$^{**) \text{ This is understood from the signs of } \beta_f \text{ and } \beta_b.}$

$^{***) \text{ Here, we do not renormalize the boson fields } \sigma, \pi \text{ and } Z \text{ to make their } Z \text{ factor unity, i.e. } Z_b = 1, \text{ since at the ultraviolet cutoff, } Z_b \text{ should vanish. Therefore, in an ordinary sense, our physical Yukawa coupling runs due to the bosonic wave-function renormalization.} \text{ A similar approximation is made in Ref. 20).}$
Some results for the effective potentials are shown in Fig. 9. Taking into account the discussions in the previous section, we take the initial condition of RG flow equations as $A_0 = 333 T_{c0}$. Indeed, with this condition, the critical line almost coincides with that from the SDE in three-dimensional GN model for $N_c = \infty$ (see Fig. 6). The chiral phase diagram on the $(T, \mu)$ plane is shown in Fig. 10. Here, the temperature and the chemical potential are normalized by the dynamical fermion mass $m_0 = y \langle \sigma \rangle$ at zero temperature and zero chemical potential. In each $N_c$ case, the phase transition is found to be of second order or very weak first order all along the critical line, from the shape of the effective potentials at the critical points. For example, an effective potential at a critical point for $N_c = 10$ is shown in Fig. 9. The $N_c$ dependence of the critical line on the $(T, \mu)$ plane is not small. The critical temperature and the critical chemical potential become small in units of $m_0$ if $N_c$ is decreased. This is due to the boson fluctuations existing for finite $N_c$. Intuitively, we can understand this as follows. The fermionic negative corrections ($\beta_f$) to the effective potential are suppressed by a high temperature and/or a large chemical potential, but the bosonic positive corrections ($\beta_b$)
are less suppressed due to the existence of the Matsubara zero mode. Hence, if the bosonic part becomes large, that is, if \( N_c \) is small, the chiral symmetry is restored at lower temperature (smaller chemical potential) in comparison with the large \( N_c \) case. The \( N_c \) dependence of the critical temperature at \( \mu = 0 \) (\( T_{c0} \)) and that of the critical chemical potential at \( T = 0 \) (\( \mu_{c0} \)) are described in Table I.  

### Table I. The critical temperature \( T_{c0} \) and the chemical potential \( \mu_{c0} \) of the four-dimensional NJL model.

<table>
<thead>
<tr>
<th>( N_c )</th>
<th>10</th>
<th>20</th>
<th>40</th>
<th>80</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_{c0}/m_0 )</td>
<td>1.07</td>
<td>1.21</td>
<td>1.30</td>
<td>1.35</td>
<td>1.38</td>
</tr>
<tr>
<td>( \mu_{c0}/m_0 )</td>
<td>2.58</td>
<td>2.70</td>
<td>2.81</td>
<td>2.83</td>
<td>2.83</td>
</tr>
<tr>
<td>( \mu_{c0}/T_{c0} )</td>
<td>2.41</td>
<td>2.23</td>
<td>2.16</td>
<td>2.10</td>
<td>2.05</td>
</tr>
</tbody>
</table>

\[ \]

§5. Summary and discussion

We performed a non-perturbative analysis using the (Wilsonian) non-perturbative renormalization group in the NJL/GN model at finite temperature and chemical potential and explored the chiral phase structure on the \((T,\mu)\) plane.

First, we explored the phase structure of the GN model at finite temperature and chemical potential in the large \( N_c \) limit. We showed that results consistent with those for the SDE could be obtained in the framework of the NPRG method. We investigated how a large ultraviolet cutoff \( A_0 \) is needed to obtain scaled quantities. We were able to calculate the scaled critical temperature/chemical potential by setting the initial condition as \( A_0 = 333 \ T_{c0} \), and they had little cutoff-scheme dependence.

Second, we improved the approximation beyond the \( 1/N_c \) leading order in the four-dimensional NJL model. For finite \( N_c \), the bosonic fluctuations have to be incorporated. In §4, we introduced the mesonic auxiliary fields \( \sigma \) and \( \pi \), and we derived the flow equation for the effective potential of these fields \( V(\sigma^2 + \pi^2) \). The RG flow equation is then a non-linear partial differential equation for this potential. Taking account of the possibility of a first order phase transition like that in Fig. 8 and the cubic term \( \rho^{3/2} \), \( 21 \) we solved the partial differential equation for \( V(\sigma^2 + \pi^2) \) without the polynomial expansion employed in Ref. 3). For several \( N_c \), we obtained the chiral phase structure on the \((T,\mu)\) plane. For \( N_c = 10 \sim \infty \), the phase transition is second order or very weakly first order all along the critical line. The critical temperatures/chemical potentials depend strongly on \( N_c \), and become small as \( N_c \) is decreased. For \( N_c = 10 \), \( T_{c0} \) is 78% of that for \( N_c = \infty \).

Generalization of this analysis to more realistic models, QCD, seems to be straightforward, except for the treatment of the gauge invariance. By introducing the instanton induced multi-fermi operator, \( 24 \) the NPRG method can also be applied to the color-superconductor. \( 27 \)

---*\) The numerical calculations for \( N_c < 10 \) need a much finer mesh in \( t \) direction because \( \beta_b \) becomes large. This point will be the subject of a forthcoming paper.
Acknowledgements

We would like to thank W. Souma for informative discussions on the numerical computations, and K-I. Aoki, T. Suzuki, and H. Terao for useful comments. Part of the numerical computations in this work were carried out at the Yukawa Institute computer facility.

Appendix A

Threshold Functions

In this appendix we give the explicit form of the threshold functions $I(a, b; \hat{T}, \hat{\mu})$ appearing in Eq. (3.1). In $d$ dimensions, the threshold function at zero temperature and zero chemical potential is

$$I(a, b; 0, 0) = \frac{1}{2^{d-1} \pi^{d/2} \Gamma(d/2)} \int_0^\infty dq \, q^{d-3/2} \left( 1 - f^2(q) \right) f^2(q).$$  \hfill (A.1)

For finite temperature and/or chemical potential, the threshold function can be obtained by the replacements $\int dq/(2\pi)^d \to \hat{T} \sum_n \int dq/(2\pi)^{d-1}$ and $q_0 \to \hat{\omega}_{f,n} - i\hat{\mu} = (2n + 1)\pi \hat{T} - i\hat{\mu}$:

$$I(a, b; \hat{T}, \hat{\mu}) = \frac{1}{2^{d-2} \pi^{(d-1)/2} \Gamma((d-1)/2)} \hat{T} \sum_{n=0}^\infty \int_0^\infty dq \, q^{d-2/2} a \, b \left( \frac{\hat{\omega}_{f,n}^2 + q^2}{\hat{\omega}_{f,n}^2 + (q + \hat{\mu})^2} \right).$$ \hfill (A.2)

The $\hat{T} \to 0$, $\hat{\mu} \to 0$ limit of Eq. (A.2) coincides with Eq. (A.1).

Appendix B

Renormalization and the Continuum Limit for $T, \mu \neq 0$

Let us discuss renormalization and the continuum limit for finite temperature and finite chemical potential in this appendix. In the large $N_c$ limit, the RG $\beta$ functions are given by

$$\frac{d}{dt} \hat{G}_s = -(d-2) \hat{G}_s + I(\hat{T}, \hat{\mu}) \hat{G}_s^2,$$ \hfill (B.1)

$$\frac{d}{dt} \hat{T} = \hat{T},$$ \hfill (B.2)

$$\frac{d}{dt} \hat{\mu} = \hat{\mu}.$$ \hfill (B.3)

The fixed points of the above RG equations at $\hat{T} = \hat{\mu} = 0$ are the Gaussian fixed point $\hat{G}_s = 0$ and Gross-Neveu point $\hat{G}_s = (d-2)/I(\hat{T} = 0, \hat{\mu} = 0) \equiv \hat{G}_s^*$. Note that $\hat{T} = \infty$ and/or $\hat{\mu} = \infty$ are also fixed points. If $\hat{T} = \infty$ and/or $\hat{\mu} = \infty$, then only the Gaussian fixed point can be found for finite $\hat{G}_s$, since $I(\hat{T}, \hat{\mu})$ vanishes. The difference
from an ordinary zero temperature and zero chemical potential field theory is the $\tilde{T}, \tilde{\mu}$ dependence of the threshold function $I$. As will be explained below, this does not affect the ‘renormalization’.

We linearize the RG equations around the GN point.\(^1\) Letting $\hat{G}_s = \hat{G}^*_s + \delta \hat{G}_s$, we have

$$
\frac{d}{dt} \hat{G}_s = -(d - 2) \delta \hat{G}_s + 2I_0 \hat{G}^*_s \delta \hat{G}_s + \left( \tilde{T} I_0^T + \hat{\mu} I_0^\mu \right) \hat{G}^*_s,
$$

(\text{B.4})

$$
\frac{d}{dt} \tilde{T} = \hat{T},
$$

(\text{B.5})

$$
\frac{d}{dt} \hat{\mu} = \hat{\mu},
$$

(\text{B.6})

where $I_0 = I(\tilde{T} = 0, \hat{\mu} = 0)$, $I_0^T = (\partial I/\partial \tilde{T})(\tilde{T} = 0, \hat{\mu} = 0)$ and $I_0^\mu = (\partial I/\partial \hat{\mu})(\tilde{T} = 0, \hat{\mu} = 0)$. By Lorentz symmetry at $\tilde{T} = \hat{\mu} = 0$, we have $I_0^T = 0$. We also find $I_0^\mu = 0$, since the difference between $I(\tilde{T}, 0)$ and $I_0$ is $\mathcal{O}(\tilde{T}^2)$.\(^{**}\) Consequently, we find

$$
\frac{d}{dt} \begin{pmatrix} \delta \hat{G}_s \\ \tilde{T} \\ \hat{\mu} \end{pmatrix} = \begin{pmatrix} -(d - 2) + 2I_0 \hat{G}^*_s & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \delta \hat{G}_s \\ \tilde{T} \\ \hat{\mu} \end{pmatrix}.
$$

(\text{B.7})

The eigenvalues are found to be $\nu_1 = d - 2$ and $\nu_2 = \nu_3 = 1$. Here we have used $\hat{G}^*_s = (d - 2)/I_0$. We derive the relations between the bare coupling and the physical quantities using standard procedure. Let us focus on an infinitesimal neighborhood of the GN point, i.e. we consider the continuum limit of the theory $\lambda_0 \to \infty$. The critical temperature $T_c$ is completely determined by three mutually independent variables: the bare four-fermi coupling $\delta \hat{G}_s$, bare chemical potential $\hat{\mu}_0$, and the ultraviolet cutoff $\lambda_0$. The extra $\hat{\mu}_0$ dependence of the critical temperature, i.e. $T_c(\delta \hat{G}_s, \lambda_0; \hat{\mu}_0)$, does not affect the following considerations. Now, it is convenient to think that the critical temperature depends on the ‘bare’ chemical potential $\hat{\mu}_0$ through the fixed as well as RG invariant dimensionless ratio $\hat{\mu}_0/\tilde{T}_0$. As is well known, the RG preserves physical quantities. Therefore, we can realize by solving the RG equations from $\lambda_0$ to $\lambda = \lambda_0$ that the critical temperature satisfies

$$
T_c(\delta \hat{G}_s, \lambda_0; \hat{\mu}_0/\tilde{T}_0) = T_c(\lambda^{-(d-2)} \delta \hat{G}_s, \lambda_0; \hat{\mu}_0/\tilde{T}_0),
$$

(\text{B.8})

where we have used the solution

$$
\delta \hat{G}_s = Ce^{(d-2)t} = C(\lambda_0/\lambda)^{d-2}.
$$

(\text{B.9})

Here $C$ is the ‘bare’ $\delta \hat{G}_s(\lambda_0)$. The relation (\text{B.8}) can be rewritten with the dimensionless critical temperature $\tilde{T}_c(\delta \hat{G}_s; \hat{\mu}_0/\tilde{T}_0) = T_c(\delta \hat{G}_s, \lambda_0; \hat{\mu}_0/\tilde{T}_0)/\lambda_0^{***}$ as

$$
\tilde{T}_c(\delta \hat{G}_s; \hat{\mu}_0/\tilde{T}_0) = \lambda \tilde{T}_c(\lambda^{-(d-2)} \delta \hat{G}_s; \hat{\mu}_0/\tilde{T}_0).
$$

(\text{B.10})

\(^1\) For an infinitesimal neighborhood of the fixed point, this linearization is valid.

\(^{**}\) $I(\tilde{T}, 0)$ is like an approximation of the integral $I_0$ using a histogram whose band width is $2T_0$. Because the trapezoidal-rule estimate differs from the integral by the quantity on the order of the square of the width,\(^{22}\) $I(\tilde{T}, 0)$ also differs from $I_0$ by $\mathcal{O}(\tilde{T}^2)$.

\(^{***}\) By dimensional analysis, $\tilde{T}_c$ is independent of $\lambda_0$.
Hence,

\[ \hat{T}_c(\delta \hat{G}_s, \hat{\mu}_0/\hat{T}_0) \propto \delta \hat{G}_s^{1/(d-2)}. \]  

(B.11)

If one starts from the temperature dependent critical chemical potential, then one finds

\[ \hat{\mu}_c(\delta \hat{G}_s, \hat{\mu}_0/\hat{T}_0) \propto \delta \hat{G}_s^{1/(d-2)}. \]  

(B.12)

One can also find a similar relation for the fermion dynamical mass at \( T = \mu = 0 \),

\[ m_0(\delta \hat{G}_s, \hat{T}_0) \propto \delta \hat{G}_s^{1/(d-2)}. \]

We can ‘renormalize’ these quantities by taking

\[ \delta \hat{G}_s(\Lambda_0) = \frac{M}{\Lambda_0^{d-2}}, \]

where \( M \) is some finite reference mass scale. Letting \( \Lambda_0 \to \infty \), we find the continuum limit of the GN model for finite temperature and/or finite chemical potential.

The above observations can be straightforwardly generalized to other models.

References


22) For Crank-Nicholson method and the trapezoidal rule, see for example,
26) S. Seide and C. Wetterich, HD-THEP-98-20, cond-mat/9806372. 
J. Alexandre, V. Branchina and J. Polonyi, cond-mat/9803007.