Some Remarks on Null-Plane Quantization

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Some features of quantum field theory on a null plane are investigated in comparison with those of conventional formalism on a space-like hyperplane. It is shown that even in a free field theory the light-like Hamiltonian should not be introduced in a whole space from the beginning as is usually done. It is also shown that, though the cluster theorem holds good in a sense along the light-like direction, it is impossible to prove Haag's strong convergence asymptotic condition for $\tau(=x') \to \pm \infty$. In conclusion it seems impossible to construct a consistent scattering theory in the context of quantum field theory on a null plane.

1) The aim of this paper is to investigate some features of quantum field theory on a null plane which provides a possible theoretical basis for the wide applications of infinite-momentum technique to current algebra, deep inelastic lepton scattering and others. Although the work is also relevant to the recent interest in the existence of light-like charges and chiral symmetry generated by them, these topics will be studied in a separate paper.

In recent years Klauder et al., Kogut et al. and Neville et al. have investigated quantization on a null plane. Chang et al. have shown, at least formally, its equivalence to the conventional quantization on a space-like hyperplane. They have argued that Schwinger's quantum action principle provides a natural framework for the procedure, and that it supplies correct commutation relations even in the presence of interactions. They have also claimed that the covariant perturbation theory with new quantization gives the same $S$-matrix as the one in the ordinary formulation to all orders. Yabuki has also proved the equivalence based on the quantization in the Hamiltonian form of the Feynman path integral modified to take into account Dirac's second class constraints. However $S$-matrix formalism needs the existence of time translation operator, that is, Hamiltonian. In the next paragraph we shall show briefly that the light-like Hamiltonian should not be introduced in a whole space from the beginning as is done in a conventional formalism. The Hamiltonian should be defined even in a free field theory in such a way that it is introduced in a finite volume as the first step, and that one takes the limit of infinite volume after all calculations are carried out.

Besides $S$-matrix formalism needs the existence of asymptotic states which is assumed in the above papers. In paragraph 3, we shall discuss the problem and show that, if a Wightman function is smeared by a function $u \in S(\mathbb{R}^n)$, the

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cluster theorem holds good also in the light-like direction. However, one cannot prove Haag's strong convergence asymptotic condition\(^{19}\) for \(\tau(=x^+ = (x^0 + x^3)/\sqrt{2}) \rightarrow \pm \infty\), since only equal \(\tau\) Wightman functions are relevant to the proof. In conclusion one may argue that scattering theory should be formulated in the context of the conventional quantum field theory on the space-like hyperplane after all.

2) Let us first study a "Hamiltonian" operator in a free field theory. Consider a Lagrangian density

\[
\mathcal{L}(\phi, \partial^\mu \phi) = \frac{1}{2} (\partial_{\mu} \phi \partial^{\mu} \phi - m^2 \phi^2),
\]

where \(\phi\) denotes a real scalar field. If we define the stress tensor as

\[
T_{\mu\nu} = -\eta_{\mu\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi)} \partial^\nu \phi,
\]

then we obtain the energy momentum operators in the conventional field theory:

\[
P^\mu = \int d^3 x : T^{\mu\nu} :.
\]

With the help of canonical commutation relations, one can ascertain that \(P^\mu\) has the property satisfied by an infinitesimal operator of the translation.

In the case of quantization on a null plane, however, energy momentum operators are defined in place of (3) as

\[
P^\mu = \int d^2 x : T_{0}\bar{0} + \bar{0} T_{0} :,
\]

where light-like variables are expressed as

\[
x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^3), \quad x^{\perp} = (x^1, x^2),
\]

\[
\partial^\pm \phi (x) = \frac{\partial}{\partial x^\pm} \phi (x).
\]

The metric tensor is chosen as \(g^{\pm \pm} = 1\) and \(g^{\pm \perp} = g^{\perp \pm} = 0\). From (2), one has

\[
P^\mu = \int d^2 x : d^2 x^{\perp} : [ (\nabla^{\perp} \phi)^2 + m^2 \phi^2 ] :,
\]

where \(\nabla^{\perp}\) stands for a Laplacian in two dimensions. When use is made of the Klein-Gordon equation and the "canonical" commutation relations on a null plane

\[
[\phi (x), \phi (y)]_{x^-=y^-} = -\frac{i}{4} (\partial^- \delta (x^- - y^-) \partial^\perp (x^\perp - y^\perp) + \partial^\perp \delta (x^- - y^-) \partial^- (x^\perp - y^\perp)),
\]

and

\[
[\phi (x), \partial^\perp \phi (y)]_{x^-=y^-} = \frac{i}{2} \delta (x^- - y^-) \partial^\perp (x^\perp - y^\perp),
\]
one obtains
\[ i [ P_{\text{op}}, \phi(x) ] = - \frac{i}{2} \int \partial^+ \partial- \phi(y) \varepsilon(y^- - x^-) \partial^2(x^+ - y^+) dy^- d^2 y^+ . \] (9)
The right-hand side of this relation should become \( \partial^- \phi(x) \) if \( P_{\text{op}} \) is an infinitesimal operator of the translation along the direction \( x^+ \), that is, "Hamiltonian". However, unless
\[ \partial^- \phi(x^+, x^- \to \infty, x^+) + \partial^- \phi(x^+, x^- \to -\infty, x^+) = 0 , \] (10)
the integral in (9) does not reduce to \( \partial^- \phi(x) \). Can we adopt such a kind of cyclic boundary condition? From (8),
\[ \lim_{A \to 0} \left[ \phi(x^- = A + \varepsilon), \phi(y^- = A) + \phi(y^- = -A) \right]_{x^+ = y^+} = \begin{cases} - (i/2) \partial^2(x^+ - y^+) & \text{for } \varepsilon > 0 , \\ 0 & \text{for } \varepsilon < 0 , \end{cases} \] (11)
so that the quantity \( \phi(x^- \to +\infty) + \phi(x^- \to -\infty) \) and its derivative with respect to \( x^+ \) cannot be assumed to be zero operators in a consistent way. This conclusion is due to the property of the function \( \varepsilon(x^- - y^-) \).

It is evident that, if \( P_{\text{op}} \) is introduced in a finite volume as the first step, it satisfies the required relation.* It is well-known that the space-cutoff Hamiltonian must be introduced even in the space-like quantization. Can we apply such a manipulation consistently to quantization on a null plane? We have no answer at present, but it is worth while to note that there are two different features. In the space-like quantization, the space-cutoff is necessary only for an interaction Hamiltonian and the free Hamiltonian can be defined rigorously in an axiomatic sense, while in the light-like case even the free Hamiltonian must be modified. Furthermore, \( P_{\text{op}} \) defined in (3) satisfies commutation relations as an infinitesimal operator of the translation, while those in (4) does not.

3) It is very important, especially in scattering problems, how to construct a particle image in the framework of quantum field theory of an interacting system. Conventional scattering theory is formulated in terms of asymptotic fields. Haag and Ruelle have proved for the limit \( t \to \pm \infty \) the existence of both the asymptotic field \( \phi^{ex}(f) \) (ex = in or out) and the basis vector \( \Phi^{ex}(f_1, \cdots, f_n) \). The field operator \( \phi^{ex}(f) \) is expressed as
\[ \phi^{ex}(f) = i \int_{x^+ = t} f'(x) \tilde{\alpha}_{\phi^{ex}(x)} d^3 x , \] (12)
where \( \phi^{ex}(x) \) denotes a free neutral scalar field with mass \( m \), and \( f(x) \) is a solution of the Klein-Gordon equation with mass \( m \). Clearly \( \phi^{ex}(f) \) does not depend

* It has been pointed out by Maskawa that even the commutation relation between field variables takes the form
\[ [\phi(x), \phi(y)]_{x^+ = y^+} = - \frac{i}{4} \left[ \delta(x^+ - y^+) - \frac{x^- - y^-}{L} \right] \partial^2(x^+ - y^+) , \]
where \( L \) is a dimension of the volume.
on time $t$. The above theorem has been proved on the basis of the cluster theorem and the asymptotic behavior of solutions of the Klein-Gordon equation.

First we shall discuss the cluster theorem on a null plane for vacuum expectation values. Let us prove the following relation:

$$\lim_{\rho \to 0} \rho^N \int \{ W_0, \phi(x_1) \cdots \phi(x_J) \int_0^{x_J+\rho a} \cdots \int_0^{x_n+\rho a} W_0 \} u(x_1 \cdots x_n)$$

$$\times dx_1 \cdots dx_n = 0$$

(13)

for arbitrary positive integer $N$ and $a^+ = 0$, where $u \in S(R^n)$. In contrast to the hyperplane $a^2 = 0$, the null plane contains the tangential line on the light cone together with the space-like region. As far as the space-like separation on the null plane is concerned, one can simply prove the rapidly decreasing asymptotic behavior. However in regard to the characteristic light-like separation ($a^2 = 0$), such arguments cannot be applied.

Let

$$\begin{align*}
(W_0, \phi(x_1) \cdots \phi(x_J) \int_0^{x_J+\rho a} \cdots \int_0^{x_n+\rho a} W_0) \\
-W(W_0, \phi(x_1) \cdots \phi(x_J) W_0) (W_0, \phi(x_{J+1}) \cdots \phi(x_n) W_0)
\end{align*}$$

$$= T(\xi_1, \xi_2, \cdots, \xi_{J-1}, \xi_{J+1}, \cdots, \xi_{n-1})$$

$$= T(\xi, \xi),$$

(14)

where $\xi_i = x_i - x_{i+1}$, $\xi = x_i - x_{J+1}$ and $\xi = (\xi_1, \cdots, \xi_{J-1}, \xi_{J+1}, \cdots, \xi_{n-1})$. The Fourier transformation gives

$$F(\rho a) = \int \cdots \int T(\xi - \rho a, \xi) u_i(\xi, \xi) d\xi$$

$$= \int \cdots \int \widetilde{T}(p, p) \tilde{u}_i(p, p) e^{ip\rho a} dp d\rho,$$

(15)

where

$$u_i(\xi, \xi) = \int u(x_1 \cdots x_n) dx_n,$$

$$\widetilde{T}(p, p) = \frac{1}{(2\pi)^{n-2}} \int \exp \left\{ i \sum_{k=1}^{n-1} p_k \xi_k + i p \xi \right\} T(\xi, \xi) d\xi$$

$$\times d\xi,$$

(16)

$$\tilde{u}_i(p, p) = \frac{1}{(2\pi)^{n-2}} \int \exp \left\{ -i \sum_{k=1}^{n-1} p_k \xi_k - i p \xi \right\} u_i(\xi, \xi) d\xi d\xi.$$

Since $T(\xi, \xi)$ is a tempered distribution, it may be written as

$$T(\xi, \xi) = Dg(\xi, \xi),$$

(17)

where $D$ is a monomial of derivative and $g(\xi, \xi)$ is a continuous function with at most polynomial increase. Thus
\[ F(\rho a) = \int \cdots \int g(\xi - \rho a, \xi) D' u_i(\xi, \xi) \, d\xi \, d\xi \]
\[ = \int \cdots \int \tilde{g}(p, p) \tilde{v}(p, p) e^{i(p - a) \cdot \rho} \, dp \, dp, \quad (18) \]

where \( D' = (-1)^{[p]} D \), and \( \tilde{g}(p, p) \) \((\tilde{v}(p, p))\) is the Fourier transform of \( g(\xi, \xi) \) \((D' u_i(\xi, \xi))\). We parametrize \( \tilde{v}(p, p) \) as follows:

\[ \tilde{v}(p, p) = \tilde{f}_d(p^-) \tilde{f}_s(p^+) \tilde{f}_a(p^+) \tilde{v}(p) \in S(\mathbb{R}^{4n-4}), \quad (19) \]

where

\[ \lim_{d \to 0} \tilde{f}_d(p^-) \tilde{f}_s(p^+) \tilde{f}_a(p^+) \tilde{v}(p) = \delta(p^+) \delta^3(p^+). \quad (20) \]

We choose\(^*)\) for simplicity

\[ f_s(x^-) = \begin{cases} 1, & |x^-| < \rho, \\ 0, & |x^-| > \rho (1 + c), \end{cases} \quad (21) \]

and then the Fourier transform \( \tilde{f}_s(p^+) \) has the property

\[ \tilde{f}_s(p^+) = \rho \tilde{f}_1(\rho p^+). \quad (22) \]

Furthermore, since \( \tilde{g}(p, p) \) is a function with at most polynomial increase, we have

\[ |\tilde{g}(p, p)| < c (1 + \| (p,p) \|)^{n/2}, \quad (23) \]

where

\[ \| (p, p) \|^2 = \| p \|^2 + \| p \|^2 = \sum_{a=1}^{n} [(p_a)^2 + \sum_{a=1}^{n} (p_a a)^2]. \quad (24) \]

From the spectral condition, the generalized function \( \hat{T}(p, p) \) should be zero for \( p \notin G_m^+ \), where \( G_m^+ = \{ p, p^+ > \sqrt{m^2 + p^2} \} \). After integrating with respect to \( p \) and changing the variables in such a way as \( (p^-, p^+) \to (s = p^+, \rho^{-1} p^+) \), we find, for some integer \( M \),

\[ |F(\rho a)| < c' \int_{m^+}^{m^-} ds \int d^2 p \| \tilde{f}_d(p^+) \| \tilde{f}_1(\rho p^+) \| \tilde{f}_a \left( \frac{p^+ + s}{2p^+ \rho} \right) \| \tilde{f}_s(p^+) \|, \quad (25) \]

As stressed in Ref. 9), the \( \rho \)-limit \((\rho \to \infty)\) must be taken for fixed \( d \neq 0 \). Since \( \tilde{f}_d(p^-) \) \((\in S(\mathbb{R}))\) decreases faster than any power of \( (p^-)^{-1} \), we finally obtain Eq. (13).

It is noted that the condition \( u_i(\xi, \xi) \in S(\mathbb{R}^{4n-4}) \) is essential to the above proof in contrast to the space-like case. Unless \( \tilde{v}(p, p) \) vanishes faster than any power of \( (p^-)^{-1} \), one cannot prove the relation (13), whereas in the space-like case one does not need such a property of \( \tilde{v}(p, p) \) along the direction \( p^0 \). In other

\(^*)\) We would like to note that arbitrary testing function \( \tilde{f}_a(p^+) \in S(\mathbb{R}) \) cannot be used generally in the proofs concerning light-like quantities as shown in Ref. 9).
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words, $T(\xi, \xi)$ at $\xi^0 = \xi_i^0 = 0$ is a rapidly decreasing generalized function, while $T(\xi, \xi)$ at $\xi^+ = \xi_i^+ = 0$ is not. We shall see that the difference will become very important in the subsequent discussion on the asymptotic condition.

Next we shall comment on the asymptotic behavior of solutions of the Klein-Gordon equation. Let $f(x)$ be a negative-frequency solution of the Klein-Gordon equation:

$$ (\Box + m^2)f(x) = 0, \quad f \in S(R^4). \quad (26) $$

Then $f(x)$ can generally be written as

$$ f(x) = \frac{1}{(2\pi)^{3/2}} \int \delta(p^2 - m^2)\theta(p^+)e^{-ipx}g(p)\,d^4p. \quad (27) $$

It has been shown that, roughly speaking, the asymptotic behavior of $f(x)$ is determined by that of the negative frequency Pauli-Jordan function $\Delta^{(-)}(x)$, where

$$ \Delta^{(-)}(x) = \frac{i}{(2\pi)^{3/2}} \int e^{-ipx}\delta(p^2 - m^2)\theta(p^+)\,d^4p. \quad (28) $$

It is well-known that $\Delta^{(-)}(x)$ falls off like

$$ |\lambda|^{-\nu/4} \exp(-m\sqrt{\lambda}) \quad \text{for} \quad \lambda = x^2(\leq 0) \to -\infty, $$

$$ \lambda^{-\nu/4} \quad \text{for} \quad \lambda > 0 \to \infty. \quad (29) $$

Therefore we obtain ($x^+ = \tau$)

$$ |f(x)| \sim \tau^{-\nu/4} \exp(-m\sqrt{|x^-|}) \quad \text{for} \quad \tau \to \infty \quad \text{and} \quad x^- < 0, $$

$$ \sim \tau^{-\nu/4} \quad \text{for} \quad \tau \to \infty \quad \text{and} \quad x^- > 0. \quad (30) $$

Finally we shall see whether Haag's strong convergence asymptotic condition holds good for $\tau(=x^+) \to \pm \infty$ or not. Let an almost localized field $A(x)$ be a Fourier transform of the field

$$ \tilde{A}(p) = h(p^+)\tilde{g}(p), \quad (31) $$

where $h(p^+)$ is infinitely differentiable with a compact support, and

$$ h(m^2) = 1, \quad h(p^+) = 0 \quad \text{for} \quad |p^2 - m^2| > m^2. \quad (32) $$

In addition let a solution of the Klein-Gordon equation $f(x)$ be infinitely differentiable with a compact support. Then one finds that an operator

$$ A(f, \tau) = i \int_{x^+ = \tau} f(x)\partial^+ A(x)\,dx^-dx^1 \quad (33) $$

is defined on the dense set $S$ of all quasi-local states. With the definition

$$ \Phi(\tau) = A(f_1, \tau)\cdots A(f_n, \tau)\Psi_0, \quad (34) $$

Haag's condition should take the form

$$ \lim_{\tau \to \infty} \Phi(\tau) = \Phi_{\text{out}} \quad \text{and} \quad \lim_{\tau \to -\infty} \Phi(\tau) = \Phi_{\text{in}}, \quad (35) $$
with respect to the new variable $\tau$ in place of $t$. For the purpose of proof of the condition it is sufficient to show that the norm of the derivative $d\Theta(\tau)/d\tau$ is integrable over the whole range of $\tau$. Then we expand $[d\Theta(\tau)/d\tau]^2$ into a sum of products of truncated vacuum expectation values. However as shown previously, the truncated vacuum expectation values at equal $\tau$ are not rapidly decreasing generalized functions in contrast to those at equal $t$, so that there does not exist any stronger restriction than Eq. (17) for the vacuum expectation values at equal $\tau$. With the asymptotic behavior of solutions of the Klein-Gordon equation in Eq. (30) one cannot prove Haag’s asymptotic condition when $\tau \to \pm \infty$. Considering further the fact that, at fixed equal $\tau$, the cluster theorem does not hold good, it is difficult to construct a particle image as the initial condition in the limit $\tau \to \pm \infty$.

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